

## INVARIANT PROJECTIONS AND CONVOLUTION OPERATORS

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**ABSTRACT.** We prove the existence of invariant projections  $\mathcal{P}$  from the Banach space  $PM_p(G)$  of  $p$ -pseudomeasures onto  $PM_p(H)$  with  $\text{supp } \mathcal{P}(T) \subset \text{supp } T$  for  $H$  closed neutral subgroup of a locally compact group  $G$ . As a main application we obtain that every closed neutral subgroup is a set of  $p$ -synthesis in  $G$  and in fact locally  $p$ -Ditkin in  $G$ . We also obtain an extension theorem concerning the Fourier algebra.

### 1. INTRODUCTION

For a locally compact group  $G$ , let  $CV_p(G)$  be the Banach algebra of all convolution operators of  $L^p(G)$  where  $1 < p < \infty$ . In 1974 Lohoué [10] proved, for an amenable closed subgroup  $H$  of  $G$  (and  $G$   $\sigma$ -compact), the existence of a projection of  $CV_p(G)$  onto  $CV_p(H)$ . We obtain the existence of a projection  $\mathcal{P}$  for the class of closed neutral subgroups. This class includes the following situations: (i) the normalizer of  $H$  in  $G$  is open in  $G$ ; (ii)  $G \in [SIN]_H$ . We write  $G \in [SIN]_H$  if there is a fundamental system of neighborhoods  $U$  of  $e$  in  $G$  such that  $hUh^{-1} = U$  for every  $h \in H$ . The class  $[SIN]_H$  has been thoroughly investigated by Henrichs [5].

The existence of such a  $\mathcal{P}$  for  $H$  normal in  $G$  is already in [1]. In fact we are now able to show that  $\text{supp } \mathcal{P}(T) \subset \text{supp } T$  and  $\mathcal{P}(uT) = (\text{Res}_H u)\mathcal{P}(T)$  for  $u \in A_p(G)$  and  $T \in CV_p(G)$ . The existence of a  $\mathcal{P}$  with these properties is new even for  $G$  abelian and  $p \neq 2$ ; if  $p = 2$  and  $G$  is abelian the result is due to C. Herz [6]. As a main application we prove that every closed neutral subgroup is a set of  $p$ -synthesis in  $G$  and locally  $p$ -Ditkin in  $G$ . A closed subset  $F$  of  $G$  is locally  $p$ -Ditkin in  $G$  (see [3], p. 102) if for every  $\varepsilon > 0$  and every  $u \in A_p(G) \cap C_{00}(G)$  with  $\text{Res}_F u = 0$  there is  $v \in A_p(G) \cap C_{00}(G)$  with  $\text{supp } v \cap F = \emptyset$  and  $\|u - uv\|_{A_p(G)} < \varepsilon$ .  $F$  is  $p$ -Ditkin in  $G$  if for every  $\varepsilon > 0$  and every  $u \in A_p(G)$  with  $\text{Res}_F u = 0$  there is  $v \in A_p(G) \cap C_{00}(G)$  with  $\text{supp } v \cap F = \emptyset$  and  $\|u - uv\|_{A_p(G)} < \varepsilon$ . The method used in the construction of  $\mathcal{P}$  gives the following extension theorem concerning the Figà-Talamanca Herz algebra of  $G$ : given  $u \in A_p(H) \cap C_{00}(H)$ ,  $\varepsilon > 0$ , and an open subset  $\Omega$  of  $G$  with  $\text{supp } u \subset \Omega$ , there exists  $v \in A_p(G) \cap C_{00}(G)$  with  $\text{Res}_H v = u$ ,  $\|v\|_{A_p(G)} \leq \|u\|_{A_p(H)} + \varepsilon$  and  $\text{supp } v \subset \Omega$ .

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2. DEFINITIONS AND PRELIMINARIES

For the precise definitions of  $CV_p(G)$ ,  $PM_p(G)$ ,  $A_p(G)$  and all the unexplained notations we refer to [1], p. 38. Let  $H$  be an arbitrary closed subgroup of  $G$ : for  $T \in CV_p(H)$ ,  $i(T)$  denotes the image of  $T$  under the inclusion  $i$  of  $H$  in  $G$ , as defined in [2], p. 76. We assume that  $G/H$  admits a  $G$ -invariant measure. For  $k, \ell \in C_{00}(G)$  the relation

$$\langle \Lambda_{k,\ell}(T)\varphi, \psi \rangle_{L^p(H), L^{p'}(H)} = \langle T\tau_p(k *_{H} \tau_p\varphi), \tau_{p'}(\ell *_{H} \tau_{p'}\psi) \rangle_{L^p(G), L^{p'}(G)}$$

defines a linear continuous map  $\Lambda_{k,\ell}$  from  $CV_p(G)$  into  $CV_p(H)$ . We have  $\|\Lambda_{k,\ell}\| \leq \|T_H|k|\|_p \|T_H|\ell|\|_{p'}$  and  $\Lambda_{k,\ell}(PM_p(G)) \subset PM_p(H)$  ([1], pp. 38–39).

The following unpublished result is due to Roelcke. Let  $H$  be a closed neutral subgroup of  $G$  ([12]). Then  $e$  admits a fundamental system of neighborhoods  $V$  with  $HV = VH$ . Indeed let  $U$  be an open neighborhood of  $e$  in  $G$ . There is an open  $W$  with  $e \in W$ ,  $W = W^{-1}$  and  $HWH \subset UH$ . Then for  $V = (HWH) \cap U$  one has precisely  $HV = VH$ .

A one-sided version of the following lemma is already in [2], p. 71.

**Lemma 1.** *Let  $G$  be a locally compact group,  $H$  an arbitrary closed subgroup,  $U$  a neighborhood of  $e$  in  $G$ , and  $W$  a neighborhood of  $e$  in  $H$ . Then there exists  $k \in C_{00}^+(G)$  with  $\text{supp } k \subset U$ ,  $(\text{supp } k) \cap H \subset W$  and*

$$\int_H k(h)dh = 1, \int_H k(xh)dh \leq 1, \int_H \Delta_H(h^{-1})k(hx)dh \leq 1$$

for every  $x \in G$ .

*Proof.* There is an open neighborhood  $U_1$  of  $e$  in  $G$  with  $U_1 \cap H \subset W$  and an open neighborhood  $U_2$  of  $e$  in  $G$  with  $U_2 = U_2^{-1}$  and  $U_2 \subset U \cap U_1$ . Let  $K$  be a compact neighborhood of  $e$  in  $G$  with  $K = K^{-1}$  and  $K \subset U_2$ . We choose  $\varphi' \in C_{00}^+(G)$  with  $\varphi'(K) = \{1\}$  and  $\text{supp } \varphi' \subset U_2$ . Consider also  $\psi' \in C_{00}^+(G/H)$  with  $\psi' \leq 1_{G/H}$ ,  $\psi'(H) = 1$  and  $\text{supp } \psi' \subset \omega(K)$ . ( $\omega$  is the canonical map from  $G$  to  $G/H$ .) Let

$$\varphi = \frac{\varphi' + \varphi'^{\vee}}{2} \quad (\varphi'^{\vee}(x) = \varphi'(x^{-1})) \quad \text{and} \quad k'(x) = \frac{\varphi(x)\psi'(\omega(x))}{\int_H \varphi(xh)dh} \quad \text{for } x \in AH,$$

where  $A = \{y \in G \mid \varphi(y) > 0\}$  and  $k'(x) = 0$  if  $x \in G \setminus AH$ . Taking into account that  $G = AH \cup (G \setminus HK)$ , we obtain that  $k' \in C_{00}^+(G)$ ,  $\text{supp } k' \subset \text{supp } \varphi$  and  $\int_H k'(xh)dh \leq 1$  for every  $x \in G$ .

Now let  $\omega'$  be the canonical map from  $G$  onto  $H \setminus G$ . We similarly choose  $\psi'' \in C_{00}^+(H \setminus G)$  with  $\psi'' \leq 1_{H \setminus G}$ ,  $\psi''(H) = 1$ ,  $\text{supp } \psi'' \subset \omega'(K)$  and

$$k''(x) = \frac{\varphi(x)\psi''(\omega'(x))}{\int_H \Delta_H(h^{-1})\varphi(hx)dh} \quad \text{for } x \in HA,$$

$k''(x) = 0$  for  $x \in G \setminus HA$ . We obtain that  $k = \min\{k', k''\}$  satisfies all the required properties.

3. PROJECTIONS OF  $CV_p(G)$  ONTO  $CV_p(H)$ 

According to Leischner and Roelcke [9]  $H$  is said to be locally neutral in  $G$  if there is a compact neighborhood  $U_0$  of  $e$  such that for every neighborhood  $U$  of  $e$  there is a neighborhood  $V$  of  $e$  with  $(HVH) \cap U_0 \subset UH$ .

**Proposition 2.** *Let  $G$  be a locally compact group and  $H$  a closed subgroup locally neutral in  $G$ . We assume that  $G/H$  admits an invariant measure. Let  $(r_n^{(j)})_{n=1}^\infty$  be  $m$  sequences of  $L^p(H)$  and  $(s_n^{(j)})_{n=1}^\infty$  be  $m$  sequences of  $L^{p'}(H)$ . We assume that for every  $1 \leq j \leq m$ ,  $\sum_{n=1}^\infty \|r_n^{(j)}\|_p \|s_n^{(j)}\|_{p'} < \infty$ . Then, for every  $\varepsilon > 0$  and every open neighborhood  $U$ , there is  $k, \ell \in C_{00}^+(G)$  with  $\text{supp } k, \text{supp } \ell \subset U$ ,  $\|\Lambda_{k,\ell}\| \leq 1$  and*

$$\sum_{n=1}^\infty \left| \langle \Lambda_{k,\ell}(i(S)r_n^{(j)}, s_n^{(j)}) \rangle_{L^p(H), L^{p'}(H)} - \langle Sr_n^{(j)}, s_n^{(j)} \rangle_{L^p(H), L^{p'}(H)} \right| \leq \varepsilon \|S\|_p$$

for every  $S \in CV_p(H)$  and every  $1 \leq j \leq m$ .

*Proof.* To avoid unessential technical difficulties, we suppose  $m = 1$  and  $r_n^{(1)}, s_n^{(1)} \in C_{00}(H)$  for every  $n \in \mathbb{N}$ . We put  $r_n = r_n^{(1)}$  and  $s_n = s_n^{(1)}$ . There is  $N \in \mathbb{N}$  such that  $\sum_{n=1+N}^\infty \|r_n\|_p \|s_n\|_{p'} < \frac{\varepsilon_1}{8}$  where  $0 < \varepsilon_1 < \min\{1, \varepsilon\}$ . We can find a relatively compact neighborhood  $V$  of  $e$  in  $H$  with  $\|r_n - (r_n)_{h^{-1}} \Delta_H(h^{-1})\|_p, \|s_n - (s_n)_{h^{-1}} \Delta_H(h^{-1})\|_{p'} < \frac{\varepsilon_2}{2}$  for every  $h \in V$  and every  $1 \leq n \leq N$ . We have chosen  $\varepsilon_2$  such that

$$0 < \varepsilon_2 < \min \left\{ \frac{\varepsilon_1}{2^{j+2}(1 + \|r_j\|_p + \|s_j\|_{p'})} \mid 1 \leq j \leq N \right\}.$$

Let  $U_0$  be a compact neighborhood of  $e$  in  $G$  with  $U_0 = U_0^{-1}$  which guarantees the local neutrality of  $H$  in  $G$ : for every neighborhood  $U'$  of  $e$  in  $G$ , there is a neighborhood  $V'$  of  $e$  in  $G$  with  $U_0 \cap (HV'H) \subset U'H$ .

Let

$$0 < \varepsilon_3 < \frac{\varepsilon_1}{40(1 + \sum_{n=1}^\infty \|r_n\|_p \|s_n\|_{p'})}.$$

There is a compact neighborhood  $U_1$  of  $e$  in  $G$  with  $\Delta_G(x) < 1 + \varepsilon_3$  for every  $x \in U_1$ . We choose an open neighborhood  $U_2$  of  $e$  in  $G$  with  $U_2^2 \subset U_1$ . According to Lemma 1, there is  $k' \in C_{00}^+(G)$  with  $\int_H k'(h) dh = 1$ ,  $\int_H k'(hx) \Delta_H(h^{-1}) dh \leq 1$  for every  $x \in G$ ,  $\text{supp } k' \subset U_0 \cap U_2 \cap U^{-1}$  and  $\text{supp } k' \cap H \subset V$ . This implies, for every  $1 \leq n \leq N$ ,  $\|r_n - \text{Res}_H(r_n *_{H} k')\|_p \leq \frac{\varepsilon_2}{2}$  and  $\|s_n - \text{Res}_H(s_n *_{H} k')\|_{p'} \leq \frac{\varepsilon_2}{2}$ . There is a relatively compact open neighborhood  $U_3$  of  $e$  in  $G$  such that, for every  $x \in U_3$  and every  $1 \leq n \leq N$ ,  $\|(r_n *_{H} k')_{x,H} - \text{Res}_H(r_n *_{H} k')\|_p < \frac{\varepsilon_2}{2}$  and  $\|(s_n *_{H} k')_{x,H} - \text{Res}_H(s_n *_{H} k')\|_{p'} < \frac{\varepsilon_2}{2}$ . (We recall that, for a function  $f$  on  $G$  and for  $x \in G$ ,  $f_{x,H}$  denotes the function defined on  $H$  by  $f_{x,H}(h) = f(xh)$ .)

Now let  $A$  be an arbitrary open neighborhood of  $e$  in  $G$  with  $A \subset U_3$ . Then, for every  $S \in CV_p(H)$  and every  $1 \leq n \leq N$ , we have

$$\begin{aligned} & \frac{\langle i(S)1_{AH}(r_n *_{H} k'), 1_{AH}(s_n *_{H} k') \rangle_{L^p(G), L^{p'}(G)}}{\dot{m}(\omega(A))} - \langle Sr_n, s_n \rangle_{L^p(H), L^{p'}(H)} \\ &= \frac{1}{\dot{m}(\omega(A))} \int_G 1_{AH}(x)\beta(x) \left( \langle (S(r_n *_{H} k'))_{x,H}, (s_n *_{H} k')_{x,H} \rangle_{L^p(H), L^{p'}(H)} \right. \\ & \qquad \qquad \qquad \left. - \langle Sr_n, s_n \rangle_{L^p(H), L^{p'}(H)} \right) dx, \end{aligned}$$

where  $\dot{m}(\omega(A)) = \int_{\omega(A)} d\dot{x}$  and  $\beta$  is a Bruhat function for  $H, G$ .

Taking into account that we have, for  $x = uh$  with  $u \in A, h \in H$ ,

$$\begin{aligned} & \langle (S(r_n *_{H} k'))_{x,H}, (s_n *_{H} k')_{x,H} \rangle_{L^p(H), L^{p'}(H)} \\ &= \langle (S(r_n *_{H} k'))_{u,H}, (s_n *_{H} k')_{u,H} \rangle_{L^p(H), L^{p'}(H)}, \end{aligned}$$

we obtain

$$\begin{aligned} & \left| \frac{\langle i(S)1_{AH}(r_n *_{H} k'), 1_{AH}(s_n *_{H} k') \rangle_{L^p(G), L^{p'}(G)}}{\dot{m}(\omega(A))} - \langle Sr_n, s_n \rangle_{L^p(H), L^{p'}(H)} \right| \\ & \leq \|S\|_p \varepsilon_2 (1 + \|r_n\|_p + \|s_n\|_{p'}) \end{aligned}$$

for every  $1 \leq n \leq N$ .

Let  $K$  be a compact subset of  $H$  with  $e \in K$  and  $\text{supp } r_n \cup \text{supp } s_n \subset K$  for every  $1 \leq n \leq N$ . There is an open neighborhood  $U_4$  of  $e$  in  $G$  such that  $U_4^{-1} = U_4$  and  $U_4 \subset U_2 \cap U_3$ . There is an open neighborhood  $U_5$  of  $e$  in  $G$  such that  $kU_5k^{-1} \subset U_4$  for every  $k \in K$ . By assumption we can find an open neighborhood  $U_6$  of  $e$  in  $G$  such that  $(HU_6H) \cap U_0 \subset U_5H$ . This implies  $(HU_6H) \cap KU_0H \subset U_4H$ . Consider  $U_8 = (HU_7H) \cap U_4$  where  $U_7$  is an open neighborhood of  $e$  in  $G$  with  $U_7^{-1} = U_7$  and  $U_7 \subset U_6$ . Roelcke's argument gives  $(KU_0K^{-1}) \cap U_8H = (KU_0K^{-1}) \cap HU_8$ . If we take into account that  $\text{supp}(r_n *_{H} k'), \text{supp}(s_n *_{H} k') \subset KU_0K^{-1}$  ( $1 \leq n \leq N$ ), we obtain, for  $1 \leq n \leq N$ ,

$$1_{U_8H}(r_n *_{H} k') = 1_{HU_8}(r_n *_{H} k') \quad \text{and} \quad 1_{U_8H}(s_n *_{H} k') = 1_{HU_8}(s_n *_{H} k').$$

Let

$$k'' = \frac{\tau_p(1_{HU_8}k')}{\dot{m}(\omega(U_8))^{1/p}} \quad \text{and} \quad \ell'' = \frac{\tau_{p'}(1_{HU_8}k')}{\dot{m}(\omega(U_8))^{1/p'}}.$$

Then, for  $1 \leq n \leq N$ ,

$$\begin{aligned} & \langle \Lambda_{k'', \ell''}(i(S)r_n, s_n) \rangle_{L^p(H), L^{p'}(H)} \\ &= \frac{\langle i(S)1_{U_8H}(r_n *_{H} k'), 1_{U_8H}(s_n *_{H} k') \rangle_{L^p(G), L^{p'}(G)}}{\dot{m}(\omega(U_8))}. \end{aligned}$$

From the inequality

$$\begin{aligned} & \sum_{n=1}^{\infty} \left| \langle Sr_n, s_n \rangle_{L^p(H), L^{p'}(H)} - \langle \Lambda_{k'', \ell''}(i(S))r_n, s_n \rangle_{L^p(H), L^{p'}(H)} \right| \\ & \leq \sum_{n=1}^N \|S\|_p \varepsilon_2 (1 + \|r_n\|_p + \|s_n\|_{p'}) + \sum_{n=N+1}^{\infty} \left| \langle Sr_n, s_n \rangle_{L^p(H), L^{p'}(H)} \right| \\ & \quad + \sum_{n=N+1}^{\infty} \left| \langle \Lambda_{k'', \ell''}(i(S))r_n, s_n \rangle_{L^p(H), L^{p'}(H)} \right|, \end{aligned}$$

we get

$$\begin{aligned} & \sum_{n=1}^{\infty} \left| \langle Sr_n, s_n \rangle_{L^p(H), L^{p'}(H)} - \langle \Lambda_{k'', \ell''}(i(S))r_n, s_n \rangle_{L^p(H), L^{p'}(H)} \right| \\ & \leq \frac{3\varepsilon_1 \|S\|_p}{8} + \frac{\varepsilon_1 \|S\|_p \|T_H k''\|_p \|T_H \ell''\|_{p'}}{8}. \end{aligned}$$

To estimate  $\|T_H k''\|_p \|T_H \ell''\|_{p'}$ , observe that, for  $x = uh'$  with  $u \in U_8, h' \in H,$   
 $\int_H (\tau_p k')(xh)dh = \int_H k'(hu^{-1})\Delta_G(hu^{-1})^{1/p}\Delta_H(h^{-1})dh.$  But  $u^{-1} \in U_1$  implies

$$\int_H (\tau_p k')(xh)dh \leq (1 + \varepsilon_3)^{1/p} \int_H k'(hu^{-1})\Delta_G(h)^{1/p}\Delta_H(h^{-1})dh.$$

From  $\int_H k'(hu^{-1})\Delta_G(h)^{1/p}\Delta_H(h^{-1})dh = \int_{U_1 \cap H} k'(hu^{-1})\Delta_G(h)^{1/p}\Delta_H(h^{-1})dh,$  we therefore get  $\|T_H k''\|_p \leq (1 + \varepsilon_3)^{2/p}$  and similarly  $\|T_H \ell''\|_{p'} \leq (1 + \varepsilon_3)^{2/p'}$ . This gives, for every  $S \in CV_p(H),$

$$\sum_{n=1}^{\infty} \left| \langle Sr_n, s_n \rangle_{L^p(H), L^{p'}(H)} - \langle \Lambda_{k'', \ell''}(i(S))r_n, s_n \rangle_{L^p(H), L^{p'}(H)} \right| \leq \frac{7\varepsilon_1}{8} \|S\|_p.$$

Consider now  $f, g \in C_{00}^+(G/H)$  with  $\left\| f - \frac{1_{\omega(U_8)}}{\hat{m}(\omega(U_8))^{1/p}} \right\|_p, \left\| g - \frac{1_{\omega(U_8)}}{\hat{m}(\omega(U_8))^{1/p'}} \right\|_{p'}$   
 both smaller than  $\frac{\varepsilon_4}{(1 + \|T_H \tau_p k'\|_{\infty})(1 + \|T_H \tau_{p'} k'\|_{\infty})},$  where

$$0 < \varepsilon_4 < \frac{\varepsilon_3}{(1 + 2^{2/p} + 2^{2/p'}) \left( 1 + \sum_{n=1}^{\infty} \|r_n\|_p \|s_n\|_{p'} \right)}.$$

Putting  $k''' = (f \circ \omega) \tau_p k'$  and  $\ell''' = (g \circ \omega) \tau_{p'} k',$  we have successively

$$\|T_H(|k''' - k''|)\|_p, \|T_H(|\ell''' - \ell''|)\|_{p'} < \varepsilon_4$$

$\|T_H k'''\|_p < \varepsilon_4 + (1 + \varepsilon_3)^{2/p}$  and  $\|T_H \ell'''\|_{p'} < \varepsilon_4 + (1 + \varepsilon_3)^{2/p'}$ . Finally it suffices to choose  $k = \frac{k'''}{\varepsilon_4 + (1 + \varepsilon_3)^{2/p}}$  and  $\ell = \frac{\ell'''}{\varepsilon_4 + (1 + \varepsilon_3)^{2/p'}}$  to obtain

$$\sum_{n=1}^{\infty} \left| \langle Sr_n, s_n \rangle_{L^p(H), L^{p'}(H)} - \langle \Lambda_{k, \ell}(i(S))r_n, s_n \rangle_{L^p(H), L^{p'}(H)} \right| \leq \varepsilon \|S\|_p$$

with  $\|\Lambda_{k, \ell}\| \leq 1$  and  $\text{supp } k, \text{supp } \ell \subset U.$

We are now ready to prove our first result concerning projections of  $CV_p(G)$  onto  $CV_p(H)$ .

**Theorem 3.** *Let  $G$  be a locally compact group and  $H$  a closed subgroup, locally neutral in  $G$ . We assume that  $G/H$  admits an invariant measure. Then there is a linear contraction  $Q$  from  $\mathcal{L}(L^p(G))$  (the Banach space of all bounded operators of  $L^p(G)$ ) into  $\mathcal{L}(L^p(H))$  such that:*

- (1)  $Q(T) \in CV_p(H)$  for every  $T \in CV_p(G)$ ,
- (2)  $\text{supp } Q(T) \subset \text{supp } T$  for every  $T \in CV_p(G)$ ,
- (3)  $Q(i(S)) = S$  for every  $S \in CV_p(H)$ ,
- (4)  $Q(T) \in PM_p(H)$  for every  $T \in PM_p(G)$ .

*Proof.* Let  $\mathcal{A}$  be the set of all pairs  $((r_n)_{n=1}^\infty, (s_n)_{n=1}^\infty)$  where  $(r_n)_{n=1}^\infty$  is a sequence of  $L^p(H)$  and  $(s_n)_{n=1}^\infty$  is a sequence of  $L^{p'}(H)$  with  $\sum_{n=1}^\infty \|r_n\|_p \|s_n\|_{p'} < \infty$ . We denote by  $\mathcal{E}$  the set of all maps  $F$  from  $\mathcal{L}(L^p(G)) \times L^p(H) \times L^{p'}(H)$  to  $\mathbb{C}$ , linear in the first two variables, conjugate linear in the third one, and for which there is a positive real number  $C$  with  $|F(T, \varphi, \psi)| \leq C \|T\|_p \|\varphi\|_p \|\psi\|_{p'}$ . For  $F \in \mathcal{E}$  we put  $\|F\| = \sup\{|F(T, \varphi, \psi)| \mid \|T\|_p \leq 1, \|\varphi\|_p \leq 1, \|\psi\|_{p'} \leq 1\}$ . For  $k, \ell \in C_{00}(G)$ ,  $F_{k,\ell}(T, \varphi, \psi) = \langle \Lambda_{k,\ell}(T)\varphi, \psi \rangle_{L^p(H), L^{p'}(H)}$  is an element of  $\mathcal{E}$  with  $\|F_{k,\ell}\| \leq \|T_H|k|\|_p \|T_H|\ell|\|_{p'}$ .

Let  $A$  be a finite subset of  $\mathcal{A}$ ,  $B$  a finite subset of  $CV_p(H)$ ,  $U$  an open neighborhood of  $e$  in  $G$  and  $\varepsilon > 0$ . Proposition 2 implies precisely that the set

$$K_{A,B,U,\varepsilon} = \left\{ F_{k,\ell} \mid k, \ell \in C_{00}^+(G) \quad \|F_{k,\ell}\| \leq 1, \text{supp } k, \text{supp } \ell \subset U, \right. \\ \left. \sum_{n=1}^\infty |F_{k,\ell}(i(S), r_n, s_n) - \langle S r_n, s_n \rangle_{L^p(H), L^{p'}(H)}| < \varepsilon \right. \\ \left. \text{for every } ((r_n)_{n=1}^\infty, (s_n)_{n=1}^\infty) \in A \text{ and every } S \in B \right\}$$

is nonempty. Let  $\overline{K}_{A,B,U,\varepsilon}$  be the closure of  $K_{A,B,U,\varepsilon}$  with respect to the topology  $\sigma(\mathcal{E}, \mathcal{L}(L^p(G)) \times L^p(H) \times L^{p'}(H))$ . The set  $\cap \{\overline{K}_{A,B,U,\varepsilon} \mid A \text{ finite subset of } \mathcal{A}, B \text{ finite subset of } CV_p(H), 0 < \varepsilon < 1, U \text{ open neighborhood of } e \text{ in } G\}$  is not empty. Choose  $J$  in this set. There is a linear map  $Q$  from  $\mathcal{L}(L^p(G))$  to  $\mathcal{L}(L^p(H))$  with  $J(T, \varphi, \psi) = \langle Q(T)\varphi, \psi \rangle_{L^p(H), L^{p'}(H)}$  for  $T \in \mathcal{L}(L^p(G))$ ,  $\varphi \in L^p(H)$ ,  $\psi \in L^{p'}(H)$ . Clearly  $Q$  satisfies conditions (1) to (4).

Let  $H$  be a closed subgroup of  $G$  for which there is a linear map  $Q$  from  $CV_p(G)$  onto  $CV_p(H)$  satisfying conditions (3) and (4) of Theorem 3. Then  $H$  is a set of  $p$ -synthesis in  $G$ . Indeed let  $T \in PM_p(G)$  with  $\text{supp } T \subset H$  and  $u \in A_p(G)$  with  $\text{Res}_H u = 0$ . According to Lohoué ([11], Théorème 5, p. 190), there is an  $S \in CV_p(H)$  with  $i(S) = T$ . From  $Q(i(S)) \in PM_p(H)$  we deduce that  $S \in PM_p(H)$  and therefore  $\langle u, T \rangle_{A_p(G), PM_p(G)} = \langle \text{Res}_H u, S \rangle_{A_p(H), PM_p(H)} = 0$ .

**Corollary 4.** *Let  $G$  be a locally compact group and  $H$  a closed subgroup, locally neutral in  $G$ , for which  $G/H$  admits an invariant measure. Then  $H$  is a set of  $p$ -synthesis of  $G$ .*

The following extension theorem was proved by C. Herz for  $G$  second countable and  $H$  normal in  $G$  ([7], p. 115).

**Corollary 5.** *Let  $G$  be a locally compact group and  $H$  a closed subgroup as in Theorem 3. Given  $u \in A_p(H) \cap C_{00}(H)$ ,  $\varepsilon > 0$  and an open subset  $\Omega$  of  $G$  with  $\text{supp } u \subset \Omega$ , there exists  $v \in A_p(G) \cap C_{00}(G)$  with  $\text{Res}_H v = u$ ,  $\|v\|_{A_p(G)} \leq \|u\|_{A_p(H)} + \varepsilon$  and  $\text{supp } v \subset \Omega$ .*

*Proof.* According to [7], p. 115, it suffices to find  $v \in A_p(G) \cap C_{00}(G)$  with  $\text{supp } v \subset \Omega$ ,  $\|v\|_{A_p(G)} \leq \|u\|_{A_p(H)}$  and  $\|u - \text{Res}_H v\|_{A_p(H)} < \varepsilon$ .

There is  $(r_n)_{n=1}^\infty$  and  $(s_n)_{n=1}^\infty$ , two sequences of  $C_{00}(H)$ , such that  $u = \sum_{n=1}^\infty \bar{r}_n * \check{s}_n$

and  $\sum_{n=1}^\infty \|r_n\|_{L^p(H)} \|s_n\|_{L^{p'}(H)} < \infty$ . There also exists an open neighborhood  $U$  of  $e$  in  $G$  such that  $U \text{supp } u U^{-1} \subset \Omega$ . There is  $k, \ell \in C_{00}^+(G)$  with  $\text{supp } k, \text{supp } \ell \subset U$ ,  $\|\Lambda_{k,\ell}\| \leq 1$  and

$$\sum_{n=1}^\infty \left| \langle \Lambda_{k,\ell}(i(S)) \tau_p r_n, \tau_{p'} s_n \rangle_{L^p(G), L^{p'}(G)} - \langle S \tau_p r_n, \tau_{p'} s_n \rangle_{L^p(H), L^{p'}(H)} \right| \leq \varepsilon \|S\|_p$$

for every  $S \in CV_p(H)$ . There exists a unique  $v \in A_p(G)$  such that

$$\langle v, i(S) \rangle_{A_p(G), PM_p(G)} = \langle u, \Lambda_{k,\ell}(S) \rangle_{A_p(H), PM_p(H)}$$

for every  $S \in PM_p(H)$ . From

$$\begin{aligned} & \left| \langle u, S \rangle_{A_p(H), PM_p(H)} - \langle \text{Res}_H v, S \rangle_{A_p(H), PM_p(H)} \right| \\ & \leq \sum_{n=1}^\infty \left| \langle S \tau_p r_n, \tau_{p'} s_n \rangle_{L^p(H), L^{p'}(H)} - \langle \Lambda_{k,\ell}(i(S)) \tau_p r_n, \tau_{p'} s_n \rangle_{L^p(G), L^{p'}(G)} \right| \leq \varepsilon \|S\|_p \end{aligned}$$

we get  $\|u - \text{Res}_H v\|_{A_p(H)} \leq \varepsilon$  with  $\text{supp } v \subset \text{supp } k \text{supp } u (\text{supp } \ell)^{-1}$ .

*Remark.* Suppose  $G$  is abelian. According to J. Inoue [8] for every neighborhood  $U$  of  $e$  in  $G$  there is a linear isometric map  $\Omega$  of  $A_2(H)$  into  $A_2(G)$  with  $\text{Res}_H \circ \Omega = \text{id}_{A_2(H)}$  (such a map is called a linear lifting) and  $\text{supp } \Omega(u) \subset (\text{supp } u)U$ . By duality we easily derive the existence of a projection of  $PM_2(G)$  onto  $PM_2(H)$  as in Theorem 3. On the other hand B. Forrest [4] has shown that for  $G$  amenable and  $H$  closed abelian normal subgroup of  $G$  a linear lifting does not always exist. Consequently the map  $Q$  of Theorem 3 can be considered as a substitute, for  $G$  nonabelian, to the nonexistence of linear liftings of  $A_2(H)$  into  $A_2(G)$ .

#### 4. INVARIANT PROJECTIONS

In [1] we proved for  $H$  normal in  $G$  the existence of a projection  $\mathcal{P}$  of  $CV_p(G)$  onto  $CV_p(H)$  satisfying the condition  $\mathcal{P}(uT) = (\text{Res}_H u)\mathcal{P}(T)$ ; however condition (2) of Theorem 3 was out of our reach. A projection  $\mathcal{P}$  with  $\mathcal{P}(uT) = (\text{Res}_H u)\mathcal{P}(T)$  will be called an invariant projection.

**Theorem 6.** *Let  $G$  be a locally compact group and  $H$  a closed normal subgroup of  $G$ . There is an invariant projection  $\mathcal{P}$  of  $CV_p(G)$  onto  $CV_p(H)$  which also satisfies all conditions of Theorem 3.*

*Proof.* Let  $X$  be the set of all maps  $f$  from  $\mathcal{L}(L^p(G)) \times L^p(G) \times L^{p'}(G)$  to  $\mathbb{C}$  which are linear in the first two variables and conjugate linear in the third one and for which there is a positive real number  $C$  with  $|f(T, \varphi, \psi)| \leq C \|T\|_p \|\varphi\|_p \|\psi\|_{p'}$ .

For  $a, b \in S(G/H)$  ( $S(G/H)$  is the set of all bounded measurable functions on  $G/H$  with compact support),  $T \in \mathcal{L}(L^p(G))$ ,  $\varphi, \psi \in C_{00}(G)$  we define

$$g_{a,b}(T, \varphi, \psi) = \int_{G/H} \langle TB(\varphi, a)(t), B(\psi, b)(t) \rangle_{L^p(G), L^{p'}(G)} dt$$

with  $B(\varphi, a)(t)(x) = \varphi(x)a(x^{-1}t)$  for  $x, t \in G$ . Let  $K$  be a compact subset of  $G$  with  $K \cap H = \emptyset$  and  $\varepsilon > 0$ . At first we show that the set  $D_{K,\varepsilon} = \left\{ g_{a,b} \mid a, b \in S(G/H), a, b \geq 0, \|a\|_p \|b\|_{p'} < 1 + \varepsilon, \text{ there is an open neighborhood } U \text{ of } e \text{ in } G \text{ with } \text{Res}_{HU} v = 1, \text{ supp } v \cap KU = \emptyset \text{ where } v(x) = \int_{G/H} a(xy)b(y)dy \right\}$  is nonempty. We indeed choose an open neighborhood  $U_1$  of  $e$  in  $G$  with  $U_1 = U_1^{-1}$  and  $KU_1 \cap HU_1^2 = \emptyset$ . There is an open neighborhood  $U_2$  of  $e$  in  $G$  with  $U_2^{-1} = U_2$ ,  $U_2 \subset U_1$  and  $U_2H = HU_2$ . Let  $U_3$  be an open neighborhood of  $e$  in  $G$  relatively compact with  $\overline{U_3} \subset U_2$  and  $U_3H = HU_3$ . This implies that  $\overline{U_3}H = H\overline{U_3}$ . There is an open subset  $\dot{U}_4$  of  $G/H$  with  $\dot{U}_4 \supset \omega(\overline{U_3})$  and  $\dot{m}(\dot{U}_4 - \omega(\overline{U_3})) < ((1 + \varepsilon)^{p'} - 1)\dot{m}(\omega(\overline{U_3}))$ . Let  $U_5 = \omega^{-1}(\dot{U}_4) \cap U_2$ . We consider an open neighborhood  $U_6$  of  $e$  with  $U_6^{-1} = U_6$  and  $U_6\overline{U_3} \subset U_5$ . Let  $v(x) = \int_{G/H} a(xy)b(y)dy$  with  $a = \frac{1_{\omega(\overline{U_3})}}{\dot{m}(\omega(\overline{U_3}))}$  and  $b = 1_{\omega(U_6\overline{U_3})}$ .

We have

$$\|a\|_p \|b\|_{p'} \leq \left( \frac{\dot{m}(\omega(U_5))}{\dot{m}(\omega(\overline{U_3}))} \right)^{\frac{1}{p'}} < 1 + \varepsilon.$$

Suppose  $v(x) \neq 0$ ; there is  $y \in G/H$  with  $xy \in \omega(\overline{U_3})$  and  $y \in \omega(U_6\overline{U_3})$ . This implies  $x \in \overline{U_3}HU_5^{-1}$  and consequently  $x \in HU_1^2$ . Let  $x \in HU_6$  for every  $y \in \omega(\overline{U_3})$   $x^{-1}y \in \omega(U_6H\overline{U_3})$  but  $\omega(U_6H\overline{U_3}) = \omega(U_6\overline{U_3})$ ; this implies precisely  $v(x) = 1$ . Let  $g$  be an element of  $\cap \{ \overline{D}_{K,\varepsilon} \mid K \text{ compact subset of } G \text{ with } K \cap H = \emptyset, 0 < \varepsilon < 1 \}$  where  $\overline{D}_{K,\varepsilon}$  is the closure of  $D_{K,\varepsilon}$  in  $X$  with respect to the topology  $\sigma(X, \mathcal{L}(L^p(G)) \times L^p(G) \times L^{p'}(G))$ . Let  $P$  be the corresponding map of  $\mathcal{L}(L^p(G))$  to itself. It suffices to consider  $Q \circ P$  where  $Q$  is the map of Theorem 3.

**Corollary 7.** *Let  $G$  be a locally compact group,  $H$  a closed neutral subgroup of  $G$  and  $F$  a closed subset of  $H$ . If  $F$  is locally  $p$ -Ditkin ( $1 < p < \infty$ ) with respect to  $H$ , then  $F$  is locally  $p$ -Ditkin with respect to  $G$ .*

*Proof.* Let  $u \in A_p(G)$ ,  $T \in CV_p(G)$  with  $\text{Res}_F u = 0$  and  $\text{supp}(uT) \subset F$ . It suffices to show that  $uT = 0$ . There is  $S \in CV_p(H)$  such that  $uT = i(S)$ . Let  $\mathcal{P}$  be the invariant projection of Theorem 6. From  $\text{supp } \mathcal{P}(uT) \subset \text{supp } uT$  and  $\mathcal{P}(uT) = (\text{Res}_H u)\mathcal{P}(T) = S$  we deduce that  $S = 0$  and therefore  $uT = 0$ .

*Remark.* For  $H$  normal in  $G$ , this corollary is already in [3], p. 103. Our proof there was completely different: it was based on the use of  $A_p(G/H)$ . The present approach is not only more conceptual but permits us to treat the case of certain interesting nonnormal subgroups:  $H$  compact or  $G \in [SIN]_H$ .

#### REFERENCES

1. J. Delaporte and A. Derighetti, *p-pseudomeasures and closed subgroups*, Mh. Math. **119** (1995), 37–47. MR **97a**:43001
2. A. Derighetti, *Relations entre les convolutes d'un groupe localement compact et ceux d'un sous-groupe fermé*, Bull. Sc. math. 2<sup>o</sup> Série, **106** (1982), 69–84. MR **83j**:43008

3. A. Derighetti, *Quelques observations concernant les ensembles de Ditkin d'un groupe localement compact*, Mh. Math. **101** (1986), 95–113. MR **88c**:43006
4. B. Forrest, *Complemented ideals in the Fourier algebra and the Radon Nikodym property*, Trans. Amer. Math. Soc. **333** (1992), 689–700. MR **92m**:43003
5. R. W. Henrichs, *Ueber Fortsetzung positiv definiten Funktionen*, Math. Ann. **232** (1978), 131–150. MR **58**:2022
6. C. S. Herz, *The spectral theory of bounded functions*, Trans. Amer. Math. **94** (1960), 181–232. MR **24**:A1627
7. ———, *Harmonic synthesis for subgroups*, Ann. Inst. Fourier **23** (3) (1973), 91–123. MR **50**:7956
8. J. Inoue, *On a lifting problem of Fourier-Stieltjes transforms of measures*, Hokkaido Math. J. **12** (1983), 112–118. MR **84f**:43004
9. M. Leischner and W. Roelcke, *On neutral subgroups of topological groups*, Math. Ann. **282** (1988), 113–129. Math. 266. MR **90c**:22005
10. N. Lohoué, *Sur les convoluteurs de  $L^p(G)$* , C.R. Acad. Sc. Paris **278** (1974), 1543–1545. MR **49**:11151
11. ———, *Estimations  $L^p$  des coefficients de représentation et opérateurs de convolution*, Adv. Math. **38** (1980), 178–221. MR **82m**:43004
12. W. Roelcke and S. Dierolf, *Uniform structures on topological groups and their quotients*, McGraw–Hill, New York, 1981. MR **83j**:22001

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