

## STURMIAN SEQUENCES AND THE LEXICOGRAPHIC WORLD

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**ABSTRACT.** In this paper, we give a complete description for the lexicographic world  $\mathcal{L} = \{(x, y) \in \Sigma \times \Sigma : \Sigma_{xy} \neq \emptyset\} = \{(x, y) : y \geq \phi(x)\}$ , where  $\Sigma = \{0, 1\}^{\mathbf{N}}$ ,  $\Sigma_{ab} = \{x \in \Sigma : a \leq \sigma^i(x) \leq b, \text{ for all } i \geq 0\}$ ,  $\phi : \Sigma \rightarrow \Sigma$  is defined by  $\phi(a) = \inf\{b : \Sigma_{ab} \neq \emptyset\}$  and the order  $\leq$  is the lexicographic order on  $\Sigma$ . The main result is that  $b = \phi(a)$  for some  $a = 0x$  if and only if  $b$  is the Sturmian sequence  $b$  such that  $\text{Orb}(b) \subset [0x, 1x]$  and  $\sigma^i(b) \leq b$  for all  $i \geq 0$ . At the same time, a new description of Sturmian minimal sets is given. A minimal set  $M$  is a Sturmian minimal set if and only if, for some  $x \in \Sigma$ ,  $M \subset [0x, 1x]$ . Moreover, for any  $x \in \Sigma$ , there exists a unique Sturmian minimal set in  $[0x, 1x]$ .

### 1. INTRODUCTION AND THE DEFINITION OF LEXICOGRAPHIC WORLD

Let  $\Sigma = \{0, 1\}^{\mathbf{N}}$  and denote by  $\sigma$  the left (one-sided) shift on  $\Sigma$ .

First, let us define the lexicographic order on  $\Sigma$ . For any  $x, y \in \Sigma$ ,  $x < y$  iff  $x \neq y$  and for some  $n \in \mathbf{N}$ ,  $x_i = y_i$  for  $i < n$  and  $x_n = 0$ ,  $y_n = 1$ . Note that  $\Sigma$  is well-ordered and the order topology given by the above order is the same as the usual topology on  $\Sigma$ .

For any  $a, b \in \Sigma$  define  $\Sigma_{ab}$  as

$$(1) \quad \begin{aligned} \Sigma_{ab} &= \{x \in \Sigma : a \leq \sigma^i(x) \leq b, \text{ for all } i \geq 0\}, \\ \mathcal{L} &= \{(x, y) \in \Sigma \times \Sigma : \Sigma_{xy} \neq \emptyset\}. \end{aligned}$$

For any  $a, b \in \Sigma$  denote  $\{x : a \leq x \leq b\}$  by  $[a, b]$ , which will be called a (closed) interval. And for any interval  $I = [a, b]$ , denote  $\Sigma_{ab}$  by  $\Sigma_I$ . If  $\Sigma_I \neq \emptyset$ ,  $I$  will be called an  $\mathcal{L}$ -interval. Now we define a map  $\phi : \Sigma \rightarrow \Sigma$  as

$$(2) \quad \phi(a) = \inf\{b \in \Sigma : \Sigma_{a,b} \neq \emptyset\}.$$

Then  $\mathcal{L} = \{(x, y) : y \geq \phi(x)\}$  (Lemma 2.1).  $\mathcal{L}$  is called the lexicographic world and it is closely related to the bifurcation of a Lorenz-like map (see [2]). In a talk, Labarca raised the question of studying  $\phi$ . In this paper, we give a complete description for  $\phi$ . The main result is

**Theorem 1.1.**  *$b = \phi(a)$  for some  $a = 0x$  if and only if  $b$  is the Sturmian sequence  $b$  such that  $\text{Orb}(b) \subset [0x, 1x]$  and  $\sigma^i(b) \leq b$  for all  $i \geq 0$ .*

At the same time, we obtain a new description for Sturmian minimal sets.

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**Theorem 1.2.** *A minimal set  $M$  is a Sturmian minimal set if and only if, for some  $x \in \Sigma$ ,  $M \subset [0x, 1x]$ . Moreover, for any  $x \in \Sigma$ , there exists a unique Sturmian minimal set in  $[0x, 1x]$ .*

This paper is organized as following. In §2, some basic properties of  $\phi$  are given. And then we introduce the concept of order minimality in §3. It is shown that  $b = \phi(a)$  if and only if  $b$  is order minimal and  $\sigma^i(b) \leq b$  for  $i \geq 0$ . In §4, we show that  $b = \phi(a)$  is a Sturmian sequence. The main result and the characterization of Sturmian minimal sets are given in §5.

2. SOME BASIC PROPERTIES OF  $\phi$

**Lemma 2.1.**  $\Sigma_{a\phi(a)} \neq \emptyset$ . Therefore,  $\mathcal{L} = \{(x, y) \in \Sigma \times \Sigma : y \geq \phi(x)\}$ .

*Proof.* Let  $b^i \in \{b \in \Sigma : \Sigma_{a,b} \neq \emptyset\}$  and  $b^i \searrow \phi(a)$ . This means  $\exists x^i \in \Sigma_{ab^i}$ , i.e.,

$$a \leq \sigma^n(x^i) \leq b^i, \quad \forall i, n.$$

We may assume  $x^i \rightarrow x$ . Letting  $i$  tend to  $\infty$  we get

$$a \leq \sigma^n(x) \leq \phi(a).$$

This means that  $x \in \Sigma_{a\phi(a)}$ . □

**Lemma 2.2.**  $\phi$  is non-decreasing.

*Proof.* If  $a \leq b \in \Sigma$ , then  $\Sigma_{b\phi(b)} \neq \emptyset$  implies  $\Sigma_{a\phi(b)} \neq \emptyset$ . So  $\phi(a) \leq \phi(b)$ . □

**Lemma 2.3.**  $\phi$  is continuous from the left, i.e., if  $a^i$  tends to  $a$  monotonically increasingly, then  $\phi(a^i)$  tends to  $\phi(a)$ .

*Proof.* Let  $\phi(a^i) \nearrow y$  and  $x^i \in \Sigma_{a^i\phi(a^i)}$ ,  $x^i \rightarrow x$ . Then

$$a^i \leq \sigma^n(x^i) \leq \phi(a^i), \quad \forall i, n.$$

Letting  $i \rightarrow \infty$ ,

$$a \leq \sigma^n(x) \leq y.$$

So  $\Sigma_{ay} \neq \emptyset$ . This implies  $\phi(a) \leq y$ . But obviously  $\phi(a) \geq y$  since  $a \geq a^i$  and  $\phi(a) \geq \phi(a^i)$ . This proves  $\phi(a) = y$ . □

**Lemma 2.4.** 1.  $\phi(0^\infty) = 0^\infty$ .

2.  $\phi(01^\infty) = 1^\infty$ .

3.  $\phi(1x) = 1^\infty$ , for any  $x \in \Sigma$ .

*Proof.* 1. Since  $0^\infty \in \Sigma_{0^\infty, 0^\infty}$ ,  $\phi(0^\infty) = 0^\infty$ .

2. Assume  $01^\infty \leq \sigma^n(x) \leq 1^\infty, \forall n$  and  $x = 0^{n_1}1^{l_1}0^{n_2}1^{l_2} \dots$ . If  $n_1 > 0$ , then  $n_1 = 1$  and  $n_i = 0$  for  $i > 1$ . If  $n_1 = 0$  ( $l_1 > 0$ ), then either  $n_i = 0$  for all  $i$  or  $n_1 = n_2 = \dots = n_k = 0$  and  $n_{k+1} > 0$ , which implies  $n_{k+i} > 0$  for all  $i > 1$ .

So for some  $m$  such that  $\sigma^m(x) = 1^\infty$ . This means  $\phi(01^\infty) = 1^\infty$ .

3. This is a corollary of 2. and Lemma 2.2. □

**Lemma 2.5.** Let  $a = w01^\infty$  and  $a' = w10^\infty$ . Then  $\phi(a) = \phi(a')$ . Moreover if  $b = w10^m1v$  and  $m > |w|$ , then  $\phi(b) = \phi(a)$ .

*Proof.* We only prove the second conclusion. The first is just a consequence of the second.

Take  $x \in \Sigma_{a\phi(a)}$ . If  $\sigma^l(x) \geq b$  for all  $l \geq 0$ , then  $x \in \Sigma_{b\phi(a)}$ . Hence  $\phi(b) = \phi(a)$ . We will show that if there is  $l \geq 0$  such that  $a \leq \sigma^l(x) < b$ , then with  $k = |w|$  and  $n = l + 2(k + 1)$   $\sigma^n(x) = 1^\infty$ . This implies  $\phi(a) = 1^\infty$  and so  $\phi(a) = \phi(b)$ .

If  $\sigma^l(x) = a$ , then  $\sigma^n(x) = 1^\infty$ . On the other hand if  $\sigma^l(x) > a$ , then  $\sigma^l(x) = w10^m u$   $\sigma^{l+k+1}(x) = 0^m u \geq a = w01^\infty$ . Since  $m > k$ , we have  $m = k + 1$  and  $u = 1^\infty$  and hence  $\sigma^n(x) = u = 1^\infty$ .  $\square$

**Corollary 2.6.**  $\phi$  is continuous at  $a = w01^\infty$  and  $a' = w10^\infty$ .

### 3. ORDERING MINIMALITY AND THE IMAGE OF $\phi$

For any  $x \in \Sigma$  denote  $I(x) = [\underline{i(x)}, \overline{s(x)}] = \{y \in \Sigma : i(x) \leq y \leq s(x)\}$ , where  $i(x) = \inf \overline{\text{Orb}(x)}$  and  $s(x) = \sup \overline{\text{Orb}(x)}$  and the infimum and supremum are taken with respect to the order  $\leq$ . Obviously,  $I(x)$  is an  $\mathcal{L}$ -interval. A point  $x \in \Sigma$  is called order minimal if  $I(x) \supset I(y)$  then  $I(x) = I(y)$ . An  $\mathcal{L}$ -interval is called minimal if it has no proper  $\mathcal{L}$ -subinterval. Note that if  $I$  is a minimal  $\mathcal{L}$ -interval, then every point in  $\Sigma_I$  is order minimal.

**Lemma 3.1.** Every  $\mathcal{L}$ -interval contains a minimal  $\mathcal{L}$ -interval. Therefore, for any  $x \in \Sigma$  there is an order minimal  $y \in \Sigma$  such that  $I(y) \subset I(x)$ .

*Proof.* The inclusion relation on subsets gives a natural partial order on the set of all  $\mathcal{L}$ -intervals. For any totally ordering  $\mathcal{L}$ -intervals  $\{I_\alpha\}$  let  $I = \bigcap_\alpha I_\alpha$ .

$$\begin{aligned} \Sigma_I &= \bigcap_l \sigma^{-l} I = \bigcap_l \sigma^{-l} \bigcap_\alpha I_\alpha \\ &= \bigcap_\alpha \bigcap_l \sigma^{-l} I_\alpha = \bigcap_\alpha \Sigma_{I_\alpha} \neq \emptyset. \end{aligned}$$

So  $I$  is also an  $\mathcal{L}$ -interval.

According to Zorn's Lemma, the first conclusion of the lemma is correct.

Since  $I(x)$  is an  $\mathcal{L}$ -interval,  $I(x)$  contains a minimal  $\mathcal{L}$ -interval  $I$ . Then every point  $y \in \Sigma_I$  is order minimal such that  $I(y) = I \subset I(x)$ .  $\square$

**Lemma 3.2.** If  $b = \phi(a)$ , then  $\sigma^j(b) \leq b, \forall j$ .

*Proof.* Since  $b = \phi(a)$ , there exists  $x \in \Sigma_{ab}$ . Let  $I(x) = [c, d]$ . Then obviously,  $c, d \in \Sigma_{ab}$ . If  $d < b$ , then  $\phi(a) \leq d < b$ . So  $b = d \in \Sigma_{ab}$ .  $\square$

**Lemma 3.3.** If  $b = \phi(a)$ , then  $\overline{\text{Orb}(b)}$  is a minimal set.

*Proof.* For any  $x \in \overline{\text{Orb}(b)}$ , write  $I(x) = [c, d]$ . According to the definition of  $\phi$ ,  $b = d$ . This means that  $b \in \overline{\text{Orb}(x)}$ . And hence  $\overline{\text{Orb}(b)} \subset \overline{\text{Orb}(x)}$ . This proves the result.  $\square$

**Theorem 3.4.**  $b = \phi(a)$  if and only if  $\sigma^j(b) \leq b, j \geq 0$  and  $b$  is order minimal.

*Proof.* “ $\Rightarrow$ ”. We only need to show  $b$  is order minimal. Let  $I(b) = [c, b]$ . Assume  $I(b) \supset I(h) = [d, e]$  and  $I(b) \neq I(h)$ . Since  $\overline{\text{Orb}(b)}$  is a minimal set,  $c < d$  and  $b > e$ . This is contradictory to  $\phi(a) = b$ .

“ $\Leftarrow$ ”. Let  $I(b) = [c, b]$ . Then  $\phi(c) \leq b$ . If  $\phi(c) = d < b$ , then  $I(d) \subset I(b)$  but  $I(d) \neq I(b)$ . This is contradictory to the order minimality of  $b$ .  $\square$

Let  $X \subset \Sigma$  be a subshift, i.e., a closed and invariant subset of  $\Sigma$ . Denote by

$$B_n(X) = \{w \in \{0, 1\}^n : w \text{ occurs in some element of } X\}.$$

Similarly, for any  $x \in \Sigma$ , we can define

$$B_n(x) = \{w \in \{0, 1\}^n : w \text{ occurs in } x\}.$$

**Lemma 3.5.** *Assume that  $X$  is a minimal set. If there exists  $n > 0$  such that  $\#B_n = \#B_{n+1}$ , then  $X$  is trivial, i.e.,  $X$  is a periodic orbit.*

*Proof.* See [1], Theorem 2.11. □

**Proposition 3.6.** *Assume that  $X$  is a minimal set. If  $X$  is nontrivial, then  $\sigma : X \rightarrow X$  is not 1-1.*

*Proof.* Since  $X$  is nontrivial,  $\#B_n \rightarrow \infty$  as  $n$  tends to  $\infty$ . According the above lemma we have  $\#B_{n+1} > \#B_n$ . Hence there exists  $w_n \in B_n$  such that  $0w_n, 1w_n \in B_{n+1}$ . So let  $x_n = 0w_nu_n$  and  $y_n = 1w_nv_n$  be in  $X$ . We may assume that  $\lim x_n = x$  and  $\lim y_n = y$ . Then  $x \neq y$  but  $\sigma(x) = \sigma(y)$ . □

#### 4. THE IMAGE OF $\phi$ AND STURMIAN SEQUENCES

**Lemma 4.1.** *For any subsets  $A, B \subset \Sigma$  we have  $\sigma(A \cap \sigma^{-1}(B)) = \sigma(A) \cap B$ .*

*Proof.* Trivial. □

**Lemma 4.2.** *For any  $x \in \Sigma$ ,  $I = [0x, 1x]$  is an  $\mathcal{L}$ -interval.*

*Proof.* We only have to show that  $\Lambda_n = \bigcap_{i \geq 0}^n \sigma^{-i}I$  is nonempty. By induction we may assume that  $\Lambda_n \neq \emptyset$ . Now let us show  $\Lambda_{n+1} \neq \emptyset$ .

$$\begin{aligned} \Lambda_{n+1} &= \bigcap_{i \geq 0}^{n+1} \sigma^{-i}I \\ &= I \cap \sigma^{-1}\left(\bigcap_{i=0}^n \sigma^{-i}I\right) \\ &= I \cap \sigma^{-1}\Lambda_n. \end{aligned}$$

So, by the above lemma,

$$\begin{aligned} \sigma(\Lambda_{n+1}) &= \sigma(I) \cap \Lambda_n \\ &= \sigma([0x, 01^\infty] \cup [10^\infty, 1x]) \cap \Lambda_n \\ &= ([x, 1^\infty] \cup [0^\infty, x]) \cap \Lambda_n \\ &= \Sigma \cap \Lambda_n = \Lambda_n \\ &\neq \emptyset. \end{aligned}$$

□

**Corollary 4.3.** *If  $x$  is order minimal, then*

(BC) *for any  $n$  and  $B \in \{0, 1\}^n$ ,  $0B0$  and  $1B1$  cannot both occur in  $b$ . In particular, only one of  $00$  and  $11$  can occur in  $b$ .*

The condition (BC) in the above corollary will be called the Block Condition. Note that according to [1], Lemma 3.06, (BC) is equivalent to the Sturmian block condition there. We have the following simple property for the BC.

**Lemma 4.4.**  $b \in \Sigma$  satisfies the BC if and only if there exists  $z \in \Sigma$  such that  $\text{Orb}(b) \subset [0z, 1z]$ .

*Proof.* Assume that  $\text{Orb}(b) \subset [0z, 1z]$  for some  $z \in \Sigma$ . If both  $0B0$  and  $1B1$  occur in  $b$  for some  $B \in \{0, 1\}^n$ , then there exist  $x, y \in \Sigma$  such that  $0B0x, 1B1y \in \text{Orb}(b) \subset [0z, 1z]$ . Hence,  $B0x \geq z \geq B1y$ , which is absurd.

Now, assume that  $b$  satisfies the BC. Suppose that  $I(b) = [0x, 1y]$  and  $x < y$ . Let  $i$  be the smallest integer such that  $x_i = 0, y_i = 1$ . Let  $B = x_1 \cdots x_{i-1} = y_1 \cdots y_{i-1}$ . Then both  $0B0$  and  $1B1$  occur in  $b$ , which is contradictory to the BC.  $\square$

Then let us recall the definition of Sturmian sequence.

For any  $\alpha \in (0, 1)$ , let  $J_\alpha = [0, \alpha)$  and  $\bar{J}_\alpha = (0, \alpha]$ . For any  $x \in [0, 1]$  define  $x^\alpha, \bar{x}^\alpha \in \Sigma$  as follows:

$$\begin{aligned} x^\alpha(i) &= \begin{cases} 1, & x + (i - 1)\alpha \in J_\alpha \pmod{1}, \\ 0, & \text{otherwise;} \end{cases} \\ \bar{x}^\alpha(i) &= \begin{cases} 1, & x + (i - 1)\alpha \in \bar{J}_\alpha \pmod{1}, \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

As usual let  $M_\alpha = \{x^\alpha, \bar{x}^\alpha : x \in [0, 1]\}$ , and then let  $M_0 = \{0^\infty\}$  and  $M_1 = \{1^\infty\}$ . For any  $\alpha \in [0, 1]$ , a sequence in  $M_\alpha$  is called a Sturmian sequence.

**Lemma 4.5** ([1]). 1)  $M_\alpha$  is a minimal set for any  $\alpha \in [0, 1]$ .

2)  $x \in \Sigma$  is a Sturmian sequence if and only if  $\text{Orb}(x)$  is a minimal set and  $x$  satisfies the block condition.

So it is clear that

**Theorem 4.6.**  $b \in \Sigma$  is order minimal and  $\sigma^i b \leq b$  for  $i \geq 0$ . Then  $b$  is a Sturmian sequence. Hence if  $b = \phi(a)$ , then  $b$  is a Sturmian sequence.

**Corollary 4.7.**  $\Lambda$  is a minimal set and contained in  $[0x, 1x]$  for some  $x \in \Sigma$  if and only if  $\Lambda$  is a Sturmian minimal set.

### 5. STURMIAN SEQUENCES AND THE IMAGE OF $\phi$ (CONTINUED)

In the last section we have shown that elements in the image of  $\phi$  are all Sturmian sequences. In this section, we show the converse: for every  $M_\alpha$ , the largest element  $\bar{\alpha}^\alpha$  is in the image of  $\phi$ . See Theorem 5.5.

For any  $n \geq 1$  define

$$\begin{aligned} \Lambda_n^0 &= \{x \in \Sigma : x = 10^{n_1}10^{n_2}1 \cdots, n_i \in \{n, n + 1\}\}, \\ \Lambda_n^1 &= \{x \in \Sigma : x = 1^{n_1}01^{n_2}0 \cdots, n_i \in \{n, n + 1\}\}. \end{aligned}$$

And

$$\Lambda_n = \Lambda_n^0 \cup \Lambda_n^1, \quad \Lambda = \bigcup_{n \geq 1} \Lambda_n.$$

And define a natural mapping  $f_n^i : \Lambda_n^i \rightarrow \Sigma$  as follows:

$$\begin{aligned} f_n^0(10^{n_1}10^{n_2}1 \cdots) &= (n + 1 - n_1)(n + 1 - n_2) \cdots, \\ f_n^1(1^{n_1}01^{n_2}0 \cdots) &= (n_1 - n)(n_2 - n) \cdots. \end{aligned}$$

When no confusion will result, we may write  $f_n^i$  as  $f_n$  or simply  $f$ . Note that there is one possible confusion for  $f$ , i.e.,  $f_n(1^{n_1}01^{n_2}0 \cdots) = 00 \cdots$  but  $f_{n-1}(1^{n_1}01^{n_2}0 \cdots) = 11 \cdots$ .

*Remark 5.1.*  $f_n^i$  is an order-preserving homeomorphism (1-1, onto and continuous) and maps periodic points to periodic points (but strictly decreases the periods).

**Lemma 5.2.** *If  $b$  satisfies BC,  $b \neq 0^\infty, 10^\infty, 1^\infty$  and  $\sigma^i b \leq b$  for  $i \geq 0$ , then  $b \in \Lambda$ .*

*Proof.* We may assume that 00 does not occur in  $b$ . (The other case can be discussed similarly.)

Since  $b \neq 1^\infty$ ,  $\sigma^i b \leq b$  for all  $i \geq 0$  and 00 does not occur in  $b$ ,  $b$  has the form  $1^{n_1}01^{n_2}0 \dots$ ,  $n_i > 0$ . Obviously,  $n_1 \geq n_i$  for all  $i > 0$ . Furthermore, we have  $n_i \geq n_1 - 1$ . In fact, if  $n_i \leq n_1 - 2$ , then  $01^{n_i}0$  and  $11^{n_i}1 \in B_{n_i+2}(b)$ , which is impossible since  $b$  satisfies the BC.  $\square$

We need the following lemma (see [3], Theorem 8.1).

**Lemma 5.3.**  *$f$  maps Sturmian sequences to Sturmian sequences.*

**Lemma 5.4.** *For any  $x \in \Sigma$ , there exists a unique Sturmian minimal set  $M_\alpha \subset [0x, 1x]$ .*

*Proof.* The existence is a consequence of Lemma 3.1, Lemma 4.2 and Theorem 4.6.

If there are two different Sturmian minimal sets  $M_\alpha, M_\beta \subset [0x, 1x]$  with  $\alpha \neq \beta$ , let  $a = \sup M_\alpha, b = \sup M_\beta$  so that  $\sigma^i a \leq a$  and  $\sigma^i b \leq b$  for  $i \geq 0$ . We may assume that  $a < b$  (since  $M_\alpha \cap M_\beta = \emptyset, a \neq b$ ). If  $b = 1z$ , then  $z \leq x$  and so  $0z \leq 0x$ . So, without loss of generality, let  $b = 1x$ . Since  $a < b$ ,  $\overline{\text{Orb}(a)} \subset [0x, 1x]$ . So if  $1y \in \overline{\text{Orb}(a)}$ , then  $1y < 1x$  and hence  $0y < 0x$  which implies  $0y \notin \overline{\text{Orb}(a)}$  and by Proposition 3.6  $\text{Orb}(a)$  is a periodic orbit. Since  $a < b, b \neq 1^\infty$ . We may assume that 00 does not occur in  $b$  (another case can be treated similarly). Let  $b = 1^{n_1}01^{n_2}0 \dots$ . If  $n_1 = 1$ , then 00 must occur in  $a$  and since  $a$  is periodic, this will be contradictory to  $\sigma^i(a) \geq 0x$ . So  $n = n_1 - 1 > 0$ . We claim that  $a, b \in \Lambda_n^1$ .

In fact, since  $\sigma^j b \leq b$  for  $j \geq 0, n_i \leq n + 1$ . And since  $\sigma^j b \geq 0x$  for  $j \geq 0, n_i \geq n$ . So  $b \in \Lambda_n^1$ . Since  $\sigma^i a \geq 0x, 00$  could not occur in  $a$ . Since  $a$  is periodic,  $a$  has the form  $1^{m_1}01^{m_2}0 \dots$ . Since  $\sigma^i a \leq a, m_i \leq m_1$  for  $i \geq 1$ . And  $a < b$  implies  $m_1 \leq n + 1$  and hence  $m_i \leq n + 1$  for  $i \geq 1$ . And  $\sigma^i a > 0x$  implies  $m_i \geq n$  for  $i \geq 2$ . And hence  $m_1 \geq n$ .

Let  $f = f_n^1 : \Lambda_n^1 \rightarrow \Sigma$ . Since  $f$  maps Sturmian sequences to Sturmian sequences and periodic sequences to periodic sequences,  $f(a), f(b)$  are Sturmian sequences and  $f(a)$  is periodic.

For any  $z = 1^{s_1}01^{s_2}0 \dots \in \Lambda_n^1$ , denote by  $l_i(z) = s_1 + s_2 + \dots + s_i + i$  for  $i \geq 1$  and  $l_0(z) = 0$ .

Let  $f(b) = 1y$ . Then  $f(x) = 0y$ . In the following we will show that for  $i \geq 0, \sigma^i f(a), \sigma^i f(b) \in [0y, 1y]$ .

Since  $f$  is order-preserving,  $\sigma^i f(b) = f(\sigma^{l_i(b)} b) \leq f(b)$  for  $i \geq 0$ . Since  $\sigma^{l_i(b)-1} b \geq 0x, \sigma^{l_i(b)} b \geq x$ , which implies  $\sigma^i f(b) = f(\sigma^{l_i(b)} b) \geq f(x) = 0y$  for  $i > 0$ . But, obviously, we have  $b > x (\in \Lambda_n^1)$  and hence  $f(b) > f(x) = 0y$ .

Similarly, we can show that  $\sigma^i f(a) \leq 1y$  for  $i \geq 0$  and  $\sigma^i f(a) \geq 0y$  for  $i > 0$ . But since  $f(a)$  is periodic, we have  $f(a) \geq 0y$ .

Since the period of  $f(a)$  is strictly less than that of  $a$ , if we take the pair  $(a, b)$  such that the period of  $a$  is the smallest, this will be a contradiction.  $\square$

**Theorem 5.5.** *Let  $b \in \Sigma$  be a Sturmian sequence. Then  $b$  is order minimal.*

*Proof.* Since  $b$  satisfies the Block Condition,  $I(b) \subset [0z, 1z]$  for some  $z \in \Sigma$ . Assume that  $b$  is not order minimal. Then there is an order minimal  $c \in \overline{\text{Orb}(b)}$  such that

$\sigma^i c \leq c$  for  $i \geq 0$ ,  $I(c) \subset I(b)$  and  $I(c) \neq I(b)$ . So  $c$  is a Sturmian sequence according to Theorem 4.6. According to the above lemma, this is impossible.  $\square$

**Corollary 5.6.** *Assume that  $b \in \Sigma$ . The following conditions are equivalent.*

- 1)  $b = \phi(a)$  for some  $a \in \Sigma$ .
- 2)  $b$  is order minimal and  $\sigma^i b \leq b$  for all  $i \geq 0$ .
- 3)  $b$  is Sturmian sequence and  $\sigma^i b \leq b$  for all  $i \geq 0$ .

So now the mapping  $\phi$  can be calculated as following.

**Theorem 5.7.** *If  $a = 1x \in \Sigma$ , then  $\phi(a) = 1^\infty$ . If  $a = 0x$ , then  $\phi(a)$  is equal to the Sturmian sequence  $b$  such that  $\text{Orb}(b) \subset [0x, 1x]$  and  $\sigma^i(b) \leq b$  for all  $i \geq 0$ .*

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