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COTLAR-STEIN LEMMA AND THE Tb THEOREM

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ABSTRACT. In this note we give a generalization of the Cotlar-Stein lemma and using this lemma we give a new proof of a special case of the Tb theorem which, in general, was proved by David, Journé and Semmes.

1. INTRODUCTION

Let Ω denote the subset of $\mathbf{R}^n \times \mathbf{R}^n$ defined by $x \neq y$ for all $x, y \in \mathbf{R}^n$. A continuous linear operator $T : C_0^\eta(\mathbf{R}^n) \to (C_0^\eta)'(\mathbf{R}^n)$ for $\eta > 0$ is said to be a singular integral if there exist an exponent $\varepsilon \in (0, 1]$, a constant C and a function K defined on Ω, such that

$$|K(x,y)| \leq C|x-y|^{-n}, \tag{1.1}$$

$$|K(x,y) - K(x,y')| \leq C|y-y'|^\varepsilon |x-y|^{-n-\varepsilon} \quad \text{for } |y-y'| \leq \frac{1}{2}|x-y|, \tag{1.2}$$

$$|K(x,y) - K(x',y)| \leq C|x-x'|^\varepsilon |x-y|^{-n-\varepsilon} \quad \text{for } |x-x'| \leq \frac{1}{2}|x-y|, \tag{1.3}$$

and, finally,

$$T(f)(x) = \int_{\mathbf{R}^n} K(x,y) f(y)\, dy \tag{1.4}$$

for every function $f \in C_0^\eta(\mathbf{R}^n)$ and every $x \notin$ support of f.

Since the Fourier transform and Plancherel's formula are strictly limited to convolution operators, a basic problem asked first by Zygmund was how should one attack these nonconvolution operators? M. Cotlar provided an interesting approach to such problems. This approach is the following lemma. The first version of this lemma is due to M. Cotlar and the second version was found independently by M. Cotlar and E. Stein. We now state the well-known Cotlar-Stein lemma.

Cotlar-Stein lemma. *Let $T_j : L^2(\mathbf{R}^n) \to L^2(\mathbf{R}^n)$, $j \in \mathbf{Z}$, be bounded linear operators with adjoints T_j^*. Suppose that there exists a sequence $\omega(j) \geq 0$, $j \in \mathbf{Z}$, such that $\sum_{-\infty}^{\infty} (\omega(j))^{\frac{1}{2}} < \infty$, and*

$$\|T_j^* T_k\| \leq \omega(j-k), \quad \textit{for all } j, k \in \mathbf{Z}, \tag{1.5}$$

$$\|T_j T_k^*\| \leq \omega(j-k), \quad \textit{for all } j, k \in \mathbf{Z}. \tag{1.6}$$

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Then the series $\sum_{-\infty}^{\infty} T_j$ *converges in* $L^2(\mathbf{R}^n)$ *and* $\|T\| \leq \sum_{-\infty}^{\infty}(\omega(j))^{\frac{1}{2}}$, *where* $T = \sum_{-\infty}^{\infty} T_j$.

Using the first version of this lemma, Cotlar first gave a new proof of the boundedness of the Hilbert transform on $L^2(\mathbf{R})$ without using the Fourier transform and Plancherel's formula ([C]). A second application of this lemma came in [KS] when Knapp and Stein were studying singular integral operators on nilpotent Lie groups. The Fourier method did not work but the Cotlar-Stein lemma was a successful tool.

A third application was obtained by A. Calderón and R. Vaillancourt ([CV]). They studied pseudo-differential operators with the symbol $\sigma(x,\xi)$. They cut $\sigma(x,\xi)$ into elementary blocks defined by $\sigma(x,\xi)\varphi_j(x,\xi)$, $j \in \mathbf{Z}$. The cut-off functions $\phi_j(x,\xi) \in C_0^\infty(\mathbf{R}^n \times \mathbf{R}^n)$ have the sharpest localization in the phase space. Then the Cotlar-Stein lemma applies to the corresponding operators defined by these elementary symbols $\sigma(x,\xi)\varphi_j(x,\xi)$.

The proof of the remarkable $T1$ theorem of David and Journé was another application of the Cotlar-Stein lemma ([DJ]). Using paraproduct operators, they reduced the proof of the original $T1$ theorem to the case where T satisfies the conditions of the $T1$ theorem and, moreover, $T1 = T^*1 = 0$. To be precise, let P_j be operators defined by convolution with the kernel $\theta_j(x) = 2^{nj}\theta(2^j x)$ where $\theta \in \mathcal{S}(\mathbf{R}^n)$ and $\int \theta(x)\,dx = 1$. Put $Q_j = P_{j+1} - P_j$. They decomposed T into

$$T = \sum_{-\infty}^{\infty}(P_{j+1}TP_{j+1} - P_jTP_j) = \sum_{-\infty}^{\infty} Q_jTP_j + \sum_{-\infty}^{\infty} P_jTQ_j + \sum_{-\infty}^{\infty} Q_jTQ_j. \tag{1.7}$$

Now the Cotlar-Stein lemma applies to $\sum_{-\infty}^{\infty} Q_jTP_j$, $\sum_{-\infty}^{\infty} P_jTQ_j$ and $\sum_{-\infty}^{\infty} Q_jTQ_j$, respectively.

The $T1$ theorem was generalized by the so-called Tb theorem ([MM], [DJS]). However, the proof of the Tb theorem was very complicated. One does not know how the Cotlar-Stein lemma can be applied to prove the Tb theorem, and one does not even know if it is possible.

In this note, we first give a generalization of the Cotlar-Stein lemma. Using this generalization of the Cotlar-Stein lemma, we give a new proof of a special case of the Tb theorem.

2. A generalized Cotlar-Stein lemma

The generalized Cotlar-Stein lemma. *Suppose that* b *is a bounded real-valued function and* $b \geq \delta > 0$. *Let* $T_j : L^2(\mathbf{R}^n) \to L^2(\mathbf{R}^n)$, $j \in \mathbf{Z}$, *be bounded linear operators with adjoints* T_j^*, *and let* M_b *be the multiplication operator by the function* b. *Suppose that there exists a sequence* $\omega(j) \geq 0$, $j \in \mathbf{Z}$, *such that* $\sum_{-\infty}^{\infty}(\omega(j))^{\frac{1}{2}} < \infty$, *and*

$$\|T_j^* M_b T_k\| \leq \omega(j-k), \quad \textit{for all } j,k \in \mathbf{Z}, \tag{2.1}$$

$$\|T_j M_b T_k^*\| \leq \omega(j-k), \quad \textit{for all } j,k \in \mathbf{Z}. \tag{2.2}$$

Then the series $\sum_{-\infty}^{\infty} T_j$ *converges in* $L^2(\mathbf{R}^n)$ *and* $\|T\| \leq \delta^{-1}\sum_{-\infty}^{\infty}(\omega(j))^{\frac{1}{2}}$ *where* $T = \sum_{-\infty}^{\infty} T_j$.

Proof. We first consider the finite sums $S = S_N = \sum_{-N}^{N} T_j$. Since b is a real-valued function and $b \geq \delta > 0$, it is easy to see that

$$\|S\|^2 = \|S^*S\| \leq \delta^{-1}\|S^*M_bS\|.$$

Since S^*M_bS is a self-adjoint operator, for every integer $n \geq 1$,

$$\|S\|^{2^n} \leq (\delta^{-1})^{(2^n-1)}\|S^*M_bSM_bS^*M_bS\cdots M_bS^*M_bSM_bS^*M_bS\|. \tag{2.3}$$

We write

$$S^*M_bSM_bS^*M_bS\cdots M_bS^*M_bSM_bS^*M_bS = \sum_{j_1=-N}^{N}\sum_{k_1=-N}^{N}\sum_{j_2=-N}^{N}\sum_{k_2=-N}^{N}\sum_{j_{2^{n-1}}=-N}^{N}\sum_{k_{2^{n-1}}=-N}^{N} T^*_{j_1}M_bT_{k_1}M_bT^*_{j_2}M_bT_{k_2}M_b \cdots T^*_{j_{2^{n-1}}}M_bT_{k_{2^{n-1}}}.$$

We now follow [S] and estimate $\|T^*_{j_1}M_bT_{k_1}M_bT^*_{j_2}M_bT_{k_2}M_b\cdots T^*_{j_{2^{n-1}}}M_bT_{k_{2^{n-1}}}\|$ in two ways, and then take the geometric mean of the two estimates to get

$$\begin{aligned}\|T^*_{j_1}M_bT_{k_1}M_bT^*_{j_2}M_bT_{k_2}M_b\cdots T^*_{j_{2^{n-1}}}M_bT_{k_{2^{n-1}}}\| &\leq (\omega(0))^{\frac{1}{2}}(\omega(j_1-k_1))^{\frac{1}{2}}(\omega(k_1-j_2))^{\frac{1}{2}} \\ &\quad \cdots(\omega(k_{2^{n-1}-1}-j_{2^{n-1}}))^{\frac{1}{2}}(\omega(j_{2^{n-1}}-k_{2^{n-1}}))^{\frac{1}{2}}.\end{aligned} \tag{2.4}$$

Let $\sigma = \sum_{-\infty}^{\infty}(\omega(j))^{\frac{1}{2}}$. Putting (2.3) and (2.4) together gives

$$\|S\|^{2^n} \leq \delta^{(1-2^n)}(2N+1)(\omega(0))^{\frac{1}{2}}\sigma^{2^n-1}$$

or

$$\|S\|^{2^n} \leq \delta^{(2^{-n}-1)}(2N+1)^{2^{-n}}(\omega(0))^{2^{-n+1}}\sigma^{1-2^{-n}}.$$

Letting n tend to infinity, we obtain

$$\|S\| \leq \delta^{-1}\sigma.$$

This proves the generalized Cotlar-Stein lemma.

3. A special case of the Tb theorem

A special case of the Tb theorem. *Suppose that b is a bounded real-valued function, $b \geq \delta > 0$ and $T : C_0^\eta(\mathbf{R}^n) \to (C_0^\eta)'(\mathbf{R}^n)$ for $\eta > 0$ is a singular integral. Suppose further that $T(b) = T^*(b) = 0$ and the operator M_bTM_b satisfies the following weak boundedness property:*

$$|\langle M_bTM_b(f), g\rangle| \leq Ct^{n+2\eta}\|f\|_\eta\|g\|_\eta \tag{3.1}$$

for all $\eta > 0$, all cubes Q with diameter at most $t > 0$ and all $f, g \in C_0^\eta(\mathbf{R}^n)$ supported in Q, where

$$\|f\|_\eta = \sup_{x\neq y}\frac{|f(x)-f(y)|}{|x-y|^\eta}$$

and $C_0^\eta(\mathbf{R}^n)$ is a collection of all f with $\|f\|_\eta < \infty$ and having compact support.

Then T is bounded on $L^2(\mathbf{R}^n)$.

To prove the above special case of the Tb theorem, we follow [MC] and first prove the following lemma.

Lemma (3.2). *Suppose that b is a bounded real-valued function and $b \geq \delta > 0$. Let $\omega(x,y) = C(1+|x-y|)^{-n-\varepsilon}$, for some $C, \varepsilon > 0$, and $\omega_j(x,y) = 2^{jn}\omega(2^j x, 2^j y)$. We suppose that*

$$T_j(f)(x) = \int_{\mathbf{R}^n} T_j(x,y) f(y)\, dy$$

where

$$|T_j(x,y)| \leq \omega_j(x,y). \tag{3.3}$$

Furthermore, there is an exponent $0 < \beta < \varepsilon$, such that

$$|T_j(x,y) - T_j(x,y')| \leq 2^{j\beta}|y-y'|^\beta \{\omega_j(x,y) + \omega_j(x,y')\}, \tag{3.4}$$

$$|T_j(x,y) - T_j(x',y)| \leq 2^{j\beta}|x-x'|^\beta \{\omega_j(x,y) + \omega_j(x',y)\}, \tag{3.5}$$

and $T_j(x,y)$ also satisfy

$$\int T_j(x,y) b(y)\, dy = 0 \quad \text{for all } x \in \mathbf{R}^n, \tag{3.6}$$

$$\int T_j(x,y) b(x)\, dx = 0 \quad \text{for all } y \in \mathbf{R}^n. \tag{3.7}$$

Then there exists a constant $C(\beta, \varepsilon, n, b)$ such that

$$\left\| \sum_{-\infty}^{\infty} T_j \right\| \leq C(\beta, \varepsilon, n, b). \tag{3.8}$$

To prove Lemma (3.2), first, it is easy to see that the kernel of $T_j^* M_b T_k$ is

$$A_{j,k}(x,y) = \int \overline{T}_j(z,x) b(z) T_k(z,y)\, dz,$$

while that of $T_j M_b T_k^*$ is

$$B_{j,k}(x,y) = \int T_j(x,z) b(z) \overline{T}_k(y,z)\, dz.$$

We now claim that

$$\|T_j^* M_b T_k\| \leq C 2^{-(\varepsilon-\beta)|j-k|}, \tag{3.9}$$

and

$$\|T_j M_b T_k^*\| \leq C 2^{-(\varepsilon-\beta)|j-k|}. \tag{3.10}$$

Assuming (3.9) and (3.10) for the moment, the proof of Lemma (3.2) now follows from the generalized Cotlar-Stein lemma with $\omega(j) = C2^{-(\varepsilon-\beta)|j|}$ for $0 < \beta < \varepsilon$.

We now prove (3.9) and (3.10). In fact, we have the following estimates:

$$|A_{j,k}(x,y)| \leq C 2^{-(\varepsilon-\beta)|j-k|} \frac{2^{-(j\wedge k)(\varepsilon-\beta)}}{(2^{-(j\wedge k)} + |x-y|)^{n+(\varepsilon-\beta)}}, \tag{3.11}$$

$$|B_{j,k}(x,y)| \leq C 2^{-(\varepsilon-\beta)|j-k|} \frac{2^{-(j\wedge k)(\varepsilon-\beta)}}{(2^{-(j\wedge k)} + |x-y|)^{n+(\varepsilon-\beta)}} \tag{3.12}$$

where $a \wedge b$ denotes the minimum of a and b.

Now (3.9) and (3.10) follow from (3.11) and (3.12), respectively. The proofs of (3.11) and (3.12) are the same as the proofs given in Lemma 2.10 of [H] (see pages 66-67 of [H]).

We return to the proof of a special case of the Tb theorem. To establish the L^2 boundedness of T, we must expand T appropriately as $T = \sum_{-\infty}^{\infty} T_j$ and then Lemma (3.2) can be applied to T_j. To do this, we follow Coifman's idea (see [DJS]). Let $\theta(x)$, $x \in \mathbf{R}^n$, be a positive C^∞ function depending only on $|x|$, whose support is contained in $|x| \leq 1$ and $\int \theta(x)\,dx = 1$. Let S_j be the operators defined by convolution with $\theta_j(x) = 2^{jn}\theta(2^j x)$. Put $P_j = S_j\{S_j M_b\}^{-1} S_j$ and $Q_j = P_{j+1} - P_j$. It is easy to check that $P_j \to \frac{1}{b} I$ as $j \to \infty$ and $P_j \to 0$ as $j \to -\infty$ in the strong operator topology, where I is the identity operator. Now the weak boundedness property implies that

$$
\begin{aligned}
T &= \sum_{-\infty}^{\infty} (P_{j+1} M_b T M_b P_{j+1} - P_j M_b T M_b P_j) \\
&= \sum_{-\infty}^{\infty} Q_j M_b T M_b P_j + \sum_{-\infty}^{\infty} P_j M_b T M_b Q_j + \sum_{-\infty}^{\infty} Q_j M_b T M_b Q_j.
\end{aligned} \tag{3.13}
$$

Now Lemma (3.2) applies to each of the above three series as the same line given in [MC]. In fact, a more precise result is valid: the condition $T(b) = 0$ together with the weak boundedness property of T is enough to deal with the first series; $T^*(b) = 0$ and the weak boundedness property of T are needed for the second; the final series requires only the weak boundedness property of T.

We now consider the first series. Let $P_j(x, y)$ be the kernel of P_j and $Q_j(x, y) = P_{j+1}(x, y) - P_j(x, y)$. The kernel $T_j(x, y)$ of the operator $Q_j M_b T M_b P_j$ can be calculated by

(3.14)
$$
T_j(x, y) = \iint Q_j(x, u) b(u) K(u, v) b(v) P_j(v, y)\,du\,dv = \langle Q_j(x, \cdot), M_b T M_b P_j(\cdot, y)\rangle
$$

where $K(u, v)$ is the distribution kernel of T.

Notice that $P_j(x, y)$ satisfies the following conditions:

$$
P_j(x, y) = 0 \text{ if } |x - y| \geq 2^{1-j} \text{ and } \|P_j\|_\infty \leq C 2^{jn}, \tag{3.15}
$$

$$
|P_j(x, y) - P_j(x, y')| \leq C 2^{j(n+1)} |y - y'|, \tag{3.16}
$$

$$
|P_j(x, y) - P_j(x', y)| \leq C 2^{j(n+1)} |x - x'|, \tag{3.17}
$$

$$
\int P_j(x, y) b(y)\,dy = \int P_j(x, y) b(x)\,dx = 1. \tag{3.18}
$$

We first check that $T_j(x, y)$ satisfies (3.3). In fact, if $|x - y| \leq 82^{-j}$, then by the weak boundedness property of $M_b T M_b$ we obtain

$$
\begin{aligned}
|T_j(x, y)| &= |\langle Q_j(x, \cdot), M_b T M_b P_j(\cdot, y)\rangle| \leq C 2^{-j(n+2)} \|Q_j(x, \cdot)\|_{\mathrm{Lip}_1} \|P_j(\cdot, y)\|_{\mathrm{Lip}_1} \\
&\leq C 2^{-j(n+2)} 2^{j(1+n)} 2^{j(1+n)} = C 2^{jn} \leq \omega_j(x, y)
\end{aligned}
$$

since $|x - y| \leq 82^{-j}$. If $|x - y| \geq 82^{-j}$, then by the condition that $Q_j(b) = 0$,

$$
\begin{aligned}
|T_j(x, y)| &= \left| \iint Q_j(x, u) b(u) [K(u, v) - K(x, v)] b(v) P_j(v, y)\,du\,dv \right| \\
&\leq C \iint |Q_j(x, u)| \frac{|x - u|^\varepsilon}{|x - y|^{n+\varepsilon}} |P_j(v, y)|\,du\,dv
\end{aligned}
$$

since $|x-y| \geq 82^{-j}$, $|x-u| \leq 22^{-j}$ and $|v-y| \leq 22^{-j}$ imply $|x-y| \sim |u-v|$

$$\leq C\frac{2^{-j\varepsilon}}{|x-y|^{n+\varepsilon}} \leq \omega_j(x,y).$$

We now check that $T_j(x,y)$ satisfies (3.4). If $|y-y'| \geq 2^{-j}$, then (3.4) follows from (3.3). So we only consider the case where $|y-y'| \leq 2^{-j}$. If $|x-y| \leq 82^{-j}$, then by the weak boundedness property of M_bTM_b we obtain for $0<\beta<\varepsilon$,

$$\begin{aligned}|T_j(x,y)-T_j(x,y')| &= \left|\iint Q_j(x,u)b(u)K(u,v)b(v)[P_j(v,y)-P_j(v,y')]\,du\,dv\right| \\ &\leq C2^{-j(n+2)}\|Q_j(x,\cdot)\|_{\mathrm{Lip}_1}\|P_j(\cdot,y)-P_j(\cdot,y')\|_{\mathrm{Lip}_1} \\ &\leq C2^{-j(n+2)}2^{j(n+1)}2^{j(n+2)}|y-y'| \\ &\leq C2^{jn}2^{j}|y-y'| \leq 2^{j\beta}|y-y'|^{\beta}\omega_j(x,y).\end{aligned}$$

If $|x-y| \geq 82^{-j}$, then by the condition that $Q_j(b)=0$,

$$\begin{aligned}&|T_j(x,y)-T_j(x,y')| \\ &\quad= \left|\iint Q_j(x,u)b(u)[K(u,v)-K(x,v)]b(v)[P_j(v,y)-P_j(v,y')]\,du\,dv\right| \\ &\quad\leq C\int_{|x-u|\leq 2^{1-j}}\int_{|v-y|\leq 2^{1-j}} 2^{jn}\frac{|x-u|^{\varepsilon}}{|x-y|^{n+\varepsilon}}2^{j(1+n)}|y-y'|\,du\,dv \\ &\quad\leq C2^{j}|y-y'|\frac{2^{-j\varepsilon}}{|x-y|^{n+\varepsilon}} \\ &\quad\leq 2^{j\beta}|y-y'|^{\beta}\omega_j(x,y).\end{aligned}$$

The proof of (3.5) is the same. (3.6) follows from the fact that $P_j(b)=1$ and $T(b)=0$, and (3.7) follows from the fact that $Q_j^*(b)=0$. All other proofs are similar. We leave all these simple calculations to the reader.

The proof of a special case of the Tb theorem now follows from Lemma (3.2).

4. Some remarks

Using paraproduct operators and the proofs given in section 3, one can prove the following Tb theorem.

The Tb theorem. *Suppose that b is a bounded real-valued function, $b \geq \delta > 0$ and $T : C_0^{\eta}(\mathbf{R}^n) \to (C_0^{\eta})'(\mathbf{R}^n)$ for $\eta > 0$ is a singular integral. Suppose further that $T(b) \in BMO$ and $T^*(b) \in BMO$, and the operator M_bTM_b satisfies the following weak boundedness property:*

$$|\langle M_bTM_b(f),g\rangle| \leq Ct^{n+2\eta}\|f\|_{\eta}\|g\|_{\eta} \tag{3.1}$$

for all $\eta > 0$, all cubes Q with diameter at most $t > 0$ and all $f,g \in C_0^{\eta}(\mathbf{R}^n)$ supported in Q.

Then T is bounded on $L^2(\mathbf{R}^n)$.

We leave these details to the reader.

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