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ASYMPTOTIC BEHAVIOR OF FOURIER TRANSFORMS OF SELF-SIMILAR MEASURES

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Abstract. Let μ be a self-similar probability measure on $\mathbb{R}$ satisfying $\mu = \sum_{j=1}^{m} p_j \mu \circ F_j^{-1}$, where $F_j(x) = \rho x + a_j$, $0 < \rho < 1$, $a_j \in \mathbb{R}$, $p_j > 0$ and $\sum_{j=1}^{m} p_j = 1$. Let $\hat{\mu}(t)$ be the Fourier transform of μ. A necessary and sufficient condition for $\hat{\mu}(t)$ to approach zero at infinity is given. In particular, if $a_j = j$ and $p_j = 1/m$ for $j = 1, ..., m$, then $\limsup_{t\to\infty} |\hat{\mu}(t)| > 0$ if and only if ρ^{-1} is a PV-number and ρ^{-1} is not a factor of m. This generalizes the corresponding theorem of Erdös and Salem for the case $m = 2$.

1. Introduction

For $j = 1, 2, ..., m$, let $F_j(x) = \rho x + a_j$, where $0 < \rho < 1$, $a_j \in \mathbb{R}$ be a family of equicontractive similitudes and let $p_1, p_2, \ldots, p_m$ be associated probability weights. Then there is a unique self-similar probability measure μ on $\mathbb{R}$ with compact support satisfying

$$\mu(A) = \sum_{j=1}^{m} p_j \mu(F_j^{-1}(A))$$

for all measurable sets A.

The measure μ can also be generated as follows.

Let $X_0, X_1, \ldots$ be a sequence of i.i.d. random variables each taking real values $a_1, a_2, \ldots, a_m$ with probability $p_1, p_2, \ldots, p_m$ respectively. For $0 < \rho < 1$ let

$$S = \sum_{j=0}^{\infty} \rho^j X_j, \tag{1.1}$$

and let μ_ρ be the probability measure induced by S, i.e., $\mu_\rho(A) = \Pr ob\{\omega : S(\omega) \in A\}$. Then it can be verified that this measure also satisfies the above self-similar equation. By uniqueness we obtain $\mu = \mu_\rho$.

It is known that μ is either purely singular or absolutely continuous [JW]. If $0 < \rho < 1/m$, then the support of μ is a set of Cantor type; hence μ is purely singular. If $1/m \le \rho < 1$, then the distribution type of μ depends on the choice of the parameters $a_1, a_2, \ldots, a_m$ and $p_1, p_2, \ldots, p_m$, the determination of which type in general is very difficult.

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The most fascinating case is when $m = 2$ and each X_j takes two values ± 1 with equal probability $1/2$. Then μ is the so-called *infinitely convolved Bernoulli measures*. If $\rho = 1/2$, then μ is a measure proportional to the Lebesgue measure on $[-2, 2]$. It was conjectured that μ ought to be absolutely continuous for $\rho > 1/2$. However, in 1939 Erdös [E] showed that if ρ^{-1} is a PV-number and $\rho \neq 1/2$ (recall that an algebraic integer $\beta > 1$ is called a PV-number if all its conjugate roots have modulus strictly less than one), then the Fourier transform $\hat{\mu}(t) = \int_{-\infty}^{\infty} e^{itx} d\mu(x)$ does not tend to zero at infinity and thus μ is purely singular. Salem [S1] showed that for $0 < \rho < 1$ with $\rho \neq 1/2$, $\hat{\mu}(t) \not\to 0$ as $t \to \infty$ only if ρ^{-1} is a PV-number. Since μ is purely singular for $0 < \rho < 1/2$, the Erdös-Salem theorem gives a large family of singular measures whose Fourier transforms tend to zero at infinity.

In the other direction, Erdös showed that there exists a sequence $\rho_k \to 1$ such that μ has k derivatives for almost all $\rho \in (\rho_k, 1)$. The result has been generalized recently by Solomyak [So]. He proved that $\hat{\mu}(t)$ is in L^2 for almost all $1/2 < \rho < 1$ and hence μ is absolutely continuous with an L^2 density function. However, the only explicit values of ρ for which μ is known to be absolutely continuous are $\rho = 2^{-1/n}$, for $n = 1, 2, \dots$, and a family of algebraic numbers found by Garsia [G]. On the other hand, all PV-numbers are the only numbers for which μ is known to be purely singular.

The Fourier transform of the probability measure induced by $\rho^j X_j$ in (1.1) is $\sum_{k=1}^{m} p_k e^{ita_k \rho^j}$. Thus the Fourier transform of μ induced by S is

$$\hat{\mu}(t) = \prod_{j=0}^{\infty} (\sum_{k=1}^{m} p_k e^{ita_k \rho^j}). \tag{1.2}$$

The average asymptotic rate of $\hat{\mu}(t)$ with some restriction on the parameters was studied extensively by Strichartz [St1]–St3, Lau [L1]–L2, Lau and Wang [LW] and Fan and Lau [FL]. In this paper we will give a necessary and sufficient condition for $\hat{\mu}(t)$ to approach zero at infinity. The criterion can be applied to various cases for which a specific set of parameters is given. The detailed proof is given in section 2. In section 3, as an application, we generalize the above Erdös-Salem theorem to $m \geq 2$.

2. THE GENERAL THEOREMS

Let μ be the probability measure induced by S as defined by (1.1) and let $\hat{\mu}(t)$ be its Fourier transform. Replacing t by $\rho^{-n} t$ in (1.2) we obtain for $n = 1, 2, \ldots$,

$$\hat{\mu}(\rho^{-n} t) = \hat{\mu}(t) \prod_{j=1}^{n} (\sum_{k=1}^{m} p_k e^{ita_k \rho^{-j}}). \tag{2.1}$$

For $j = 0, \pm 1, \ldots$, the convex combination $\sum_{k=1}^{m} p_k e^{ita_k \rho^j}$ lies inside the closed unit disk. We write

$$d_j(t) = 1 - |\sum_{k=1}^{m} p_k e^{ita_k \rho^j}|$$

for the distance from this convex combination to the circumference of the unit circle. Note that $d_j(t)$ depends also on ρ, $a_1, a_2, \dots, a_m$ and $p_1, p_2, \dots, p_m$. This notation will be used for the rest of the paper. Furthermore it is easy to check that for $j = 0, \pm 1, ...,$

$$d_j(t) \leq diam\{e^{ita_1\rho^j}, e^{ita_2\rho^j}, ..., e^{ita_m\rho^j}\}. \tag{2.2}$$

Lemma 2.1. *Let* $\hat{\mu}(t)$ *be defined by* (1.2). *Then* $\hat{\mu}(t) = 0$ *if and only if* $\sum_{k=1}^m p_k e^{ita_k\rho^j} = 0$ *for some* $j = 0, 1, ...$

Proof. It suffices to show that if $\sum_{k=1}^m p_k e^{ita_k\rho^j} \neq 0$ for all $j = 0, 1, ...$, then $|\hat{\mu}(t)| > 0$.

Using (2.2) and noting that $ta_k\rho^j \to 0$ as $j \to \infty$ we obtain

$$\begin{aligned} \sum_{j=0}^{\infty} d_j(t) &\leq \sum_{j=0}^{\infty} diam\{e^{ita_1\rho^j}, e^{ita_2\rho^j}, ..., e^{ita_m\rho^j}\} \\ &\leq 2\sum_{j=0}^{\infty} \max\{|ta_1\rho^j|, |ta_2\rho^j|, ..., |ta_m\rho^j|\} \\ &< \infty. \end{aligned}$$

Since $\sum_{k=1}^m p_k e^{ita_k\rho^j} \neq 0$, we have $1 - d_j(t) > 0$ for all j. It follows that

$$|\hat{\mu}(t)| = \lim_{n\to\infty} \prod_{j=0}^{n} (1 - d_j(t)) > 0.$$

Q.E.D.

Lemma 2.2. *Let* $\hat{\mu}(t)$ *be defined by* (1.2) *and let* $t \in \mathbb{R}$ *with* $t \neq 0$ *be any point. If either* $\sum_{j=1}^{\infty} d_{-j}(t) = \infty$ *or* $\sum_{k=1}^m p_k e^{ita_k\rho^j} = 0$ *for some* $j = 0, \pm 1, ...,$ *then* $\lim_{n\to\infty} \hat{\mu}(\rho^{-n}t) = 0$. *Otherwise,* $\lim_{n\to\infty} |\hat{\mu}(\rho^{-n}t)| > 0$.

Proof. If $\sum_{k=1}^m p_k e^{ita_k\rho^j} = 0$ for some $j = 0, \pm 1, ...$, then by (2.1) $\hat{\mu}(\rho^{-n}t) = 0$ for all large n, and we have trivially $\lim_{n\to\infty} \hat{\mu}(\rho^{-n}t) = 0$.

Suppose that $\sum_{k=1}^m p_k e^{ita_k\rho^j} \neq 0$ for all $j = 0, \pm 1, ...$ Then $|\hat{\mu}(t)| > 0$ by Lemma 2.1 and $1 - d_{-j}(t) > 0$ for all $j = 1, 2, ...$ This, along with (2.1), implies that

$$\lim_{n\to\infty} |\hat{\mu}(\rho^{-n}t)| = |\hat{\mu}(t)| \lim_{n\to\infty} \prod_{j=1}^{n} (1 - d_{-j}(t)) > 0$$

if and only if $\sum_{j=1}^{\infty} d_{-j}(t) < \infty$. Q.E.D.

Theorem 2.3. *Let* $\hat{\mu}(t)$ *be defined by* (1.2). *If for every* $t \in \mathbb{R}$ *with* $t \neq 0$ *either* $\sum_{j=1}^{\infty} d_{-j}(t) = \infty$ *or* $\sum_{k=1}^m p_k e^{ita_k\rho^j} = 0$ *for some* $j = 0, \pm 1, ...,$ *then* $\lim_{x\to\infty} \hat{\mu}(x) = 0$. *Otherwise,* $\limsup_{x\to\infty} |\hat{\mu}(x)| > 0$ *and hence* μ *is purely singular.*

Proof. In view of Lemma 2.2 it suffices to show that if $\lim_{n\to\infty} \hat{\mu}(\rho^{-n}t) = 0$ for every $t \in [\rho\pi, \pi]$, then $\lim_{x\to\infty} \hat{\mu}(x) = 0$. We will show this by contradiction. Suppose that there is a sequence $t_n \to \infty$ and an integer n_0 such that

$$|\hat{\mu}(t_n)| \geq \delta > 0 \text{ for all } n \geq n_0. \tag{2.3}$$

Let $k_n, n = 1, 2, ...,$ be a sequence of positive integers such that $\rho^{k_n}t_n \in [\rho\pi, \pi]$. Without loss of generality, assume that $\lim_{n\to\infty} \rho^{k_n}t_n = t \in [\rho\pi, \pi]$. By assumption, $\lim_{n\to\infty} \hat{\mu}(\rho^{-n}t) = 0$. Hence for any $\varepsilon > 0$ there exists an integer, which may also be taken to be n_0, such that for all $n \geq n_0$

$$|\hat{\mu}(\rho^{-n}t)| < \varepsilon. \tag{2.4}$$

Note that for this fixed n_0, $\hat{\mu}(\rho^{-n_0}x)$ is a continuous function of x. Using

$$\lim_{n\to\infty} \rho^{k_n} t_n = t$$

we have for all large n

$$|\hat{\mu}(\rho^{k_n - n_0} t_n) - \hat{\mu}(\rho^{-n_0} t)| < \varepsilon. \tag{2.5}$$

By (2.1) we have $|\hat{\mu}(\rho^{-n}t)| \le |\hat{\mu}(t)|$ or equivalently $|\hat{\mu}(t)| \le |\hat{\mu}(\rho^n t)|$ for every t and for every positive integer n. It follows from (2.4) and (2.5) that for all large n

$$|\hat{\mu}(t_n)| \le |\hat{\mu}(\rho^{k_n - n_0} t_n)| < |\hat{\mu}(\rho^{-n_0} t)| + \varepsilon < 2\varepsilon.$$

This contradicts (2.3) if ε is chosen to be less than $\delta/3$. Q.E.D.

Next, we give a condition so that $\limsup_{x\to\infty} |\hat{\mu}(x)| > 0$ if ρ^{-1} is a PV-number.

For any real number a, we denote by $||a||$ the distance between a and the nearest integer. Thus $||a|| = \min |a - n|, n = 0, \pm 1, \ldots$. If k is the integer nearest to a, we write $a = k + \langle a \rangle$ so that $||a|| = |\langle a \rangle| \le 1/2$.

It is known that $\beta > 1$ is a PV-number if and only if there exists a real number $t \ne 0$ such that $\sum_{j=1}^{\infty} ||t\beta^j||^2 < \infty$. It is also known that for any given PV-number β the set

$$E(\beta) = \{t : \sum_{j=1}^{\infty} ||t\beta^j||^2 < \infty\}$$

is countable [S2, BDGPS].

If $\beta > 1$ is a PV-number, then $||\beta^j||$ tends to zero geometrically, and so does $||k\beta^j||$ for any fixed integer k; hence $k \in E(\beta)$ for every integer k. Furthermore, if $t = k\beta^{j_0}$ for some integer k and for some $j_0 = 0, \pm 1, \ldots$, then $||t\beta^j|| = ||k\beta^{j+j_0}||$ tends to zero geometrically. It follows that $k\beta^{j_0} \in E(\beta)$ for arbitrary integers k, j_0. In Lemma 3.1 we will show that if $\beta \ge 2$ is an integer, then $E(\beta)$ consists precisely these numbers.

Theorem 2.4. *Let ρ^{-1} be a PV-number and let $\hat{\mu}(t)$ be defined by* (1.2). *If $a_k \in E(\rho^{-1})$ for $k = 1, \ldots, m$ and if there exists a positive integer N such that*

$$\sum_{k=1}^{m} p_k e^{i2N\pi a_k \rho^j} \ne 0$$

for all $j = 0, \pm 1, \ldots$, then $\lim_{n\to\infty} |\hat{\mu}(\rho^{-n} 2N\pi)| > 0$. In particular, μ is purely singular.

Proof. Using Lemma 2.2 with $t = 2N\pi$ we need only to show that $\sum_{j=1}^{\infty} d_{-j}(2N\pi) < \infty$, where $d_{-j}(2N\pi) = 1 - |\sum_{k=1}^{m} p_k e^{i2N\pi a_k \rho^{-j}}|$. Since $a_k \in E(\rho^{-1})$ for $k = 1, \ldots, m$, we have

$$\sum_{j=1}^{\infty} \max\{||a_1\rho^{-j}||^2, ||a_2\rho^{-j}||^2, \ldots, ||a_m\rho^{-j}||^2\} < \infty.$$

In particular, $\max\{||a_1\rho^{-j}||, ||a_2\rho^{-j}||, ..., ||a_m\rho^{-j}||\} \to 0$, as $j \to \infty$. We have

$$\begin{aligned} |\sum_{k=1}^{m} p_k e^{i2N\pi a_k \rho^{-j}}| &\geq \mathrm{Re}(\sum_{k=1}^{m} p_k e^{i2N\pi a_k \rho^{-j}}) \\ &\geq \min_{1\leq k\leq m}\{\mathrm{Re}(e^{i2N\pi a_k\rho^{-j}})\} \\ &= \min_{1\leq k\leq m}\{\cos 2N\pi ||a_k\rho^{-j}||\} \\ &= \cos\theta_j \\ &\geq 1-\frac{\theta_j^2}{2}, \end{aligned}$$

where $\theta_j = 2N\pi\max\{||a_1\rho^{-j}||, ||a_2\rho^{-j}||, ..., ||a_m\rho^{-j}||\}$. Hence for large j we have $d_{-j}(2N\pi) \leq \theta_j^2/2$ and

$$\sum_{j=1}^{\infty} d_{-j}(2N\pi) \leq 2N^2\pi^2 \sum_{j=1}^{\infty} \max\{||a_1\rho^{-j}||^2, ||a_2\rho^{-j}||^2, ..., ||a_m\rho^{-j}||^2\} < \infty.$$

Q.E.D.

3. A THEOREM OF ERDÖS AND SALEM

In this section we will generalize a theorem of Erdös and Salem on the asymptotic behavior of Fourier transforms of *infinitely convolved Bernoulli measures.*

Lemma 3.1. *Let $n \geq 2$ be any given integer and let $t \in \mathbb{R}$ with $t \neq 0$. If $t = k/n^{j_0}$ for some integer k and for some $j_0 = 0, 1, ...,$ then $\langle tn^j\rangle = 0$ for all $j \geq j_0$. Otherwise,* $\limsup_{j\to\infty} ||tn^j|| > 0$.

Proof. Write $tn^j = k_j + \langle tn^j\rangle$, where k_j is the integer nearest to tn^j. Then

$$tn^{j+1} = nk_j + n\langle tn^j\rangle.$$

If $0 < |\langle tn^j\rangle| < \frac{1}{2n}$, then $0 < n|\langle tn^j\rangle| < 1/2$. Hence $|\langle tn^j\rangle| < n|\langle tn^j\rangle| = |\langle tn^{j+1}\rangle|$. It follows that the union of the two intervals $(-\frac{1}{2n}, 0) \cup (0, \frac{1}{2n})$ is a repelling region for $\langle tn^j\rangle$. Therefore $\limsup_{j\to\infty} ||tn^j|| > 0$ unless $\langle tn^j\rangle = 0$ for all $j \geq j_0$ and for some j_0, which is equivalent to $tn^{j_0} = k$ for some integer k and for some $j_0 = 0, 1, ...$ Q.E.D.

Theorem 3.2 (A generalized theorem of Erdös-Salem)**.** *Let S be defined by* (1.1) *with each X_j taking m values $0, 1, ..., m-1$ with equal probability $1/m$. Let μ be the probability measure induced by S and let $\hat{\mu}$ be its Fourier transform. If ρ^{-1} is a PV-number and ρ^{-1} is not a factor of m, then* $\limsup_{t\to\infty} |\hat{\mu}(t)| > 0$. *Otherwise,* $\lim_{t\to\infty} \hat{\mu}(t) = 0$.

Proof. Suppose that ρ^{-1} is a PV-number which is not a factor of m. Since all integers are contained in $E(\rho^{-1})$, using Theorem 2.4 with $N = 1$ we need to verify that

$$\sum_{k=0}^{m-1} e^{i2k\pi\rho^j} \neq 0 \text{ for all } j = 0, \pm 1, ...$$

We will show that if $\sum_{k=0}^{m-1} e^{i2k\pi\rho^j} = 0$ for some j and if ρ^{-1} is a PV-number, then it must be a factor of m. Clearly $\sum_{k=0}^{m-1} e^{i2k\pi\rho^j} = 0$ implies that $e^{i2\pi\rho^j} \neq 1$; otherwise

$\sum_{k=0}^{m-1} e^{i2k\pi\rho^j} = m \neq 0$. Note that $(1-e^{i2\pi\rho^j})\sum_{k=0}^{m-1} e^{i2k\pi\rho^j} = 1-e^{i2m\pi\rho^j}$. Hence $\sum_{k=0}^{m-1} e^{i2k\pi\rho^j} = 0$ for some j if and only if $e^{i2\pi\rho^j}$ is a mth root of the unity, i.e., $\langle\rho^j\rangle = k/m$ for some positive integer j and for some $k = 0, 1, ..., m-1$. Since the only rational PV-numbers are integers greater than or equal to 2, it implies that $\rho^j = k/m$ must be a reciprocal of an integer. Hence ρ^{-j} is a factor of m and so is ρ^{-1}. Theorem 2.4 thus shows $\limsup_{t\to\infty} |\hat{\mu}(t)| > 0$.

Next, suppose that ρ^{-1} is not a PV-number. Then $\sum_{j=1}^{\infty} ||t\rho^{-j}||^2 = \infty$ for every $t \in \mathbb{R}$ with $t \neq 0$.

Case 1. m is odd. For simplicity let X_j take m values in

$$A = \pm 2\pi\{0, 1, ..., (m-1)/2\}$$

with equal probability $1/m$.

Fix any $t \in \mathbb{R}$ with $t \neq 0$. Since ρ^{-1} is not an integer, then $\langle t\rho^{-j}\rangle = 0$ occurs for at most one index j. Without loss of generality, assume that $\langle t\rho^{-j}\rangle \neq 0$ for all j. Using the formula $\frac{1}{2} + \cos x + \cos 2x + \cdots + \cos nx = \frac{\sin(n+\frac{1}{2})x}{2\sin\frac{x}{2}}$ for $x \neq 2k\pi$ we have

$$\begin{aligned} |\frac{1}{m}\sum_{a\in A} e^{iat\rho^{-j}}| &= \frac{1}{m}|1 + 2\sum_{k=1}^{(m-1)/2} \cos 2k\pi t\rho^{-j}| \\ &= \frac{1}{m}|\frac{\sin(\frac{m-1}{2}+\frac{1}{2})2\pi t\rho^{-j}}{\sin \pi t\rho^{-j}}| \\ &= \frac{1}{m}\frac{|\sin m\pi\langle t\rho^{-j}\rangle|}{\sin \pi\langle t\rho^{-j}\rangle}. \end{aligned} \tag{3.1}$$

If $\langle t\rho^{-j}\rangle \geq \frac{1}{2m}$, then

$$(3.1) \leq \frac{1}{m\sin(\pi/2m)}.$$

Consider the Taylor expansion $\frac{\sin mx}{\sin x} = m - \frac{(m^3-m)x^2}{3!} + (\frac{m^5-m}{5!} - \frac{m^3-m}{3!3!})x^4 - \cdots$. If $\langle t\rho^{-j}\rangle < \frac{1}{2m}$, then

$$\begin{aligned} (3.1) &= 1 - \frac{m^2-1}{3!}\pi^2\langle t\rho^{-j}\rangle^2 + (\frac{m^4-1}{5!} - \frac{m^2-1}{3!3!})\pi^4\langle t\rho^{-j}\rangle^4 - \cdots \\ &\leq 1 - c_m\langle t\rho^{-j}\rangle^2, \end{aligned}$$

where $c_m > \frac{m^2-1}{3!}\pi^2$ is some constant depending only on m such that $1 - c_m\langle t\rho^{-j}\rangle^2 > 0$. Using the hypothesis $\sum_{j=1}^{\infty} ||t\rho^{-j}||^2 = \infty$ we obtain

$$\begin{aligned} \sum_{j=1}^{\infty} d_{-j}(t) &= \sum_{j=1}^{\infty}(1 - |\frac{1}{m}\sum_{a\in A} e^{iat\rho^{-j}}|) \\ &\geq \sum_{\langle t\rho^{-j}\rangle < \frac{1}{2m}} c_m\langle t\rho^{-j}\rangle^2 + \sum_{\langle t\rho^{-j}\rangle \geq \frac{1}{2m}} (1 - \frac{1}{m\sin(\pi/2m)}) \\ &= \infty. \end{aligned}$$

Hence $\lim_{t\to\infty} \hat{\mu}(t) = 0$ by Theorem 2.3.

Case 2. m is even. For simplicity we let X_j take m values in

$$A = \pm 2\pi\{1, 3, 5, ..., m-1\}$$

with equal probability $1/m$.

Fix any $t \in \mathbb{R}$ with $t \neq 0$. Using the formula $\cos x + \cos 3x + \cdots + \cos(2n-1)x = \frac{\sin nx \cos nx}{\sin x}$ we have

$$\begin{aligned} |\frac{1}{m}\sum_{a\in A} e^{iat\rho^{-j}}| &= \frac{2}{m}|\sum_{k=1}^{m/2}\cos 2(2k-1)\pi t\rho^{-j}| \\ &= \frac{2}{m}|\frac{\sin m\pi t\rho^{-j}\cos m\pi t\rho^{-j}}{\sin 2\pi t\rho^{-j}}| \\ &= \frac{1}{m}|\frac{\sin 2m\pi\langle t\rho^{-j}\rangle}{\sin 2\pi\langle t\rho^{-j}\rangle}|, \end{aligned}$$

which is similar to (3.1). Applying the same argument we obtain $\lim_{t\to\infty}\hat{\mu}(t) = 0$.

Finally, we will show that if ρ^{-1} is a positive integer which is a factor of m, then $\lim_{t\to\infty}\hat{\mu}(t) = 0$. We let X_j take m values in $2\pi\{0, 1, 2, ..., (m-1)\}$ with equal probability $1/m$. By Theorem 2.3 it suffices to show that for every $t \in \mathbb{R}$ with $t \neq 0$ either $\sum_{j=1}^{\infty} d_{-j}(t) = \infty$ or $\sum_{k=0}^{m-1} e^{i2k\pi t\rho^j} = 0$ for some integer $j = 0, \pm 1, ...$

Suppose that $t = k_0\rho^{j_0}$ for some integer k_0 and for some $j_0 = 0, \pm 1, ...$ Without loss of generality, assume that $k_0\rho$ is not an integer. Otherwise, we write $t = k_1\rho^{j_0 - 1}$, where $k_1 = k_0\rho$. Let $j = 1 - j_0$. Then $t\rho^j = k_0\rho$ is not an integer and hence $e^{i2\pi t\rho^j} \neq 1$. Since ρ^{-1} is a factor of m, i.e., $m\rho$ is an integer; hence $e^{i2m\pi t\rho^j} = e^{i2m\pi k_0\rho} = 1$. It follows that

$$\sum_{k=0}^{m-1} e^{i2k\pi t\rho^j} = \frac{1 - e^{i2m\pi t\rho^j}}{1 - e^{i2\pi t\rho^j}} = 0.$$

If $t \neq k_0\rho^{j_0}$ for any integer k_0 and for any $j_0 = 0, 1, ...,$ then by Lemma 3.1 $||t\rho^{-j}||$ does not tend to zero as $j \to \infty$. This certainly implies that $\sum_{j=1}^{\infty}||t\rho^{-j}||^2 = \infty$. So $\sum_{j=1}^{\infty} d_{-j}(t) = \infty$ by above argument and hence $\lim_{t\to\infty}\hat{\mu}(t) = 0$. Q.E.D.

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