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THE ABELIANIZATION OF ALMOST FREE GROUPS

CHARLY BITTON

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ABSTRACT. We will construct an almost free non-free group G of cardinality $\aleph_1$ such that G/G' is a free abelian group.

1. Introduction

In this paper, we will deal with almost free non-free groups and almost free non-free abelian groups. A group G of cardinality λ is said to be almost free if every subgroup of G of cardinality less than λ is free. Similarly, an abelian group A of cardinality λ is said to be almost free if every subgroup of A of cardinality less than λ is free abelian. In 1951, Higman [2] constructed an almost free non-free group of cardinality $\aleph_1$; since then, almost free groups and almost free abelian groups have been investigated by a number of authors. In particular, Shelah [6] proved that if there exists an almost free non-free abelian group of cardinality λ, then there also exists an almost free non-free group of cardinality λ. The converse of this result is still open and only partial results are known in this direction (see [1]). There is a natural way to approach this question. Suppose that G is an almost free non-free group of cardinality λ. Then an easy argument shows that the quotient group G/G' is an almost free abelian group of cardinality λ. (Here G' denotes the commutator subgroup of G.) If it could be shown that G/G' is necessarily non-free, then the problem would be solved. In this paper, we will construct an almost free non-free group G such that G/G' is a free abelian group. Consequently, this naive approach to the problem cannot succeed.

2. The abelianization of almost free non-free groups

In this section, we will construct an almost free non-free group G of cardinality $\aleph_1$ such that G/G' is a free abelian group of cardinality $\aleph_1$.

Before starting the proof, we will review some group theoretic results and establish our notation. Suppose that G is a free group. Then it is well known that every subgroup H of G is free. We will write $H \mid G$ if H is a free factor of G; i.e., if there exists a basis X of G and a subset $X' \subseteq X$ such that X' is a basis of H. In this case, if E is the subgroup generated by $X \setminus X'$, then $G = H * E$ is the free product of H and E. If Y is any subset of G, then $\langle Y \rangle$ denotes the subgroup of G which is

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generated by Y. If H is any subgroup of G, then H^G denotes the normal closure of H in G; i.e., the smallest normal subgroup of G which contains H.

We will make repeated use of the following result.

Lemma 2.1. *Suppose that G is a free group and that $K \mid G$. If H is any subgroup of G, then $H \cap K \mid H$.*

Proof. For example, see Problem 32 in Section 2.4 of [4]. □

We will also make use of the following characterization of the commutator subgroup of a free group.

Lemma 2.2. *Suppose that the group A is freely generated by the set $\mathcal{A}$. For each $x \in \mathcal{A}$, let $\ell_x^A : A \to \mathbb{Z}$ be the function such that $\ell_x^A(w)$ is the sum of the exponents of the occurrences of x in the word $w \in A$. Then $w \in A'$ iff $\ell_x^A(w) = 0$ for every $x \in \mathcal{A}$.*

Proof. For example, see Problem 2 in Section 2.2 of [4]. □

Definition 2.3. If κ is an infinite cardinal, then we say that the principle $F(\kappa)$ holds iff there is a free group A of rank κ and a free subgroup B of rank κ such that:

(I) If C is a free factor of B of rank less than κ, then C is also a free factor of A.
(II) B is not a free factor of A.
(III) $B \cap A' = B'$.

In [2], Higman used a pair of free groups of rank $\aleph_0$ satisfying conditions (I) and (II) of principle $F(\aleph_0)$ to construct an almost free non-free group of cardinality $\aleph_1$. We will essentially give his construction in the proof of Lemma 2.5. We have added condition (III) in order to ensure that B/B' is naturally embedded into A/A' via the map $bB' \mapsto bA'$. Later we will construct a pair of free groups (B, A) satisfying the conditions of principle $F(\aleph_0)$ in such a way that the natural embedding $B/B' \hookrightarrow A/A'$ is surjective. In this case, we will write $B/B' = A/A'$.

Theorem 2.4. *There exists an almost free non-free group G of cardinality $\aleph_1$ such that G/G' is a free abelian group of cardinality $\aleph_1$.*

We will break the proof of Theorem 2.4 into a series of lemmas. We will use the next lemma to construct the group G from a suitably chosen pair (A, B) of free groups exemplifying principle $F(\aleph_0)$.

Lemma 2.5. *Principle $F(\aleph_0)$ implies that there exists an almost free non-free group G of cardinality $\aleph_1$.*

Proof. Suppose that the free groups $B \leqslant A$ exemplify principle $F(\aleph_0)$. We will define a smooth strictly increasing chain of free group G_α by induction on $\alpha < \aleph_1$. Let G_0 be a free group of rank $\aleph_0$. Now suppose inductively that $\alpha > 0$ and that G_β has been defined for all $\beta < \alpha$. Suppose further that if $\gamma < \beta < \alpha$, then every finitely generated free factor of G_γ is also a free factor of G_β. First suppose that α has the form $\beta + 2$. In this case, we let $G_\alpha = G_{\beta+1} * \langle x_{\beta+1} \rangle$, where $\langle x_{\beta+1} \rangle$ is an infinite cyclic group such that $G_{\beta+1} \cap \langle x_{\beta+1} \rangle = \{e\}$. Next suppose that $\alpha = \delta + 1$ for some limit ordinal δ. Then we let G_α be a free group such that $G_\delta \leqslant G_\alpha$ and the following diagram commutes:

$$\begin{array}{ccc} G_\delta & \xrightarrow{\text{isomorphism}} & B \\ \big\downarrow & & \big\downarrow \\ G_\alpha = G_{\delta+1} & \xrightarrow{\text{isomorphism}} & A \end{array}$$

Finally if α is a limit ordinal, then we let $G_\alpha = \bigcup_{\beta<\alpha} G_\beta$. In this case, it is necessary to check that G_α is a free group. To see this, first let $\langle g_i \mid i < \omega\rangle$ be an enumeration of the elements of G_α; then choose inductively an increasing sequence of ordinals

$$\beta_0 \leq \beta_1 \leq \cdots \leq \beta_n \leq \cdots < \alpha$$

and an increasing sequence of subgroups of G_α

$$F_0 \leqslant F_1 \leqslant \cdots \leqslant F_n \leqslant \cdots$$

such that $\{g_0, \dots, g_n\} \subseteq F_n$ and F_n is a finitely generated free factor of G_{β_n}. By Lemma 2.1, F_n is a free factor of F_{n+1} for each $n < \omega$. Since $G_\alpha = \bigcup_{n<\omega} F_n$, it follows that G_α is a free group.

Now let $G = \bigcup_{\alpha<\aleph_1} G_\alpha$. Then it is clear that G is almost free. Suppose that G is free. Then we can express $G = \bigcup_{\alpha<\aleph_1} K_\alpha$ as the union of a smooth strictly increasing chain of countable subgroups such that $K_\alpha \mid G$ for all $\alpha < \aleph_1$. There exists a club C such that $K_\gamma = G_\gamma$ for all $\gamma \in C$; so by Lemma 2.1, $G_\gamma \mid G_{\gamma+1}$ for all $\gamma \in C$. But by taking $\gamma \in C$ to be a limit ordinal, we obtain a contradiction. □

Lemma 2.6. *In the above construction, $G_\alpha \cap G' = G'_\alpha$.*

Proof. An easy induction shows that if $\alpha < \beta$, then $G_\alpha \cap G'_\beta = G'_\alpha$. (The only point where the induction could conceivably break down is when α is a limit ordinal and $\beta = \alpha + 1$. However, condition (III) of principle $F(\aleph_0)$ deals with this case.) The result follows. □

Now we want to define a particular pair of free groups (B, A) exemplifying principle $F(\aleph_0)$ such that if G is the group constructed in Lemma 2.5, then G/G' is a free abelian group of cardinality $\aleph_1$. We shall make use of the following result of Levi. Recall that if H is any group, then the n^{th} derived subgroup $H^{(n)}$ of H is defined inductively by $H^{(0)} = H$ and $H^{(n+1)} = (H^{(n)})'$.

Lemma 2.7. *If H is a free group, then $\bigcap_{n<\omega} H^{(n)} = \{e\}$.*

Proof. This is an immediate consequence of Corollary 2.12 of [4]. □

Definition 2.8. Let A be the group freely generated by the set $\{a_n \mid n \geq 1\}$ and let

$$B = \langle a_1^{-1}[a_2, a_3], a_2^{-1}[a_3, a_4], a_3^{-1}[a_4, a_5], \dots, a_n^{-1}[a_{n+1}, a_{n+2}], \dots\rangle.$$

Lemma 2.9. (1) *Every finitely generated free factor of B is also a free factor of A.*
(2) *B is not a free factor of A.*
(3) *$B \cap A' = B'$.*
(4) *$B/B' = A/A'$.*

Proof. First we will show that every finitely generated free factor of B is also a free factor of A. For each $n > 2$, define $A_n = \langle a_1, \dots, a_n\rangle$ and

$$B_n = \langle a_1^{-1}[a_2, a_3], a_2^{-1}[a_3, a_4], \dots, a_{n-2}^{-1}[a_{n-1}, a_n]\rangle.$$

Then it is easily checked that $A_n = B_n * \langle a_{n-1}, a_n\rangle$. Hence B_n is a free factor of A. If C is a finitely generated free factor of B, then there exists an integer $n > 2$ such that $C \leqslant B_n$. By Lemma 2.1, C is a free factor of B_n. Since B_n is a free factor of A, it follows that C is also a free factor of A.

Next we will prove that B is not a free factor of A. To prove this, it is enough to show that $H = A/B^A$ is not a free group. (Recall that B^A denotes the normal closure of B in A.) Note that H is generated by the elements $\{a_n \mid n \geq 1\}$ subject to the relations

$$a_1 = [a_2, a_3],\ a_2 = [a_3, a_4],\ a_3 = [a_4, a_5],\ \dots\ , a_n = [a_{n+1}, a_{n+2}],\ \cdots .$$

So we see that

$$a_1 = [a_2, a_3] \in H^{(1)}$$

and then that

$$a_1 = [a_2, a_3] = [[a_3, a_4], [a_4, a_5]] \in H^{(2)}.$$

Continuing in this fashion, we see that $a_1 \in \bigcap_{n<\omega} H^{(n)}$. It is easily seen that H is isomorphic to the direct limit of the system

$$H_2 \xrightarrow{h_2} H_3 \xrightarrow{h_3} \dots \xrightarrow{h_{n-1}} H_n \xrightarrow{h_n} H_{n+1} \xrightarrow{h_{n+1}} \cdots$$

where H_n is the free group $\langle a_{n-1}, a_n\rangle$ and $h_n : H_n \to H_{n+1}$ is the embedding such that $a_{n-1} \mapsto [a_n, a_{n+1}]$ and $a_n \mapsto a_n$. The element $a_1 \in H$ corresponds to the constant sequence $(a_1, a_1, \cdots)$ in the direct limit and so a_1 is a nonidentity element of H. By Lemma 2.7, H is not a free group.

Finally consider the canonical homomorphism $\pi : B \to A/A'$ such that for each $n \geq 1$,

$$\pi(a_n^{-1}[a_{n+1}, a_{n+2}]) = a_n^{-1}[a_{n+1}, a_{n+2}]A' = a_n^{-1}A'.$$

Clearly π is surjective and $\ker \pi = B \cap A'$. Note that

$$\begin{aligned} w \in \ker\pi \text{ iff } & \ell_x^A(w) = 0 \text{ for all } x \in \{a_n \mid n \geq 1\}, \\ \text{iff } & \ell_y^B(w) = 0 \text{ for all } y \in \{a_n^{-1}[a_{n+1}, a_{n+2}] \mid n \geq 1\}, \\ \text{iff } & w \in B'. \end{aligned}$$

(The last equivalence is a consequence of Lemma 2.2.) Thus $B \cap A' = B'$ and $B/B' = A/A'$. □

Lemma 2.10. *G/G' is free abelian of cardinality $\aleph_1$.*

Proof. For each $\alpha < \aleph_1$, let $A_\alpha = G_\alpha/G'_\alpha$. By Lemma 2.6, we have that

$$\begin{aligned} G/G' &= \bigcup_{\alpha<\aleph_1} G_\alpha/G_\alpha \cap G' \\ &= \bigcup_{\alpha<\aleph_1} G_\alpha/G'_\alpha = \bigcup_{\alpha<\aleph_1} A_\alpha. \end{aligned}$$

If α is a limit ordinal, then $G_\alpha/G'_\alpha = G_{\alpha+1}/G'_{\alpha+1}$ and so $A_\alpha = A_{\alpha+1}$. On the other hand, if α is a successor ordinal, then $A_{\alpha+1} = A_\alpha \oplus \langle \overline{x}_\alpha\rangle$, where $\overline{x}_\alpha$ denotes the image of the element $x_\alpha \in G_{\alpha+1}$ in the quotient group $A_{\alpha+1}$. It follows that if $\alpha < \beta$, then A_α is a direct summand of A_β. It is now clear that G/G' is a free abelian group of cardinality $\aleph_1$. □

This completes the proof of Theorem 2.4.

3. Concluding remarks

Finally we will say a few words about extending our main result to cardinalities $\lambda > \aleph_1$. We need only consider regular cardinals λ, since Shelah [5] has proved that if G is an almost free group of singular cardinality, then G is free. (The corresponding result is also true for almost free abelian groups of singular cardinality.) In [6], Shelah discovered that the purely combinatorial principle $NPT(\lambda)$ is equivalent to the existence of an almost free non-free abelian group; in [3], it was shown in ZFC that $NPT(\lambda)$ holds for an unbounded set of cardinals below the first cardinal fixed point. If $NPT(\lambda)$ holds, then it is possible to generalize the proof of Theorem 2.4 and construct an almost free non-free group G of cardinality λ such that G/G' is a free abelian group. Of course, the real problem is to produce such a group when $NPT(\lambda)$ fails.

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Institute of Mathematics, Hebrew University, Jerusalem, Israel

Current address: Department of Mathematics, University of California, Irvine, California 92679

E-mail address: `cbitton@math.uci.edu`