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CONTINUOUS SELECTIONS AND REFLEXIVE BANACH SPACES

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ABSTRACT. Every l.s.c. mapping Φ from a space X into the non-empty closed convex subsets of a reflexive Banach space Y admits a continuous selection provided it satisfies a "weak" u.s.c. condition. This result partially generalizes some known selection theorems. Also, it is successful in solving a problem concerning the set of proper lower semi-continuous convex functions on a reflexive Banach space.

1. INTRODUCTION

Let X be a topological space, and let Y be a normed space. We let 2^Y denote the set of all non-empty subsets of Y. Also, let

$$\mathcal{F}(Y) = \{S \in 2^Y : S \text{ is closed}\}$$

and

$$\mathcal{F}_c(Y) = \{S \in \mathcal{F}(Y) : S \text{ is convex}\}.$$

A set-valued mapping $\Phi : X \to 2^Y$ is *lower semi-continuous*, or l.s.c., if the set $\Phi^{-1}(U) = \{x \in X : \Phi(x) \cap U \neq \emptyset\}$ is open in X for every open $U \subset Y$. A map $f : X \to Y$ is a *selection* for $\Phi : X \to 2^Y$ if $f(x) \in \Phi(x),\ x \in X$.

The existence of single-valued continuous selections for l.s.c. mappings $\Phi : X \to \mathcal{F}_c(Y)$ implies some separation properties (like *paracompactness*, *collectionwise normality*, etc.) of X [16]. A starting point of this paper is the following result of [13] which dispenses with the separation properties of X by strengthening the restriction on the *continuity* of Φ.

Theorem 1.1. *Let X be a topological space, let Y be a Banach space, and let $\Phi : X \to \mathcal{F}_c(Y)$ be d-proximal continuous. Then Φ admits a single-valued continuous selection.*

Let us recall that $\Phi : X \to \mathcal{F}_c(Y)$ is *d-proximal continuous* if it is l.s.c. and the set $\Phi^{\#}(\Phi(x_0) + B_\varepsilon) = \{x \in X : \Phi(x) \subset \Phi(x_0) + B_\varepsilon\}$ is open in X for every $x_0 \in X$ and $\varepsilon > 0$. Here, $B_\varepsilon = \{y \in Y : \|y\| < \varepsilon\}$.

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The main purpose of the paper is to show that, in Theorem 1.1, we can weaken the restriction on the *continuity* of Φ provided Y is reflexive. We shall say that $\Phi : X \to \mathcal{F}_c(Y)$ is *weakly continuous* if it is l.s.c., and the set $\Phi^{\#}(Y \setminus K)$ is open in X for every weakly compact $K \subset Y$. Every d-proximal continuous $\Phi : X \to \mathcal{F}_c(Y)$ is weakly continuous, while the converse need not be true (see Propositions 2.5 and 2.6).

Theorem 1.2. *Let X be a topological space, let Y be a reflexive Banach space, and let $\Phi : X \to \mathcal{F}_c(Y)$ be weakly continuous. Then Φ admits a single-valued continuous selection.*

Theorem 1.2 works in several interesting situations. So, it applies to l.s.c. mappings $\Phi : X \to \mathcal{F}_c(Y)$ whose Graph is *weakly closed* (i.e., $\mathrm{Graph}(\Phi) = \{(x, y) : y \in \Phi(x)\}$ is closed in $X \times Y$, where Y is considered in the weak topology) because every such mapping is weakly continuous (Proposition 2.7). By way of example, the Graph of every *upper hemicontinuous* mapping is weakly closed (Proposition 2.8). Thus, Theorem 1.2 implies [1, Theorem 9.3.4] when Φ is l.s.c. and upper hemicontinuous, and Y is a Hilbert space.

Finally, and perhaps most interesting, Theorem 1.2 applies successfully in solving a problem concerning the set $\Gamma(Y)$ of all proper lower semi-continuous convex functions on a reflexive Banach space Y. More precisely, if $\mathrm{Aff}(Y)$ denotes the continuous real affine functions on Y, then we construct a map $\alpha : \Gamma(Y) \to \mathrm{Aff}(Y)$ which is continuous with respect to *Mosco convergence* in $\Gamma(Y)$ and $\mathrm{Aff}(Y)$, and is such that $\alpha(f) \le f$ (i.e. $\alpha(f)(y) \le f(y),\ y \in Y$) for every $f \in \Gamma(Y)$ (Theorem 6.1). In case of separable Y, this is proved by G. Beer [3, Theorem 3.7].

A proof of Theorem 1.2 is obtained in Section 5. A preparation for that is made in Sections 2, 3 and 4.

2. Weakly continuous set-valued mappings

Let X be a topological space, let $(Y, \|.\|)$ be a normed space, and let τ be a topology on $\mathcal{F}_c(Y)$. We shall say that $\Phi : X \to \mathcal{F}_c(Y)$ is *τ-continuous* if Φ is continuous as a mapping from X to the space $(\mathcal{F}_c(Y), \tau)$.

Now, we recall some natural ways for topologizing $\mathcal{F}_c(Y)$. The *Mosco topology* τ_M on $\mathcal{F}_c(Y)$ [2], [3], [17], [18] is the one generated by all sets of the form

$$\langle \mathcal{V} \rangle = \left\{ S \in \mathcal{F}_c(Y) : S \cap V \neq \emptyset,\ V \in \mathcal{V},\ \text{and}\ \ S \subset \bigcup \mathcal{V} \right\},$$

where $\mathcal{V}$ runs over the finite families of open subsets of Y such that $Y \setminus \bigcup \mathcal{V}$ is weakly compact. The *slice topology* τ_S on $\mathcal{F}_c(Y)$ [5], [6], [20] is the one generated by all sets of the form

$$\langle\langle \mathcal{V} \rangle\rangle = \left\{ S \in \langle \mathcal{V} \rangle : S + B_\varepsilon \subset \bigcup \mathcal{V}\ \text{ for some } \varepsilon > 0 \right\},$$

where $\mathcal{V}$ is a finite family of open subsets of Y such that $Y \setminus \bigcup \mathcal{V}$ is a bounded convex subset of Y. Finally, we need the so-called *Wijsman topology* $\tau_{W(d)}$ (see, [12], [15]) generated by all functions

$$d(y, .) : \mathcal{F}(Y) \to \mathbb{R}, \quad y \in Y,$$

where $d(y, A) = \inf\{\|y - a\| : a \in A\},\ A \in \mathcal{F}(Y)$. Namely, by $\tau_{W(d)}$ we will denote the weakest topology on $\mathcal{F}(Y)$ such that, for every $y \in Y$, the function $d(y, .)$ is continuous.

Our interest in these topologies on $\mathcal{F}_c(Y)$ is motivated by the following (usually) strong inclusion:

(2.1) $\tau_M \cup \tau_{W(d)} \subset \tau_S$.

Indeed, the inclusion $\tau_{W(d)} \subset \tau_S$ is obvious. As for $\tau_M \subset \tau_S$, see [7, Proposition 5.4.11]. On the other hand,

(2.2) $\tau_M = \tau_S$ if and only if Y is reflexive [7, Corollary 5.4.14].

(2.3) $\tau_{W(d)} = \tau_M = \tau_S$ if and only if Y is reflexive and the dual norm $\|.\|^*$ is weak* Kadec [7, Corollary 5.5.6] (in sequential form [9]).

Proposition 2.4. *Let X be a topological space, and let $(Y, \|.\|)$ be a normed space. Then $\Phi : X \to \mathcal{F}_c(Y)$ is weakly continuous if and only if it is τ_M-continuous.*

Proof. This follows from the fact that τ_M has a subbase of the form

$$\{S \in \mathcal{F}_c(Y) : S \cap V \neq \emptyset\} \quad \text{and} \quad \{S \in \mathcal{F}_c(Y) : S \cap K = \emptyset\},$$

where V runs over open and K over weakly compact subsets of Y. $\square$

Now, we present some examples of weakly continuous set-valued mappings. To this end, we need the *proximal topology* $\tau_{\delta(d)}$ on $\mathcal{F}(Y)$ [8] generated by all sets of the form $\langle\langle \mathcal{V} \rangle\rangle$, where $\mathcal{V}$ is a finite family of open subsets of Y. Also, we need the *linear topology* τ_L on $\mathcal{F}_c(Y)$ [4], [14] generated by all sets of the form $\langle\langle \mathcal{V} \rangle\rangle$, where $\mathcal{V}$ is a finite family of open subsets of Y such that $Y \setminus \bigcup \mathcal{V}$ is convex.

Proposition 2.5. *Let $(Y, \|.\|)$ be a normed space. Then every d-proximal continuous $\Phi : X \to \mathcal{F}_c(Y)$ is weakly continuous.*

Proof. Suppose $\Phi : X \to \mathcal{F}_c(Y)$ is d-proximal continuous. By [13, Proposition 2.2], Φ is $\tau_{\delta(d)}$-continuous. Hence, it is τ_S-continuous because $\tau_S \subset \tau_{\delta(d)}$. Finally, by (2.1) and Proposition 2.4, Φ is weakly continuous. $\square$

Proposition 2.6. *Let Y be a separable reflexive Banach space. If every weakly continuous $\Phi : X \to \mathcal{F}_c(Y)$ is d-proximal continuous, then the dimension of Y is at most one.*

Proof. Suppose every weakly continuous $\Phi : X \to \mathcal{F}_c(Y)$ is d-proximal continuous. Then, by (2.2), Proposition 2.4 and [13, Proposition 2.2], $\Phi : X \to \mathcal{F}_c(Y)$ is $\tau_{\delta(d)}$-continuous if and only if it is τ_S-continuous. That is, $\tau_{\delta(d)} = \tau_S$ on $\mathcal{F}_c(Y)$. Hence, $\tau_L = \tau_S$ because $\tau_S \subset \tau_L \subset \tau_{\delta(d)}$. Now, by [10], [11], Y admits an equivalent norm $\|.\|$ such that $\|.\|^*$ is weak* Kadec. In particular, this implies the metrizability of $(\mathcal{F}_c(Y), \tau_L)$ because, by (2.3), $\tau_{W(d)} = \tau_S = \tau_L$ with respect to this norm. Therefore, by [7, Proposition 4.3.10], the dimension of Y is at most one. $\square$

Proposition 2.7. *Let X be a topological space, let Y be a normed space, and let $\Phi : X \to \mathcal{F}_c(Y)$ be an l.s.c. mapping whose Graph is weakly closed. Then Φ is weakly continuous.*

Proof. Let $K \subset Y$ be weakly compact. Take a point $x_0 \in \Phi^{\#}(Y \setminus K)$ and $y \in K$. Since Graph(Φ) is weakly closed and $(x_0, y) \notin$ Graph(Φ), there exists a neighbourhood U_y of x_0 and a weak neighbourhood V_y of y such that $U_y \times V_y \cap \text{Graph}(\Phi) = \emptyset$. In this way, we get a cover $\{V_y : y \in K\}$ of K consisting of weakly open subsets of Y. Because of the weak compactness of K, there now exists a finite subset $K_0 \subset K$ such that $K \subset \bigcup\{V_y : y \in K_0\}$. Setting then $U_0 = \bigcap\{U_y : y \in K_0\}$, we get a neighbourhood U_0 of x_0 such that $\Phi(x) \cap K = \emptyset$, $x \in U_0$. So, $U_0 \subset \Phi^{\#}(Y \setminus K)$. $\square$

A mapping $\Phi : X \to \mathcal{F}_c(Y)$ is *upper hemicontinuous* if, for every continuous linear function $p \in Y^*$, the function

$$x \longrightarrow \sigma(\Phi(x), p) = \sup\{p(y) : y \in \Phi(x)\} \in \mathbb{R} \cup \{\infty\}, \quad x \in X,$$

is upper semi-continuous. The following proposition seems to be known (see, for instance, [1, Proposition 2.6.3]).

Proposition 2.8. *Let X be a topological space, let Y a Banach space, and let $\Phi : X \to \mathcal{F}_c(Y)$ be upper hemicontinuous. Then,* Graph(Φ) *is weakly closed.*

Proof. Since Φ is closed convex valued, by a version of the Hahn-Banach Separation Theorem,

$$(*) \qquad \Phi(x) = \{y \in Y : p(y) \leq \sigma(\Phi(x), p),\ p \in Y^*\}, \quad x \in X.$$

Now, let $(x_0, y_0) \notin$ Graph(Φ). By $(*)$, there exists $p_0 \in Y^*$ such that $\sigma(\Phi(x_0), p_0) < p_0(y_0) - \varepsilon$ for some $\varepsilon > 0$. Hence, there exists a neighbourhood U_0 of x_0 such that $\sigma(\Phi(x), p_0) < p_0(y_0) - \varepsilon$ for every $x \in U_0$ because $\sigma(\Phi(.), p_0)$ is upper semi-continuous. Setting then $V_0 = \{y \in Y : |p_0(y) - p_0(y_0)| < \varepsilon\}$, we get a weak neighbourhood V_0 of y_0 such that, by $(*)$, $U_0 \times V_0 \cap$ Graph$(\Phi) = \emptyset$. So, the Graph of Φ is weakly closed. □

3. A continuity criterion in reflexive Banach spaces

Let Y be a normed space. A norm $\|.\|$ on Y is called *locally uniformly rotund* if from

$$\|y_n\| = \|y\| = 1 \ \text{ for all } n, \quad \text{and} \quad \lim_{n\to\infty} \|y_n + y\| = 2,\ y, y_n \in Y,$$

it follows that $\lim_{n\to\infty} \|y_n - y\| = 0$.

Theorem 3.1. *Let $(Y, \|.\|)$ be a normed space with $\|.\|$ locally uniformly rotund, let X be a topological space, $x_0 \in X$, and let $f : X \to Y$ be such that the function $\|f(.)\| : X \to \mathbb{R}$ is continuous at x_0. Also, let $g : X \to Y$ be a map which is continuous at x_0, $g(x_0) = f(x_0)$, and $\|g(x) + f(x)\| \geq 2\|f(x)\|$ for every $x \in X$. Then f is continuous at x_0.*

In what follows, we use S_1 to denote $\{y \in Y : \|y\| = 1\}$. To prove Theorem 3.1, we need the following proposition.

Proposition 3.2. *Let $(Y, \|.\|)$ be a normed space with $\|.\|$ locally uniformly rotund, let X be a topological space, $x_0 \in X$, and let $h : X \to S_1$ be such that $\lim_{x\to x_0} \|h(x) + h(x_0)\| = 2$. Then, h is continuous at x_0.*

Proof. Since $\lim_{x\to x_0} \|h(x) + h(x_0)\| = 2$, for every positive integer n there exists a neighbourhood U_n of x_0 such that

$$(1) \qquad \big|2 - \|h(x) + h(x_0)\|\big| < 2^{-n} \quad \text{for every } x \in U_n.$$

Suppose, contrary to what we wish to prove, that h is not continuous at x_0. Then, there exists an $\varepsilon > 0$ such that for every neighbourhood U of x_0 there is a point $x_U \in U$ for which $\|h(x_0) - h(x_U)\| \geq \varepsilon$. In particular, for every n there exists $x_n = x_{U_n} \in U_n$ such that

$$(2) \qquad \|h(x_0) - h(x_n)\| \geq \varepsilon.$$

Now, by (1), $\lim_{n\to\infty} \|h(x_n) + h(x_0)\| = 2$. Then, by the property of the norm $\|.\|$, $\lim_{n\to\infty} h(x_n) = h(x_0)$, which contradicts (2). □

Proof of Theorem 3.1. In the case $\|f(x_0)\| = 0$, there is nothing to prove. So, we may suppose that $\|f(x_0)\| > 0$. Since $\|f(.)\|$ is continuous at x_0, there exists a neighbourhood U_0 of x_0 such that $\|f(x)\| > 0,\ x \in U_0$. Then, define a map $h : X \to S_1$ by

$$h(x) = \frac{f(x)}{\|f(x)\|} \text{ if } x \in U_0 \quad \text{and} \quad h(x) = h(x_0) \text{ otherwise.}$$

Also, define another map $k : X \to Y$ by

$$k(x) = \frac{g(x)}{\|f(x)\|} \text{ if } x \in U_0 \quad \text{and} \quad k(x) = h(x_0) \text{ otherwise.}$$

Now, on the one hand, $\lim_{x\to x_0} k(x) = \frac{f(x_0)}{\|f(x_0)\|} = h(x_0)$ because $\|f(.)\|$ and g are continuous at x_0, and $g(x_0) = f(x_0)$. On the other hand, $x \in U_0$ implies

$$2 - \|k(x) - h(x_0)\| \le \|h(x) + h(x_0)\| \le 2$$

because

$$\|k(x) + h(x)\| = \frac{\|g(x) + f(x)\|}{\|f(x)\|} \ge \frac{2\|f(x)\|}{\|f(x)\|} = 2\|h(x)\| = 2,\ x \in U_0.$$

So, $\lim_{x\to x_0} \|h(x) + h(x_0)\| = 2$. Hence, by Proposition 3.2, h is continuous at x_0, and therefore the same is true for f because $f|U_0 = (\|f(.)\| \cdot h)|U_0$. □

4. Norm-minimal selections

Let $(Y, \|.\|)$ be a normed linear space. A selection $f : X \to Y$ for $\Phi : X \to 2^Y$ is *minimal* with respect to the norm, or *norm-minimal* (see [1]), if

$$\|f(x)\| = \min\{\|y\| : y \in \Phi(x)\} \quad \text{for every } x \in X.$$

In what follows, we use D_ε to denote $\{y \in Y : \|y\| \le \varepsilon\}$.

Theorem 4.1. *Let X be a topological space, let $(Y, \|.\|)$ be a reflexive Banach space, endowed with a locally uniformly rotund norm $\|.\|$, and let $\Phi : X \to \mathcal{F}_c(Y)$ be l.s.c. The following two conditions are equivalent*:

(a) *$\Phi^{-1}(D_\varepsilon)$ is closed in X for every $\varepsilon > 0$.*
(b) *Φ admits a continuous norm-minimal selection.*

Proof. Since Φ is closed convex valued, Y is a reflexive Banach space, and $\|.\|$ is locally uniformly rotund, for every $x \in X$ there exists a unique point $f(x) \in \Phi(x)$ such that

$$\|f(x)\| = \min\{\|y\| : y \in \Phi(x)\}. \tag{1}$$

In this way, we get a map $f : X \to Y$ which is a selection for Φ. Since Φ is l.s.c., the statement of (a) becomes equivalent to continuity of $\|f(.)\|$. Thus, to finish the proof, it only remains to show that f is continuous provided $\|f(.)\|$ is. To see this, we repeat elements of the proof of [19, Theorem]. Namely, take an arbitrary point $x_0 \in X$. For technical reasons only, let $B_{1/0} = Y$. Next, for every integer $n \ge 0$, set

$$U_n = \Phi^{-1}(f(x_0) + B_{1/n}). \tag{2}$$

Now, for every point $x \in U_0 = X$, let $n(x) = \sup\{n \ge 0 : x \in U_n\}$. Since each $\Phi(x)$ is closed, by (2),

$$n(x) = \infty \quad \text{implies} \quad f(x_0) \in \Phi(x). \tag{3}$$

For every $x \in X$ fix a point $g_0(x) \in U_{n(x)} \cap \Phi(x)$, where $U_\infty = \{f(x_0)\}$. In this way, we get a map $g_0 : X \to Y$, which is continuous at x_0. Indeed, Φ is l.s.c., and therefore, by (2), each $U_n,\ n < \infty$, is open. Then, by (3), this follows from the fact that $g_0^{-1}(f(x_0) + B_{1/n}) = U_n$ for every n.

Since g_0 and f are selections for Φ and each $\Phi(x)$ is convex, we now get

$$\frac{g_0(x) + f(x)}{2} \in \Phi(x), \quad x \in X.$$

By (1), this implies $\|g_0(x) + f(x)\| \geq 2\|f(x)\|, \quad x \in X$. Finally, by Theorem 3.1 (with $g = g_0$), f is continuous at x_0 because so are g_0 and $\|f(.)\|$. ☐

5. Proof of Theorem 1.2

Let $A \subset X$. We shall say that a map $g : A \to Y$ is *A-regular* (see [13]) if for every locally finite cozero-set cover $\mathcal{V}$ of Y there exists a locally finite cozero-set cover $\mathcal{U}$ of X such that $g^{-1}(\mathcal{V})$ is refined by $\mathcal{U} \cap A = \{U \cap A : U \in \mathcal{U}\}$. The following important property of *A-regularity* was actually established in [13].

Proposition 5.1. *Let X be a topological space, let $A \subset X$, Y be a Banach space, and $g : A \to Y$. Then g can be extended to a continuous map $k : X \to Y$ if and only if g is A-regular.*

In the situation of Theorem 1.2, every weakly continuous mapping admits "sufficiently many" continuous selections. Namely, in this section we establish the following more general selection theorem.

Theorem 5.2. *Let X be a topological space, let Y be a reflexive Banach space, and let $\Phi : X \to \mathcal{F}_c(Y)$ be weakly continuous. If $A \subset X$, then every A-regular selection $g : A \to Y$ for $\Phi|A$ can be extended to a continuous selection for Φ.*

It should be mentioned that Theorem 5.2 partially generalizes [13, Theorem 6.1]. As for its proof, we need the following simple proposition, whose verification is left to the reader.

Proposition 5.3. *Let X be a topological space, let $(Y, \|.\|)$ be a normed space, and let $\Phi : X \to \mathcal{F}_c(Y)$ be $\tau_{W(d)}$-continuous. Then, for every continuous $h : X \to Y$, the function $d(h(.), \Phi(.)) : X \to \mathbb{R}$ is continuous.*

Proof of Theorem 5.2. Let $A \subset X$, and let $g : A \to Y$ be an A-regular selection for $\Phi|A$. By Proposition 5.1, g extends to a continuous map $k : X \to Y$. Next, define another set-valued mapping $\varphi : X \to \mathcal{F}_c(Y)$ by $\varphi(x) = \Phi(x) - k(x),\ x \in X$. Note that, for every $x \in X$,

$$(*) \qquad 0 \in \varphi(x) \quad \text{if and only if} \quad k(x) \in \Phi(x).$$

Now, by [21, Theorem 1], Y admits an equivalent norm $\|.\|$ which is locally uniformly rotund. Let us check that the mapping φ is $\tau_{W(d)}$-continuous with respect to this norm. Indeed, take a point $y_0 \in Y$. Then, for every $x \in X$,

$$d(y_0, \varphi(x)) = \inf\{\|y_0 + k(x) - y\| : y \in \Phi(x)\} = d(y_0 + k(x), \Phi(x)).$$

Hence, by Proposition 5.3 (with $h(x) = y_0 + k(x)$), $d(y_0, \varphi(.)) : X \to \mathbb{R}$ is continuous because, by (2.1), (2.2) and Proposition 2.4, Φ is $\tau_{W(d)}$-continuous.

Thus, in particular, φ is l.s.c. such that $\varphi^{-1}(D_\varepsilon)$ is closed in X for every $\varepsilon > 0$. Then, by Theorem 4.1, φ admits a continuous minimal selection $\ell : X \to Y$. Finally, by $(*)$, $f = \ell + k$ is a selection for Φ which extends g. ☐

6. Lower semi-continuous convex functions and continuous affine functions

Let $(Y, \|.\|)$ be a linear normed space. The *epigraph* of a convex function $f : Y \to [-\infty, \infty]$ is the following convex subset of $Y \times \mathbb{R}$:

$$\operatorname{epi} f = \{(y, r) \in Y \times \mathbb{R} : r \geq f(y)\}.$$

A convex function $f : Y \to [-\infty, \infty]$ is called *proper* if its epigraph is non-empty and contains no vertical lines. Let us recall that such a function is *lower semi-continuous* if and only if its epigraph is closed in $Y \times \mathbb{R}$.

Now, let $\Gamma(Y)$ denote the proper lower semi-continuous convex functions on Y, and let $\mathrm{Aff}(Y)$ denote the continuous real affine functions on Y. The fundamental notion of convergence for sequences in $\Gamma(Y)$ is Mosco convergence, introduced by U. Mosco [17]: A sequence $\{f_n\} \subset \Gamma(Y)$ is *Mosco convergent* to $f \in \Gamma(Y)$ if, for every $y \in Y$, (1) there exists a sequence $\{y_n\}$ convergent strongly to y for which $\lim f_n(y_n) = f(y)$; and (2) whenever $\{y_n\}$ converges weakly to y, then $\liminf f_n(y_n) \geq f(y)$.

In the present paper we regard another approach to Mosco convergence in $\Gamma(Y)$. Namely, identifying each function $f \in \Gamma(Y)$ with its epigraph, we may view $\Gamma(Y)$ as a subset of $\mathcal{F}_c(Y \times \mathbb{R})$. Then, the Mosco convergence in $\Gamma(Y)$ is compatible with the Mosco topology τ_M in $\mathcal{F}_c(Y \times \mathbb{R})$ [2].

In this section, we obtain the following generalization of [3, Theorem 3.7].

Theorem 6.1. *Let Y be a reflexive Banach space, $A \subset \Gamma(Y)$, and let $\beta : (A, \tau_M) \to (\mathrm{Aff}(Y), \tau_M)$ be an A-regular map such that $\beta(f) \leq f$ for every $f \in A$. Then, β can be extended to a continuous map $\alpha : (\Gamma(Y), \tau_M) \to (\mathrm{Aff}(Y), \tau_M)$ such that $\alpha(f) \leq f$ for every $f \in \Gamma(Y)$.*

Proof. To any $f \in \Gamma(Y)$, we consider its *conjugate* $f^* : Y^* \to [-\infty, \infty]$ defined by the formula:

$$f^*(p) = \sup\{p(y) - f(y) : y \in Y\}, \quad p \in Y^*. \tag{1}$$

The so-obtained map $f \to f^*$ is usually called the *Young-Fenchel transform* and, by [3, Theorem 3.1] (in sequential form [18]), it is a homeomorphism of $(\Gamma(Y), \tau_M)$ onto $(\Gamma(Y^*), \tau_M)$.

In what follows, we also need the map $\psi : Y^* \times \mathbb{R} \to \mathrm{Aff}(Y)$ defined by

$$\psi(p, r)(y) = p(y) - r, \quad y \in Y. \tag{2}$$

By [3, Theorem 3.4], this map is a homeomorphism, where Y^* is equipped with the norm topology and $\mathrm{Aff}(Y)$ with the Mosco topology.

Now, let $f^* \in Y^*$, and let us observe that, by (1) and (2),

$$(p, r) \in \operatorname{epi} f^* \quad \text{if and only if} \quad \psi(p, r) \leq f. \tag{3}$$

Since Y is reflexive, so is the space $Y^* \times \mathbb{R}$, where Y^* is equipped with the norm topology. Now, let $A^* = \{f^* : f \in A\}$, and let $\beta^*(f^*) = \psi^{-1}(\beta(f))$, $f \in A$. Thus, we get an A^*-regular map $\beta^* : A^* \to Y^* \times \mathbb{R}$ because β is an A-regular map and because the Young-Fenchel transform and ψ are homeomorphisms. By (3), $\beta^*(f^*) \in \operatorname{epi} f^*$, $f^* \in A^*$. Also, the set-valued mapping: $f^* \to \operatorname{epi} f^*$ is weakly continuous. Then, by Theorem 5.2, there exists a continuous map $\alpha^* : (\Gamma(Y^*), \tau_M) \to Y^* \times \mathbb{R}$ such that $\alpha^*|A^* = \beta^*$ and $\alpha^*(f^*) \in \operatorname{epi} f^*$, $f^* \in \Gamma(Y^*)$. Finally, by (3), $\alpha(f) = \psi(\alpha^*(f^*))$, $f \in \Gamma(Y)$, is as required. ∎

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