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A LIFTING THEOREM FOR SYMMETRIC COMMUTANTS

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Abstract. Let $T_1, \dots, T_n \in B(\mathcal{H})$ be bounded operators on a Hilbert space $\mathcal{H}$ such that $T_1T_1^* + \cdots + T_nT_n^* \le I_{\mathcal{H}}$. Given a symmetry j on $\mathcal{H}$, i.e., $j^2 = j^*j = I_{\mathcal{H}}$, we define the j-symmetric commutant of $\{T_1, \dots, T_n\}$ to be the operator space

$$\{A \in B(\mathcal{H}) :\ T_iA = jAT_i,\ i = 1, \dots, n\}.$$

In this paper we obtain lifting theorems for symmetric commutants. The result extends the Sz.-Nagy–Foiaş commutant lifting theorem ($n = 1, j = I_{\mathcal{H}}$), the anticommutant lifting theorem of Sebestyén ($n = 1, j = -I_{\mathcal{H}}$), and the noncommutative commutant lifting theorem ($j = I_{\mathcal{H}}$). Sarason's interpolation theorem for H^∞ is extended to symmetric commutants on Fock spaces.

1. Lifting theorem for symmetric commutants

Let $\mathbb{F}_n^+$ be the unital free semigroup on n generators $s_1, \dots, s_n$, and let e be its neutral element. For any $\sigma := s_{i_1} \cdots s_{i_k} \in \mathbb{F}_n^+$, we define its length $|\sigma| := k$, and $|e| = 0$. On the other hand, if $T_i \in B(\mathcal{H}),\ i = 1, \dots, n$, we denote $T_\sigma := T_{i_1} \cdots T_{i_k}$ and $T_e := I_{\mathcal{H}}$.

Let us recall from [Po1], [Po2], and [Po4] a few results concerning the noncommutative dilation theory for n-tuples of operators. A sequence of operators $\mathcal{T} := [T_1, \dots, T_n]$, $T_i \in B(\mathcal{H})$, $i = 1, \dots, n$, is called contractive (or row contraction) if $T_1T_1^* + \cdots + T_nT_n^* \le I_{\mathcal{H}}$. We say that a sequence of isometries $\mathcal{V} := [V_1, \dots, V_n]$ on a Hilbert space $\mathcal{K} \supset \mathcal{H}$ is a minimal isometric dilation of $\mathcal{T}$ if the following properties are satisfied:

(i) $V_1V_1^* + \cdots + V_nV_n^* \le I_{\mathcal{K}}$;
(ii) $V_i^*|_{\mathcal{H}} = T_i^*,\ i = 1, \dots, n$;
(iii) $\mathcal{K} = \bigvee_{\alpha \in \mathbb{F}_n^+} V_\alpha \mathcal{H}$.

Consider the full Fock space on n generators

$$F^2(H_n) := \mathbb{C}1 \oplus \bigoplus_{m \ge 1} H_n^{\otimes m},$$

where H_n is an n-dimensional complex Hilbert space with orthonormal basis $\{e_1, e_2, \dots, e_n\}$ if n is finite, and $\{e_1, e_2, \dots\}$ if $n = \infty$. For each $i = 1, 2, \dots$, let S_i be the left creation operator with e_i, i.e.,

$$S_i\xi := e_i \otimes \xi, \quad \xi \in F^2(H_n).$$

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As in [Po1], let us define $D_{\mathcal{T}} : \bigoplus_{j=1}^n \mathcal{H} \to \bigoplus_{j=1}^n \mathcal{H}$ by $D_{\mathcal{T}} := (I_{\bigoplus_{j=1}^n \mathcal{H}} - \mathcal{T}^*\mathcal{T})^{1/2}$, and set $\mathcal{D} := \overline{D_{\mathcal{T}}(\bigoplus_{j=1}^n \mathcal{H})}$. Let $D_i : \mathcal{H} \to F^2(H_n) \otimes \mathcal{D}$ be defined by

$$D_i h := 1 \otimes D_{\mathcal{T}}(\underbrace{0, \ldots, 0}_{i-1 \text{ times}}, h, 0, \ldots, 0) \oplus 0 \oplus 0 \cdots .$$

Consider the Hilbert space $\mathcal{K} := \mathcal{H} \oplus (F^2(H_n) \otimes \mathcal{D})$ and define $V_i : \mathcal{K} \to \mathcal{K}$ by

$$V_i(h \oplus (\xi \otimes d)) := T_i h \oplus (D_i h + (S_i \otimes I_{\mathcal{D}})(\xi \otimes d))$$

for any $h \in \mathcal{H}, \xi \in F^2(H_n), d \in \mathcal{D}$. Notice that

$$V_i = \begin{bmatrix} T_i & 0 \\ D_i & S_i \otimes I_{\mathcal{D}} \end{bmatrix}$$

with respect to the decomposition $\mathcal{K} = \mathcal{H} \oplus (F^2(H_n) \otimes \mathcal{D})$. It was proved in [Po1] that the sequence $\mathcal{V} := [V_1, \ldots, V_n]$ is the minimal isometric dilation of $\mathcal{T}$, and it is uniquely determined up to an isomorphism. Let $\mathcal{H}_0 := \mathcal{H}$ and

$$\mathcal{H}_k := \mathcal{H}_{k-1} \vee (\bigvee_{|\alpha|=1} V_\alpha \mathcal{H}_{k-1}) \quad \text{if } k \geq 1. \tag{1.1}$$

Notice that $\mathcal{K} = \bigvee_{k=0}^\infty \mathcal{H}_k$, $\mathcal{H}_k \subseteq \mathcal{H}_{k+1}$, and all subspaces $\mathcal{H}_k$ are invariant to each $V_i^*, i = 1, \ldots, n$. On the other hand, we have $\mathcal{H}_1 = \mathcal{H} \oplus \mathcal{D}$ and

$$\mathcal{H}_k = \mathcal{H} \oplus \bigoplus_{|\alpha| \leq k-1} e_\alpha \otimes \mathcal{D} \quad \text{if } k \geq 2, \tag{1.2}$$

where $\{e_\alpha\}_{\alpha \in \mathbb{F}_n^+}$ is the canonical basis of $F^2(H_n)$ generated by $e_1, \ldots, e_n$, i.e., $e_\alpha := e_{i_1} \otimes \cdots \otimes e_{i_k}$ if $\alpha := s_{i_1} \cdots s_{i_k} \in \mathbb{F}_n^+$, and $e_\alpha = 1$ if $\alpha = e$. Denote by $P_{\mathcal{H}_k}$ the orthogonal projection from $\mathcal{K}$ onto $\mathcal{H}_k$, and notice that

$$P_{\mathcal{H}_{k+1}} V_i = V_i P_{\mathcal{H}_k} \quad \text{for any} \quad i = 1, \ldots, n,$$

and $k = 0, 1, \ldots$.

Let us recall from [Se] the following consequence of Krein's self-adjoint extension theorem [K].

Lemma 1.1. *Let $\mathcal{M}, \mathcal{M}'$ be Hilbert spaces, $\mathcal{N} \subseteq \mathcal{M}$, $\mathcal{N}' \subseteq \mathcal{M}'$ be subspaces, and $X \in B(\mathcal{N}, \mathcal{M}'), X' \in B(\mathcal{N}', \mathcal{M})$ be bounded operators. Then there exists $Y \in B(\mathcal{M}, \mathcal{M}')$ such that $Y|_{\mathcal{N}} = X$ and $Y^*|_{\mathcal{N}'} = X'$ if and only if*

$$\langle Xh, h' \rangle = \langle h, X'h' \rangle \tag{1.3}$$

for any $h \in \mathcal{N}$, $h' \in \mathcal{N}$. Moreover, Y can be chosen of norm $\max\{\|X\|, \|X'\|\}$.

Proof. Let $A : \mathcal{N} \oplus \mathcal{N}' \to \mathcal{M} \oplus \mathcal{M}'$ be defined by

$$\begin{bmatrix} 0 & X' \\ X & 0 \end{bmatrix}$$

and notice that $\|A\| = \max\{\|X\|, \|X'\|\}$. Since (1.3) holds, A is a symmetric operator, i.e., $\langle Ax, y \rangle = \langle x, Ay \rangle$ for any $x, y \in \mathcal{N} \oplus \mathcal{N}'$. According to Krein's self-adjoint extension theorem [K], there exists $B \in B(\mathcal{M} \oplus \mathcal{M}')$ such that $B = B^*$, $B|_{\mathcal{N} \oplus \mathcal{N}'} = A$, and $\|B\| = \|A\|$. The operator $Y \in B(\mathcal{M}, \mathcal{M}')$ defined by $Y := P_{\mathcal{M}'} B|_{\mathcal{M}}$, where $P_{\mathcal{M}'}$ is the orthogonal projection from $\mathcal{M} \oplus \mathcal{M}'$ to $\mathcal{M}'$, has the required properties. $\square$

Let $\mathcal{T} := [T_1, \ldots, T_n]$, $T_i \in B(\mathcal{H})$, be a row contraction, $\mathcal{V} := [V_1, \ldots, V_n]$ be its minimal isometric dilation on $\mathcal{K} \supseteq \mathcal{H}$, and $\mathcal{H}_k \subseteq \mathcal{K}$, $k = 0, 1, \ldots,$ be defined by (1.1). Let $j \in B(\mathcal{H})$ be a symmetry on $\mathcal{H}$, i.e., $j^2 = j^*j = I_{\mathcal{H}}$.

We say that $A \in B(\mathcal{H})$ is in the j-commutant of $\{T_1, \ldots, T_n\}$ if and only if

$$T_iA = jAT_i \quad \text{for any } i = 1, \ldots, n.$$

The main result of this paper is the following lifting theorem for j-commutants.

Theorem 1.2. *Let $\mathcal{T} := [T_1, \ldots, T_n]$ be a row contraction with $T_i \in B(\mathcal{H})$ and let $\mathcal{V} := [V_1, \ldots, V_n]$ be its minimal isometric dilation on $\mathcal{K} \supseteq \mathcal{H}$. Let j be a symmetry on $\mathcal{H}$ and let $J \in B(\mathcal{K})$ be a contraction such that each $\mathcal{H}_k$, $k = 0, 1, \ldots,$ reduces J and $J|_{\mathcal{H}} = j$. If $A \in B(\mathcal{H})$ is in the j-commutant of $\{T_1, \ldots, T_n\}$, then there exists $A_J \in B(\mathcal{K})$ in the J-commutant of $\{V_1, \ldots, V_n\}$ such that*

$$A_J^*|_{\mathcal{H}} = A^* \quad \text{and} \quad \|A_J\| = \|A\|.$$

Proof. We use the notation and preliminaries preceding the theorem. Our goal is to construct a sequence $A_k \in B(\mathcal{H}_k)$, $k = 1, 2, \ldots,$ with the following properties:

(i) $A_k^*|_{\mathcal{H}_{k-1}} = A_{k-1}^*$;
(ii) $A_k|_{\bigvee_{i=1}^n V_i\mathcal{H}_{k-1}} = \left(\sum_{i=1}^n P_{\mathcal{H}_k} JV_iA_{k-1}V_i^*\right)|_{\bigvee_{i=1}^n V_i\mathcal{H}_{k-1}}$;
(iii) $P_{\mathcal{H}_k} JV_iA_{k-1}P_{\mathcal{H}_{k-1}} = A_kP_{\mathcal{H}_k}V_i$, $i = 1, \ldots, n$;
(iv) $\|A_k\| = \|A_{k-1}\| = \|A\|$.

Once this is established, since $\mathcal{K} = \bigvee_{k=0}^\infty \mathcal{H}_k$, it will be easy to see that the limit $A_J^* := \text{SOT} - \lim_{k\to\infty} A_k^*P_{\mathcal{H}_k}$ exists and A_J has the stated properties in the theorem.

Our first step is to show that A_1 exists with the above-mentioned properties, when $k = 1$. Since $P_{\mathcal{H}}J = JP_{\mathcal{H}}$, $J|_{\mathcal{H}} = j$, $A \in B(\mathcal{H})$ is in the j-commutant of $\{T_1, \ldots, T_n\}$, $j^2 = 1$, and $\mathcal{V}$ is the minimal isometric dilation of $\mathcal{T}$, we have

$$\begin{aligned}\langle(\sum_{i=1}^n P_{\mathcal{H}_1}JV_iAV_i^*)(\sum_{i=1}^n V_ih_i), h\rangle &= \langle\sum_{i=1}^n P_{\mathcal{H}}JV_iAh_i, h\rangle = \langle\sum_{i=1}^n jT_iAh_i, h\rangle \\ &= \langle\sum_{i=1}^n j^2AT_ih_i, h\rangle = \langle\sum_{i=1}^n P_{\mathcal{H}}T_ih_i, A^*h\rangle \\ &= \langle\sum_{i=1}^n V_ih_i, A^*h\rangle\end{aligned}$$

for any $h_i, h \in \mathcal{H}$. On the other hand, since $\|J\| \le 1$, $P_{\mathcal{H}_1}J = JP_{\mathcal{H}_1}$, and $V_1, \ldots, V_n$ have orthogonal ranges, we have

$$\begin{aligned}\|(\sum_{i=1}^n P_{\mathcal{H}_1}JV_iAV_i^*)(\sum_{i=1}^n V_ih_i)\|^2 &= \|\sum_{i=1}^n JV_iAh_i\|^2 \\ &\le \|\sum_{i=1}^n V_iAh_i\|^2 = \sum_{i=1}^n \|Ah_i\|^2 \\ &\le \|A\|^2\|\sum_{i=1}^n V_ih_i\|^2.\end{aligned}$$

Therefore,

$$\|(\sum_{i=1}^{n} P_{\mathcal{H}_1} JV_i A V_i^*)|_{\vee_{i=1}^{n} V_i \mathcal{H}}\| \leq \|A\|.$$

Now, we apply Lemma 1.1 in the particular case when

$$\mathcal{M} := \mathcal{M}' := \mathcal{H}_1 := \mathcal{H} \vee \bigvee_{i=1}^{n} V_i \mathcal{H},$$
$$\mathcal{N} := \bigvee_{i=1}^{n} V_i \mathcal{H}, \quad \mathcal{N}' := \mathcal{H},$$

and $X \in B(\mathcal{N}, \mathcal{M}')$ is defined by $X := (\sum_{i=1}^{n} P_{\mathcal{H}_1} JV_i A V_i^*)|_{\bigvee_{i=1}^{n} V_i \mathcal{H}}$, and $X' \in B(\mathcal{N}', \mathcal{M})$ by $X' := A^*$. Therefore, we find $A_1 \in B(\mathcal{H}_1)$ such that $A_1^*|_{\mathcal{H}} = A^*$, $\|A_1\| = \|A\|$, and $A_1|_{\bigvee_{i=1}^{n} V_i \mathcal{H}} = (\sum_{i=1}^{n} P_{\mathcal{H}_1} JV_i A V_i^*)|_{\bigvee_{i=1}^{n} V_i \mathcal{H}}$. On the other hand, since $V_i P_{\mathcal{H}} = P_{\mathcal{H}_1} V_i$ and $V_p^* V_i = \delta_{pi} I_{\mathcal{K}}$, we infer that

$$\begin{aligned} P_{\mathcal{H}_1} JV_i A P_{\mathcal{H}} &= (\sum_{p=1}^{n} P_{\mathcal{H}_1} JV_p A V_p^*) V_i P_{\mathcal{H}} \\ &= A_1 V_i P_{\mathcal{H}} = A_1 P_{\mathcal{H}_1} V_i \end{aligned} \tag{1.4}$$

for any $i = 1, \ldots, n$.

Our second step is to apply again Lemma 1.1 in order to to find $A_2 \in B(\mathcal{H}_2)$ such that $A_2^*|_{\mathcal{H}_1} = A_1^*$, $\|A_2\| = \|A_1\|$, $A_2|_{\bigvee_{i=1}^{n} V_i \mathcal{H}_1} = (\sum_{i=1}^{n} P_{\mathcal{H}_2} JV_i A_1 V_i^*)|_{\bigvee_{i=1}^{n} V_i \mathcal{H}_1}$, and, moreover,

$$P_{\mathcal{H}_1} JV_i A P_{\mathcal{H}} = A_1 P_{\mathcal{H}_1} V_i \tag{1.5}$$

for any $i = 1, \ldots, n$. Let $k, k_i, i = 1, \ldots, n$, be in $\mathcal{H}_1$. Taking into account that $JP_{\mathcal{H}_1} = P_{\mathcal{H}_1} J$, $P_{\mathcal{H}_1} V_i = V_i P_{\mathcal{H}}$, $P_{\mathcal{H}} A_1 = A P_{\mathcal{H}}$, and the relation (1.4), we infer the following

$$\begin{aligned} \langle (\sum_{i=1}^{n} P_{\mathcal{H}_2} JV_i A_1 V_i^*)(\sum_{i=1}^{n} V_i k_i), k\rangle &= \langle \sum_{i=1}^{n} P_{\mathcal{H}_2} JV_i A_1 k_i, k\rangle = \langle \sum_{i=1}^{n} JP_{\mathcal{H}_1} V_i A_1 k_i, k\rangle \\ &= \langle \sum_{i=1}^{n} JV_i P_{\mathcal{H}} A_1 k_i, k\rangle = \langle \sum_{i=1}^{n} P_{\mathcal{H}_1} JV_i A P_{\mathcal{H}} k_i, k\rangle \\ &= \langle \sum_{i=1}^{n} A_1 P_{\mathcal{H}_1} V_i k_i, k\rangle = \langle \sum_{i=1}^{n} V_i k_i, A_1^* k\rangle. \end{aligned}$$

On the other hand, since $\|J\| \leq 1$, $P_{\mathcal{H}_2} J = JP_{\mathcal{H}_2}$, and $V_1, \ldots, V_n$ have orthogonal ranges, we infer that

$$\|(\sum_{i=1}^{n} P_{\mathcal{H}_2} JV_i A_1 V_i^*)|_{\bigvee_{i=1}^{n} V_i \mathcal{H}_1}\| \leq \|A_1\|$$

(see the first step for similar computations).

Using Lemma 1.1, we find A_2 with the above-mentioned properties. On the other hand, since $V_iP_{\mathcal{H}_1} = P_{\mathcal{H}_2}V_i$ and $V_p^*V_i = \delta_{pi}I_{\mathcal{K}}$, we infer that

$$\begin{aligned} P_{\mathcal{H}_2}JV_iA_1P_{\mathcal{H}_1} &= (\sum_{p=1}^{n} P_{\mathcal{H}_2}JV_pAV_p^*)V_iP_{\mathcal{H}_1} \\ &= A_2V_iP_{\mathcal{H}_1} = A_1P_{\mathcal{H}_2}V_i \end{aligned}$$

for any $i = 1, \ldots, n$.

Iterating the process, we obtain at the kth step an operator $A_k \in B(\mathcal{H}_k)$ satisfying the properties (i), (ii), (iii), (iv). The proof is complete. □

Let us remark that in the particular case when $n = 1, j = I_{\mathcal{H}}$, and $J = I_{\mathcal{K}}$, we find again the Sz.-Nagy–Foiaş commutant lifting theorem [SzF1], [SzF2] (see also [S], [DMP]). The anticommutant lifting theorem of Sebestyén [Se] is obtained when $n = 1, j = -I_{\mathcal{H}}$, and $J = -I_{\mathcal{K}}$. When $j = I_{\mathcal{H}}$ and $J = I_{\mathcal{K}}$, we obtain a new proof of the noncommutative commutant lifting theorem [Po4]. We refer to [Po5], [Po7], [Po8], [Po9], and [APo], for applications of this theorem to interpolation in several variables.

2. Symmetric commutants on Fock spaces

For each $i = 1, 2, \ldots$, let S_i be the left creation operator with e_i, i.e.,

$$S_i\xi := e_i \otimes \xi,\ \xi \in F^2(H_n).$$

We shall denote by $\mathcal{P}$ the set of all $p \in F^2(H_n)$ which are sums of finite number of tensor monomials, i.e., $p = a_0 + \sum a_{i_1\cdots i_k} e_{i_1} \otimes \cdots \otimes e_{i_k}$, where $a_0, a_{i_1\ldots i_k} \in \mathbb{C}$. The set $\mathcal{P}$ may be viewed as the algebra of polynomials in n noncommuting indeterminates, with $p \otimes q,\ p, q \in \mathcal{P}$, as multiplication. Define F_n^∞ as the set of all $g \in F^2(H_n)$ such that

$$\|g\|_\infty := \sup\{\|g \otimes p\|_2 : p \in \mathcal{P}, \|p\|_2 \le 1\} < \infty$$

where $\|\cdot\|_2 := \|\cdot\|_{F^2(H_n)}$. The Banach algebra F_n^∞ can be viewed as a noncommutative analogue of the Hardy space H^∞; when $n = 1$ they coincide. It follows from [Po6] that the noncommutative analytic Toeplitz algebra F_n^∞ can be identified with the WOT-closed algebra generated by the left creation operators $S_1, \ldots S_n$, and the identity. The algebra F_n^∞ was introduced by the author in [Po3] in connection with a noncommutative von Neumann type inequality in several variables. In particular, we proved that there is a completely contractive homomorphism

$$\Phi : F_n^\infty \to H^\infty(\mathbb{B}_n), \quad [\Phi(f)](\lambda_1, \ldots, \lambda_n) = f(\lambda_1, \ldots, \lambda_n),$$

where $f := f(S_1, \ldots, S_n) \in F_n^\infty$ and $(\lambda_1, \ldots, \lambda_n) \in \mathbb{B}_n$, the open unit ball of $\mathbb{C}^n$. This homomorphism established a strong connection between the algebra F_n^∞ and the function theory of the open unit ball $\mathbb{B}^n$ (see [APo], [Po7]).

We recall from [Po5] the characterization of the commutant of $\{S_1, \ldots, S_n\}$. Define the flipping operator $U : F^2(H_n) \to F^2(H_n)$ by

$$U(e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_k}) = e_{i_k} \otimes \cdots \otimes e_{i_2} \otimes e_{i_1},$$

and let $\tilde{\varphi} := U\varphi$. It is easy to see that U is a unitary operator, which satisfies $U(\varphi \otimes \psi) = \tilde{\psi} \otimes \tilde{\varphi}$, and $U^2 = I$. According to Theorem 1.2 from [Po5], an operator $A \in B(F^2(H_n))$ commutes with $\{S_1, \ldots, S_n\}$ if and only if there exists $\phi \in F_n^\infty$ such that $Ah = h \otimes \tilde{\phi},\ h \in F^2(H_n)$. Notice that $A = U^*\phi(S_1, \ldots, S_n)U$, where

$\phi(S_1,\ldots,S_n)$ is the left multiplication operator by ϕ on the Fock space $F^2(H_n)$. In what follows we use the natural identification of ϕ with $\phi(S_1,\ldots,S_n)$.

As in [Po2], we say that an element $\varphi \in F_n^\infty$ is inner if φ is an isometry, and outer if it has dense range. A family of inner elements $\{\varphi_i : i \in J\}$ is called orthogonal if φ_i and φ_j have orthogonal ranges, whenever $i \neq j$. A complete description of the invariant subspace structure of F_n^∞ was obtained in [Po2] (even in a more general setting), using a noncommutative version of the Wold decomposition (see [Po1]). It follows from Theorem 2.2 of [Po2] that a subspace $\mathcal{N}$ of $F^2(H_n)$ is invariant under $S_1,\ldots,S_n$ if and only if

$$\mathcal{N} = \bigoplus_{i\in J} U^*\varphi_i U[F^2(H_n)]$$

for some family $\{\varphi_i : i \in J\}$ of orthogonal inner elements (see also [Po5]).

In what follows we extend Sarason's result [S] to symmetric commutants on Fock spaces. Let $\mathcal{K}$ be a Hilbert space and let $F_n^\infty \bar{\otimes} B(\mathcal{K})$ be the WOT-closed algebra generated by the spatial tensor product of the two algebras.

Theorem 2.1. *Let $j \in B(\mathcal{K})$ be a symmetry on a Hilbert space $\mathcal{K}$ and let $\mathcal{N} \subseteq F^2(H_n)$ be an invariant subspace for each $S_1^*,\ldots,S_n^*$. If $X \in B(\mathcal{N}\otimes\mathcal{K})$ is in the $I_{\mathcal{N}}\otimes j$-commutant of $\{P_{\mathcal{N}\otimes\mathcal{K}}(S_i\otimes I_{\mathcal{K}})|_{\mathcal{N}\otimes\mathcal{K}},\ i=1,\ldots,n\}$, then there exists $\Phi \in F_n^\infty\bar{\otimes}B(\mathcal{K})$ such that*

$$P_{\mathcal{N}\otimes\mathcal{K}}E(U^*\otimes I)\Phi(U\otimes I) = XP_{\mathcal{N}\otimes\mathcal{K}}, \quad \|\Phi\| = \|X\|, \tag{2.1}$$

where $E \in B(F^2(H_n)\otimes\mathcal{K})$ is the symmetry defined by

$$E\Big(\sum_{\alpha\in\mathbb{F}_n^+} e_\alpha\otimes k_\alpha\Big) := \sum_{\alpha\in\mathbb{F}_n^+} e_\alpha\otimes j^{|\alpha|}(k_\alpha), \tag{2.2}$$

U is the flipping operator on $F^2(H_n)$, and $P_{\mathcal{N}\otimes\mathcal{K}}$ is the orthogonal projection of $F^2(H_n)\otimes\mathcal{K}$ onto $\mathcal{N}\otimes\mathcal{K}$.

Proof. Since $[P_{\mathcal{N}}S_1|_{\mathcal{N}},\ldots,P_{\mathcal{N}}S_n|_{\mathcal{N}}]$ is a C_0-contraction, according to [Po1], its minimal isometric dilation is $[S_1,\ldots,S_n]$. Therefore, the minimal isometric dilation of $[P_{\mathcal{N}}S_1|_{\mathcal{N}}\otimes I_{\mathcal{K}},\ldots,P_{\mathcal{N}}S_n|_{\mathcal{N}}\otimes I_{\mathcal{K}}]$ is $[S_1\otimes I_{\mathcal{K}},\ldots,S_n\otimes I_{\mathcal{K}}]$. Since $I_{\mathcal{N}}\otimes j$ is a symmetry on $\mathcal{N}\otimes\mathcal{K}$, its extension $J := I_{F^2(H_n)}\otimes j$ is a contraction on $F^2(H_n)\otimes\mathcal{K}$ satisfying the conditions of Theorem 1.2. Therefore, there exists

$$A \in B(F^2(H_n)\otimes K)$$

in the $I_{F^2(H_n)}\otimes j$-commutant of $\{S_1\otimes I_{\mathcal{K}},\ldots,S_n\otimes I_{\mathcal{K}}\}$, i.e.,

$$A(S_i\otimes I) = (I_{F^2(H_n)}\otimes j)(S_i\otimes I)A, \tag{2.3}$$

such that

$$P_{\mathcal{N}\otimes\mathcal{K}}A = XP_{\mathcal{N}\otimes\mathcal{K}} \quad\text{and}\quad \|A\| = \|X\|. \tag{2.4}$$

Notice that the operator E defined by (2.2) is a symmetry on $F^2(H_n)\otimes K$ and

$$E(S_i\otimes I_{\mathcal{K}}) = (I_{F^2(H_n)}\otimes j)(S_i\otimes I)E. \tag{2.5}$$

Using (2.3) and (2.5), we infer that

$$\begin{aligned} EA(S_i\otimes I_{\mathcal{K}}) &= E(I_{F^2(H_n)}\otimes j)(S_i\otimes I)A \\ &= E^2(S_i\otimes I)EA = (S_i\otimes I)EA. \end{aligned}$$

Therefore EA is in the commutant of $\{S_1 \otimes I_{\mathcal{K}}, \dots, S_n \otimes I_{\mathcal{K}}\}$. According to [Po7], there exists $\Phi \in F^\infty(H_n)\bar{\otimes}B(\mathcal{K})$ such that $EA = (U^* \otimes I)\Phi(U \otimes I)$. Since E is a symmetry, and using (2.4), the result follows. □

Corollary 2.2. *Let $j \in B(\mathcal{K})$ be a symmetry on a Hilbert space $\mathcal{K}$. Then the $I_{F^2(H_n)} \otimes j$-commutant of $\{S_1 \otimes I_{\mathcal{K}}, \dots, S_n \otimes I_{\mathcal{K}}\}$ is*

$$\{E(U^* \otimes I)\Phi(U \otimes I) : \ \Phi \in F_n^\infty \bar{\otimes} B(\mathcal{K})\},$$

where E is the symmetry defined by (2.2).

Let us remark that Theorem 2.1 can be used to obtain versions of Nevanlinna-Pick type interpolation for F_n^∞. Since the approach is similar to [APo], [Po8], we leave this task to the reader. All the results of this paper hold true if we allow $n = \infty$.

REFERENCES

[APo] A. Arias and G. Popescu, *Noncommutative interpolation and Poisson transforms*, Israel J. Math. **115** (2000), 205–234. CMP 2000:10

[DMP] R.G. Douglas, P.S. Muhly and C. Pearcy, *Lifting commuting operators*, Michigan Math.J. **15** (1968), 385–395. MR **38:**5046

[K] M.A. Krein, *The theory of self-adjoint extensions of semi-bounded Hermitian transformations and its applications*, Math. Collection, Moscow **20** (1947), 431–495. MR **9:**515c

[Po1] G. Popescu, *Isometric dilations for infinite sequences of noncommuting operators*, Trans. Amer.Math.Soc. **316** (1989), 523–536. MR **90c:**47006

[Po2] G. Popescu, *Characteristic functions for infinite sequences of noncommuting operators*, J.Operator Theory **22** (1989), 51–71. MR **91m:**47012

[Po3] G. Popescu, *Von Neumann inequality for $(B(H)^n)_1$*, Math.Scand. **68** (1991), 292–304. MR **92k:**47073

[Po4] G. Popescu, *On intertwining dilations for sequences of noncommuting operators*, J.Math. Anal.Appl. **167** (1992), 382–402. MR **93e:**47012

[Po5] G. Popescu, *Multi-analytic operators on Fock spaces*, Math. Ann. **303** (1995), 31–46. MR **96k:**47049

[Po6] G. Popescu, *Functional calculus for noncommuting operators*, Michigan Math. J. **42** (1995), 345–356. MR **96k:**47025

[Po7] G. Popescu, *Interpolation problems in several variables*, J. Math Anal. Appl. **227** (1998), 227–250. MR **99i:**47028

[Po8] G. Popescu, *Commutant lifting, tensor algebras, and functional calculus*, Proc. Edinburg Math. Soc. (2), to appear.

[Po9] G. Popescu, *Spectral liftings in Banach algebras and interpolation in several variables*, preprint, 1998.

[S] D. Sarason, *Generalized interpolation in H^∞*, Trans. Amer. Math. Soc. **127** (1967), 179–203. MR **34:**8193

[Se] Z. Sebestyén, *Anticommutant lifting and anticommuting dilation*, Proc. Amer. Math. Soc. **121** (1994), 133–136. MR **94g:**47009

[SzF1] B.Sz.-Nagy, C. Foiaş, *Dilatation des commutants d'operateurs*, C.R. Acad. Sci. Paris, Serie A **266** (1968), 493–495. MR **38:**5049

[SzF2] B.Sz.-Nagy, C. Foiaş, *Harmonic analysis on operators on Hilbert space*, North–Holland, Amsterdam (1970). MR **43:**947

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