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THE POSTNIKOV TOWER AND THE STEENROD PROBLEM

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Abstract. The Steenrod problem asks: given a G-module, when does there exist a Moore space realizing the module? By using the equivariant Postnikov Tower, it is shown that a $\mathbb{Z}G$-module is $\mathbb{Z}G$-realizable if and only if it is $\mathbb{Z}H$-realizable for all p-Sylow subgroups H, for all primes $p||G|$.

Let G be a finite group. Let M be a finitely generated $\mathbb{Z}G$-module. We say that M is a *Steenrod representation* if there exists a Moore space X with G-action such that the homology of X is isomorphic to M as a $\mathbb{Z}G$-module. Recall that a Moore space is a topological space whose reduced homology vanishes in all dimensions except one. Without loss of generality, M will be assumed $\mathbb{Z}$-free [1] from now on. In this paper, the following statement will be proved:

> *M is a Steenrod representation as a $\mathbb{Z}G$-module if and only if M is a Steenrod representation as a $\mathbb{Z}H$-module for all p-Sylow subgroups H of G, for all primes $p||G|$.*

This statement is inspired by papers by J. Arnold [1, 2] and P. Vogel [18]. Arnold [2] showed that if G is cyclic, then every finitely generated $\mathbb{Z}G$-module is a Steenrod representation. And Vogel showed that if for any $\mathbb{Z}G$-module M, we can always find a G-Moore space X realizes M, then G has only cyclic Sylow subgroups.

Let us first consider a Moore space X whose homology is isomorphic to M as a $\mathbb{Z}$-module. Let $G(X)$ denote the space of self homotopy equivalences of X and let $B_{G(X)}$ [9] denote the classifying space of the H-space $G(X)$ for a certain fibration defined by Dold and Lashof [10]; see also Stasheff [15]. Since a homotopy equivalence of X induces an automorphism on M, there is a map $G(X) \to Aut(M)$. And the map $G(X) \to Aut(M)$ induces another map $\sigma : B_{G(X)} \to B_{Aut(M)}$. On the other hand, since M is a $\mathbb{Z}G$-module, there is a map $G \to Aut(M)$ $(= GL(n,\mathbb{Z})$ if $M = \mathbb{Z}^n)$, which induces $\varphi : B_G \to B_{Aut(M)}$. Following Cooke's theory [9], as pointed out in [12], M is a Steenrod representation as a $\mathbb{Z}G$-module if and only if the lifting φ_∞ exists in

(1)
$$\begin{array}{ccc} & & B_{(G)(X)} \\ & \overset{\varphi_\infty}{\nearrow} & \downarrow \sigma \\ B_G = K(G,1) & \xrightarrow{\ \varphi\ } & B_{Aut(M)} = K(Aut(M),1). \end{array}$$

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We will follow this path to show the lifting exists in (1) provided the lifting exists for B_H for all p-Sylow subgroups H of G, for all primes $p||G|$:

$$\begin{array}{ccccc} & & & & B_{G(X)} \\ & \nearrow^{\varphi_{\infty,H}} & & \nearrow^{\varphi_{\infty,G}} & \downarrow \\ B_H & \longrightarrow & B_G & \longrightarrow & B^{Aut(M)}. \end{array}$$

Note that we do not assume the same space X is chosen for different Sylow subgroups of G. In fact, we work on $B_{G(X)}$, the classifying space of the H-space $G(X)$ (references: [10] Dold and Lashof and [15] Stasheff). $B_{G(X)}$ is homeomorphic to $B_{G(Y)}$ if X is homotopy equivalent to Y. Following Cooke [9] (see also Kahn [12]), if the lifting

$$\begin{array}{ccc} & & B_{G(X)} \\ & \nearrow^{\varphi_\infty} & \downarrow \sigma \\ B_G & \xrightarrow{\varphi} & B_{Aut(M)} \end{array}$$

exists, then there is a space Y homotopy equivalent to X such that G acts on Y, and Y realizes M. The idea to show the lifting exists in (1) is to use the Postnikov Tower to build the lifting step by step. But the usual Postnikov Tower does not apply because in the first level $\sigma : B_{G(X)} \to B_{Aut(M)}$ of the tower the base space $B_{Aut(M)}$ has a non-trivial fundamental group $\pi_1(B_{Aut(M)}) = Aut(M)$. We modify this by working on the universal covering spaces and in the equivariant category instead. Let E_G denote the universal covering space of B_G. And let $E_{G(X)}$ denote the universal covering space of $B_{G(X)}$. In fact, $E_{G(X)}$ is the pullback $\sigma^* E_{Aut(M)}$ of $\sigma : B_{G(X)} \to B_{Aut(M)}$. Then there are induced maps on covering spaces $\widetilde{\varphi} : E_G \to E_{Aut(M)}$ and $\widetilde{\sigma} : E_{G(X)} \to E_{Aut(M)}$. We will construct an equivariant lifting $\widetilde{\varphi}_\infty : E_G \to E_{G(X)}$ in the following diagram:

(2)
$$\begin{array}{ccc} & & E_{G(X)} \\ & \nearrow^{\widetilde{\varphi}_\infty} & \downarrow \widetilde{\sigma} \\ E_G & \xrightarrow{\widetilde{\varphi}} & E_{Aut(M)}. \end{array}$$

And that is sufficient for the proof of the main statement: by taking the quotient of (2) by suitable group actions,

$$\begin{array}{ccccccc} & & E_{G(X)}/G & \to & E_{G(X)}/Aut(M) & & \\ & \nearrow & \downarrow & & \downarrow & & \Rightarrow \\ E_G/G & \longrightarrow & E_{Aut(M)}/G & \to & E_{Aut(M)}/Aut(M) & & \end{array}$$

$$\begin{array}{ccc} & & E_{G(X)}/Aut(M) = B_{G(X)} \\ & \nearrow & \downarrow \\ B_G = E_G/G & \longrightarrow & E_{Aut(M)}/Aut(M) = B_{Aut(M)} \end{array}$$

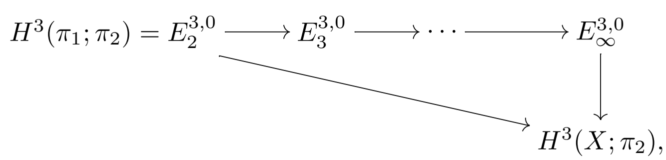
we obtain (1).

In Postnikov theory, one of the key ingredients is the identification of $H^n(X;\pi)$ with $[X; K(\pi, n)]$. There is a counterpart in the equivariant setting: In [11], Theorem 24.1, Eilenberg showed that the cohomology (a) $H^n(X;\pi)$ defined by using local coefficient (i.e. $\pi_1(X)$ acting on π) is the same as (b) $H^n(Y;\pi)$, the homology of the cochain complex $Hom_{\pi_1(X)}(C_n(Y),\pi)$, with $\pi_1(X)$ acting on both Y and π, where Y is the universal covering space of X. Bredon [5] then showed that $H^n(Y;\pi)$ can be identified with (c) $[Y;K(\pi,n)]_G$, the equivariant homotopy classes of equivariant maps; see also [6] p. 268. Note that the G-action on $K(\pi,n)$ has a fixed point. In summary, using the above notation, we have

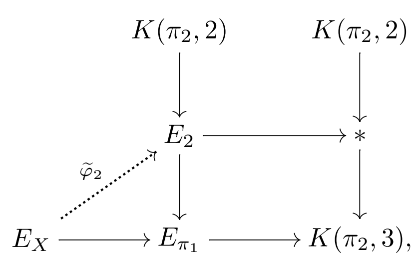
Proposition. *The following cohomology groups are isomorphic:*
(a) $H^n(X;\pi)$*,*
(b) $H^n(Y;\pi)$ *and,*
(c) $[Y;K(\pi,n)]_G$*.*

For example, the group cohomology $H^n(G;\pi) = H^n(B_G;\pi)$ is equal to $[E_G; K(\pi,n)]_G$. We will use the three equivalent cohomology groups (a), (b) and (c) interchangeably depending on the circumstances.

The arguments in the usual Postnikov theory will carry through in the equivariant setting. This is explained in the following: Let X be a topological space and $\pi_1(X) \neq 0$. Let π_n denote $\pi_n(X)$. Consider the fibration $F_1 \to X \to B_{\pi_1}$ with π_1 acting on the fibre F_1 and consider the spectral sequence associated with this fibration. The fundamental class ι_2 of $H^2(F_1;\pi_2)$ [14] is invariant under π_1, so it is an element in $H^0(\pi_1;H^2(F_1;\pi_2))$. ι_2, considered as an element in $E_3^{0,2}$, transgresses to $\tau(\iota_2) \in H^3(\pi_1;\pi_2) = E_3^{3,0}$. We can regard $\tau(\iota_2) \in H^3(\pi_1;\pi_2)$ as an element in $[E_{\pi_1}; K(\pi_2,3)]_{\pi_1}$, i.e. a π_1-equivariant map $\tau(\iota_2) : E_{\pi_1} \to K(\pi_2,3)$. Now we switch to covering spaces and the equivariant setting. As in the simply-connected case of Postnikov theory, the composition $E_X \to E_{\pi_1} \xrightarrow{\tau(\iota_2)} K(\pi_2,3)$ is equivariantly trivial, for it corresponds to the edge homomorphism



which maps $\tau(\iota_2)$ to zero in $H^3(X;\pi_2)$. So the lifting $\widetilde{\varphi}_2 : E_X \to E_2$ exists in



where $K(\pi_2,2) \to E_2 \to E_{\pi_1}$ is the pullback of $K(\pi_2,2) \to * \to K(\pi_2,3)$ and $*$ denotes an equivariantly contractible space.

It can be shown, as in the simply-connected case of Postnikov theory, that $\widetilde{\varphi}_2$ induces isomorphisms on homotopy groups for dimension ≤ 2, and zero otherwise. Now, take the quotient of $\widetilde{\varphi}_2 : E_X \to E_2$ by π_1-action, we get $\varphi_2 : X \to E_2/\pi_1$. Then consider the fibration $F_2 \to X \to E_2/\pi_1$. Now we are back to the situation where we started with the fibration $F_1 \to X \to B_{\pi_1}$. The argument can be repeated.

Summing up, we have a sequence of equivariant maps and free spaces

$$E_X \to \cdots \to E_n \to \cdots \to E_2 \to E_1 = E_{\pi_1},$$

and when we take the quotient of the sequence by π_1 action, we get the Postnikov Tower of X

$$X \to \cdots \to E_n/\pi_1 \to \cdots \to E_2/\pi_1 \to E_1/\pi_1 = B_{\pi_1}$$

in the non-simply-connected case. Note

$$\pi_i(E_n) = \begin{cases} \pi_i(E_X) = \pi_i(X) & 1 < i \leq n, \\ 0 & \text{otherwise.} \end{cases}$$

For the proof of the main theorem, we start with

$$\begin{array}{ccccc} & & & \nearrow & B_{G(X)} \\ & & & & \downarrow \\ B_H & \longrightarrow & B_G & \longrightarrow & B_{Aut(M)}. \end{array}$$

The corresponding universal covering spaces and equivariant maps are:

$$\begin{array}{ccccc} & & & \nearrow & E_{G(X)} \\ & & & & \downarrow \\ E_H & \longrightarrow & E_G & \longrightarrow & E_{Aut(M)}. \end{array}$$

Consider the spectral sequence associated with the fibration $F_1 \to B_{G(X)} \to B_{Aut(M)}$. Let π_i denote $\pi_i(B_{G(X)})$. Since $\pi_1(B_{G(X)}) = \pi_0(G(X)) = Aut(M) = \pi_1(B_{Aut(M)})$ (see [12], Theorem 1.1(a)),

$$\pi_i(F_1) = \begin{cases} 0, & i = 1, \\ \pi_i, & i \geq 2. \end{cases}$$

Let ι_2 be the fundamental class of $H^2(F_1, \pi_2)$. ι_2 is invariant under the action of $Aut(M)$, i.e. ι_2 is an element in $H^0(B_{Aut(M)}; H^2(F_1; \pi_2))$. Then the transgression $\tau(\iota_2)$ is an element in $H^3(B_{Aut(M)}, \pi_2)$, which can be regarded as an element in $[E_{Aut(M)}; K(\pi_2, 3)]_{Aut(M)}$. Since we assume M is a Steenrod representation as a $\mathbb{Z}H$-module, the lifting $\varphi_{H,\infty}$ exists.

$$\begin{array}{ccc} & \overset{\varphi_{H,\infty}}{\nearrow} & B_{G(X)} \\ & & \downarrow \\ B_H = K(H,1) & \xrightarrow{\varphi_H} & B_{Aut(M)} = K(Aut(M),1) \end{array}$$

Hence there exists a sequence of liftings $\{\widetilde{\varphi}_{n,H}\}$.

$$
\begin{array}{ccccc}
 & & E_{G(X)} & & \\
 & \overset{\widetilde{\varphi}_{\infty,H}}{\nearrow} & \vdots & & \\
E_H & \overset{\widetilde{\varphi}_{n,H}}{\longrightarrow} & E_n & \overset{\tau(\iota_{n+1})}{\longrightarrow} & K(\pi_{n+1}, n+2) \\
 & \overset{\widetilde{\varphi}_{n-1,H}}{\searrow} & \downarrow & & \\
 & & E_{n-1} & \overset{\tau(\iota_n)}{\longrightarrow} & K(\pi_n, n+1) \\
 & & \downarrow & & \\
 & & \vdots & & \\
 & & E_{Aut(M)}. & &
\end{array}
$$

The transgression $\tau(\iota_2)$ is then zero in the pullback $F_1 \to \varphi_H^*(B_{G(X)}) \to B_H$ of $\varphi_H : B_H \to B_{Aut(M)}$. Hence the composition $E_H \to E_G \to E_{Aut(M)} \to K(\pi_2, 3)$ is equivariantly homotopy equivalent to a trivial map. In other words, in the sequence of maps $[E_{Aut(M)}; K(\pi_2, 3)]_{Aut(M)} \to [E_G; K(\pi_2, 3)]_G \to [E_H; K(\pi_2, 3)]_H$, $\tau(\iota_2)$ goes to zero, i.e. the element $\tau(\iota_2) \circ \widetilde{\varphi}_G$ restricts to zero in the map $[E_G; K(\pi_2, 3)]_G \to [E_H; K(\pi_2, 3)]_H$, which is identified with $H^3(G; \pi_2) \to H^3(H; \pi_2)$. Since $\tau(\iota_2) \circ \widetilde{\varphi}_G$ restricts to zero via the restriction maps $H^3(G; \pi_2) \to H^3(H; \pi_2)$ for all p-Sylow subgroups H, $\tau(\iota_2) \circ \widetilde{\varphi}_G$ is a zero element in $H^3(G; \pi_2)$. So the pull-back $\widetilde{\varphi}_G^*(E_2)$ of the map $\tau(\iota_2) \circ \widetilde{\varphi}_G$ is a trivial bundle. Therefore the lifting $\widetilde{\varphi}_{2,G}$ exists. Note that the liftings are in one-to-one correspondence with elements in $[E_G; K(\pi_2, 2)]_G$. Let $\theta_{2,G}$ denote the element in $[E_G; K(\pi_2, 2)]_G$ corresponding to $\widetilde{\varphi}_{2,G}$, as illustrated in the following diagram:

$$
\begin{array}{ccccc}
 & & K(\pi_2, 2) & \longrightarrow & K(\pi_2, 2) \\
 & & \downarrow & & \downarrow \\
E_G \times K(\pi_2, 2) & \longrightarrow & E_2 & \longrightarrow & * \\
{\scriptstyle id \times \theta_{2,G}} \uparrow & \overset{\widetilde{\varphi}_{2,G}}{\nearrow} & \downarrow & & \downarrow \\
E_G & \overset{\widetilde{\varphi}_G}{\longrightarrow} & E_{Aut(M)} & \overset{\tau(\iota_2)}{\longrightarrow} & K(\pi_2, 3)
\end{array}
$$

The above diagram is in the equivariant category, where E_2 is the second level of the Postnikov Tower of $E_{G(X)}$.

Now the problem is to choose a right lifting $\widetilde{\varphi}_{2,G}$ (or $\theta_{2,G}$) so that the diagram

$$
\begin{array}{ccccccc}
 & & & & K(\pi_2, 2) & \longrightarrow & K(\pi_2, 2) \\
 & & & & \downarrow & & \downarrow \\
E_H \times K(\pi_2, 2) & \longrightarrow & E_G \times K(\pi_2, 2) & \longrightarrow & E_2 & \longrightarrow & * \\
{\scriptstyle id \times \theta_{2,H}} \uparrow & & {\scriptstyle id \times \theta_{2,G}} \uparrow & \overset{\widetilde{\varphi}_{2,G}}{\nearrow} & \downarrow & & \downarrow \\
E_H & \longrightarrow & E_G & \overset{\widetilde{\varphi}_G}{\longrightarrow} & E_{Aut(M)} & \overset{\tau(\iota_2)}{\longrightarrow} & K(\pi_2, 3)
\end{array}
$$

commutes for all p-Sylow subgroups H. If the above diagram commutes, then we can repeat the previous argument: $[\tau(\iota_3) \circ \widetilde{\varphi}_{2,G}] \in H^4(G; \pi_3)$ restricts to $[\tau(\iota_3) \circ \widetilde{\varphi}_{2,H}] = 0$ in $H^4(H; \pi_3)$ for all p-Sylow subgroups H. So $[\tau(\iota_3) \circ \widetilde{\varphi}_{2,G}] = 0$ in $H^4(G; \pi_3) = [E_G; K(\pi_3, 4)]_G$ and then we can go inductively. But the desired lift

$\theta_{2,G}$ may not exist, unless we have a right $\theta_{2,H}$, i.e. $\theta_{2,H}$ has to be in the image of the restriction map $H^2(G;\pi_2) \to H^2(H;\pi_2)$. Such a $\theta_{2,H}$ is said to be G-invariant in $H^2(H;\pi_2)$. In the following, we will show how to alter all the liftings $\{\widetilde{\varphi}_{n,H}\}$ to ensure $\{\theta_{n,H}\}$ be G-invariant. We take the new $\theta_{2,H}$ to be $\frac{1}{[G:H]}\Sigma_{G=\bigcup gH}\, g\theta_{2,H}g^{-1}$, i.e. $\frac{1}{[G:H]} res \circ tr\ \theta_{2,H}$. The corresponding new lift $\widetilde{\varphi}_{2,H}$

$$\begin{array}{ccccc} & & K(\pi_2,2) & \longrightarrow & K(\pi_2,2) \\ & & \downarrow & & \downarrow \\ E_H \times K(\pi_2,2) & \longrightarrow & E_2 & \longrightarrow & * \\ \uparrow & \nearrow^{\widetilde{\varphi}_{2,H}} & \downarrow & & \downarrow \\ E_H & \xrightarrow{\widetilde{\varphi}_H} & E_{Aut(M)} & \xrightarrow{\tau(\iota_2)} & K(\pi_2,3) \end{array}$$

is $(\widetilde{\varphi}_H, \frac{1}{[G:H]} res \circ tr\ \theta_{2,H})$. (The lifting $\widetilde{\varphi}_{2,H}$ that corresponds to $\theta_{2,H}$ is $(\widetilde{\varphi}_H, \theta_{2,H})$.) Now we use the fact that $E_{G(X)}$ is an $\mathcal{H}$-space (a space with homotopy mutiplication), E_n's are $\mathcal{H}$-spaces and $\tau(\iota_n)$'s are $\mathcal{H}$-maps (see [19]). Therefore $\tau(\iota_3) \circ (\widetilde{\varphi}_H, \frac{1}{[G:H]} res \circ tr\ \theta_{2,H}) = \tau(\iota_3) \circ (\frac{1}{[G:H]} res \circ tr(\widetilde{\varphi}_H, \theta_{2,H})) = \frac{1}{[G:H]} res \circ tr\ \tau(\iota_3) \circ (\widetilde{\varphi}_H, \theta_{2,H})) = 0$. In general, we will change $\widetilde{\varphi}_{n,H} = (\widetilde{\varphi}_H, \theta_{2,H}, \cdots, \theta_{n,H})$ to $(\widetilde{\varphi}_H, \frac{1}{[G:H]} res \circ tr\ \theta_{2,H}, \frac{1}{[G:H]} res \circ tr\ \theta_{n,H})$. Then the new $\theta_{n,H}$, i.e. $\frac{1}{[G:H]} res \circ tr\ \theta_{n,H}$ will be G-invariant.

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