

## MENGER CURVATURE AND $C^1$ REGULARITY OF FRACTALS

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ABSTRACT. We show that if  $E$  is an  $s$ -regular set in  $\mathbf{R}^2$  for which the triple integral  $\int_E \int_E \int_E c(x, y, z)^{2s} d\mathcal{H}^s x d\mathcal{H}^s y d\mathcal{H}^s z$  of the Menger curvature  $c$  is finite and if  $0 < s \leq 1/2$ , then  $\mathcal{H}^s$  almost all of  $E$  can be covered with countably many  $C^1$  curves. We give an example to show that this is false for  $1/2 < s < 1$ .

### 1. INTRODUCTION

The Menger curvature  $c(x, y, z)$  of three points  $x, y$  and  $z$  in the plane  $\mathbf{R}^2$  is defined as the reciprocal of the radius of the circle passing through these points. For a historical background, see [K]. In [Me] Melnikov found a remarkable connection between the Menger curvature and the Cauchy kernel  $1/z$ ,  $z \in \mathbf{C}$ . This led to a rapid development on singular integrals over 1-dimensional subsets of  $\mathbf{R}^2$  and on removable sets of bounded analytic functions; see [MV], [MMV], [D], and for a survey [M2].

Another aspect of the Menger curvature is that its integrals can be used to measure smoothness properties of subsets of  $\mathbf{R}^n$ . Note that  $c(x, y, z) = 0$  if and only if the points  $x, y$  and  $z$  lie on the same line. Let  $\mathcal{H}^s$  be the  $s$ -dimensional Hausdorff measure. For  $\mathcal{H}^s$  measurable sets  $E \subset \mathbf{R}^n$  with  $0 < \mathcal{H}^s(E) < \infty$  the proper quantity to use is

$$c^{2s}(E) = \int_E \int_E \int_E c(x, y, z)^{2s} d\mathcal{H}^s x d\mathcal{H}^s y d\mathcal{H}^s z.$$

Léger proved in [L] that if  $\mathcal{H}^1(E) < \infty$  and  $c^2(E) < \infty$ , then there are rectifiable curves  $\Gamma_1, \Gamma_2, \dots$  such that

$$\mathcal{H}^1\left(E \setminus \bigcup_{i=1}^{\infty} \Gamma_i\right) = 0.$$

Sets with this property are called 1-rectifiable in [M1] and countably  $(\mathcal{H}^1, 1)$  rectifiable in [F].

In this paper we study analogous questions for other values of  $s$ . It was shown in [Li, Theorem 1.4] that if  $E \subset \mathbf{R}^n$  is  $\mathcal{H}^s$  measurable and  $0 < \mathcal{H}^s(E) < \infty$  for

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some  $1 < s \leq n$ , then  $c^{2s}(E) = \infty$ . Hence we restrict to  $0 < s < 1$ . We also study only subsets of  $\mathbf{R}^2$ , although with slight modifications the results would extend to  $\mathbf{R}^n$ . For reasons indicated in Example 2.5 we restrict to the so-called  $s$ -regular sets. This means that there is a constant  $C$  such that

$$(1.1) \quad r^s/C \leq \mathcal{H}^s(E \cap B(x, r)) \leq Cr^s \quad \text{for } x \in E, 0 < r < d(E).$$

Here  $B(x, r)$  is the closed ball with centre  $x$  and radius  $r$ , and  $d(E)$  stands for the diameter of  $E$ .

When  $0 < s < 1$  and  $E$  is compact, (1.1) alone implies that  $E$  is contained in one rectifiable curve  $\Gamma$ ; see the proof of Theorem 4.1 in [MM]. A rectifiable curve is the same as a Lipschitz image of the interval  $[0, 1]$ . We study here whether Lipschitz images can be replaced by  $C^1$  curves. By a  $C^1$  curve we mean a curve with continuously varying tangent. It is the same as the image of an interval under a regular  $C^1$  map, that is, a  $C^1$  map with non-vanishing derivative. We shall prove in Corollary 2.2 that if  $E$  satisfies (1.1),  $c^{2s}(E) < \infty$  and  $0 < s \leq 1/2$ , then there are  $C^1$  curves  $\Gamma_1, \Gamma_2, \dots$  such that

$$\mathcal{H}^s\left(E \setminus \bigcup_{i=1}^{\infty} \Gamma_i\right) = 0.$$

We give in 2.4 an example showing that this is false if  $1/2 < s < 1$ . For  $s = 1$  it is again true due to [L], even with the weaker condition  $\mathcal{H}^1(E) < \infty$  instead of (1.1), because then covering  $\mathcal{H}^1$  almost everything with Lipschitz images or  $C^1$  curves are equivalent as a consequence of Rademacher’s theorem; see [F, 3.2.29].

There are also many other characterizations for 1-rectifiable sets. In [MM] analogous conditions for  $s$ -dimensional sets were investigated and this paper can be considered as a further contribution to that study.

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## 2. COVERING WITH $C^1$ CURVES

We begin with the following result on the existence of tangents. We say that a set  $E \subset \mathbf{R}^2$  has a tangent  $L$  at  $x$  if  $L$  is a line through  $x$  such that for any  $\alpha > 0$ ,  $E \cap B(x, r) \subset C(x, \alpha)$  for all sufficiently small  $r > 0$ , where  $C(x, \alpha)$  is the double-cone with centre  $x$ , axis  $L$  and angle  $\alpha$ .

**2.1. Theorem.** *Let  $0 < s \leq 1/2$  and let  $E \subset \mathbf{R}^2$  be  $\mathcal{H}^s$  measurable and  $s$ -regular. If  $x \in E$  and*

$$(2.1) \quad c^{2s}(E, x) := \int_E \int_E c(x, y, z)^{2s} d\mathcal{H}^s y d\mathcal{H}^s z < \infty,$$

*then  $E$  has a tangent at  $x$ .*

*Proof.* Let  $x \in E$ . By the  $s$ -regularity of  $E$  there are positive numbers  $b$  and  $d < d(E)$  such that for  $i = 1, 2, \dots$ ,

$$(2.2) \quad \mathcal{H}^s(A_i) \geq bd^{is},$$

where

$$A_i = \{y \in E : d^{i+1} < |x - y| \leq d^i\}.$$

Set

$$\gamma_i = \int_{A_i} \int_E c(x, y, z)^{2s} d\mathcal{H}^s y d\mathcal{H}^s z.$$

Then

$$(2.3) \quad \sum_i \gamma_i \leq \int_E \int_E c(x, y, z)^{2s} d\mathcal{H}^s y d\mathcal{H}^s z < \infty.$$

We shall show that for each  $i$  there is a line  $L_i$  through  $x$  such that

$$(2.4) \quad \mathcal{H}^s(A_i \cap L_i(\eta_i d^i)) \geq bd^{is}/16,$$

where

$$(2.5) \quad \eta_i = (16b^{-1})^{1/s} \gamma_i^{1/(2s)},$$

and

$$B(\delta) = \{x \in \mathbf{R}^2 : \text{dist}(x, \beta) \leq \delta\} \quad \text{for } B \subset \mathbf{R}^2, \delta > 0.$$

Suppose (2.4) fails for some  $i$ . By (2.2) there is a closed quarter-plane  $Q$  (a sector with angle  $\pi/2$ ) with vertex at  $x$  such that  $\mathcal{H}^s(A_i \cap Q) \geq bd^{is}/4$ . Further, there is a line  $L$  through  $x$  such that

$$\mathcal{H}^s(A_i \cap Q \cap H_j) \geq bd^{is}/8 \quad \text{for } j = 1, 2,$$

where  $H_1$  and  $H_2$  are the two closed half-planes whose boundary is  $L$ . Since  $\mathcal{H}^s(A_i \cap L(\eta_i d^i)) < bd^{is}/16$ , we have

$$(2.6) \quad \mathcal{H}^s(A_i \cap Q \cap H_j \setminus L(\eta_i d^i)) > bd^{is}/16 \quad \text{for } j = 1, 2.$$

Let  $x_j \in A_i \cap Q \cap H_j \setminus L(\eta_i d^i)$  for  $j = 1, 2$ . We use the following formula, which is an exercise in elementary geometry:

$$(2.7) \quad c(x, x_1, x_2) = \frac{2 \text{dist}(x_2, L_{x, x_1})}{|x - x_2| |x_1 - x_2|},$$

where  $L_{y,z}$  denotes the line through two points  $y$  and  $z$ . This gives

$$c(x, x_1, x_2) \geq \frac{2\eta_i d^i}{d^i \cdot d^i} = \frac{2\eta_i}{d^i}.$$

Thus by (2.6) and (2.5)

$$\gamma_i > \left(\frac{\eta_i}{d^i}\right)^{2s} (bd^{is}/16)^2 = (b/16)^2 \eta_i^{2s} = \gamma_i,$$

which is a contradiction proving (2.4).

Next we show that if

$$(2.8) \quad \zeta_i = \max \{12\eta_i/d, (16 \cdot 50^{2s} C b^{-1} d^{-2s} \gamma_i)^{1/(3s)}\},$$

and if  $\zeta_i < d$ , then

$$(2.9) \quad A_i \subset L_i(\zeta_i d^i).$$

Suppose this fails and let  $y_1 \in A_i \setminus L_i(\zeta_i d^i)$ . Then  $\zeta_i < 1$  and  $B(y_1, \frac{1}{2}\zeta_i d^i) \subset B(x, 2d^i) \setminus L_i(\frac{1}{2}\zeta_i d^i)$ . Thus for all  $y \in B(y_1, \frac{1}{2}\zeta_i d^i)$  and  $z \in A_i \cap L_i(\frac{1}{12}\zeta_i d^{i+1})$  we have by some elementary geometry  $d(x, L_{y,z}) \geq \frac{1}{24}\zeta_i d^{i+1}$ . Hence by (2.7),

$$c(x, y, z) \geq \frac{\zeta_i d^{i+1}/12}{d^i \cdot 2d^i} = \frac{\zeta_i d}{24d^i}.$$

Since by (1.1)  $\mathcal{H}^s(E \cap B(y_1, \frac{1}{2}\zeta_i d^i)) \geq (\frac{1}{2}\zeta_i d^i)^s / C$ , and by (2.4)  $\mathcal{H}^s(A_i \cap L_i(\frac{1}{12}\zeta_i d^{i+1})) \geq \frac{1}{16}bd^{is}$  (as  $L_i(\frac{1}{12}\zeta_i d^{i+1}) \supset L_i(\eta_i d^i)$  by (2.8)), we get

$$\begin{aligned} \gamma_i &\geq \left(\frac{\zeta_i d}{24d^i}\right)^{2s} C^{-1} \left(\frac{1}{2}\zeta_i d^i\right)^s \frac{1}{16}bd^{is} \\ &> \frac{bd^{2s}\zeta_i^{3s}}{16 \cdot 50^{2s}C} \geq \gamma_i \end{aligned}$$

which proves (2.9).

Let  $\alpha_i \in [0, \pi)$  be the angle between the lines  $L_i$  and  $L_{i+1}$ . We claim that

$$(2.10) \quad \alpha_i \leq \max \{8d^{-1}\eta_i, 8d^{-1}\eta_{i+1}, 8(16/b)^{1/s}d^{-3}(\gamma_i + \gamma_{i+1})^{1/(2s)}\}.$$

Suppose this is false. Then if  $y \in A_i \cap L_i(\eta_i d^i)$  and  $z \in A_{i+1} \cap L_{i+1}(\eta_{i+1} d^{i+1})$ , we have by simple elementary geometry  $\text{dist}(z, L_{x,y}) \geq \frac{1}{4}\alpha_i d^{i+2}$ . Hence (2.7) gives

$$c(x, y, z) \geq \frac{\frac{1}{2}\alpha_i d^{i+2}}{(2d^i)^2} = \frac{\alpha_i d^2}{8d^i}.$$

Integrating over  $A_i \cap L_i(\eta_i d^i)$  and  $A_{i+1} \cap L_{i+1}(\eta_{i+1} d^{i+1})$  and using (2.4), we obtain

$$\gamma_i + \gamma_{i+1} \geq \left(\frac{\alpha_i d^2}{8d^i}\right)^{2s} \left(\frac{1}{16}bd^{(i+1)s}\right)^2 = \frac{d^{6s} b^2 \alpha_i^{2s}}{16^2 \cdot 8^{2s}} > \gamma_i + \gamma_{i+1};$$

a contradiction proving (2.10).

By the definition (2.5) of  $\eta_i$  and by (2.10) we have for some  $C_1 < \infty$  for all  $i$ ,

$$\alpha_i \leq C_1(\gamma_i + \gamma_{i+1})$$

since  $0 < s \leq 1/2$ . Using (2.3) we find that  $\sum_i \alpha_i < \infty$ . This means that the lines  $L_i$  converge to a line  $L$  through  $x$ . Applying (2.9) and the fact that  $\zeta_i \rightarrow 0$ , we see that  $L$  is a tangent to  $E$  at  $x$ . This completes the proof.

**2.2. Corollary.** *If  $0 < s \leq 1/2$ ,  $E \subset \mathbf{R}^2$  is  $s$ -regular,  $\mathcal{H}^s$  measurable and  $c^{2s}(E) < \infty$ , then there are  $C^1$  curves  $\Gamma_1, \Gamma_2, \dots$  such that*

$$\mathcal{H}^s\left(E \setminus \bigcup_{i=1}^{\infty} \Gamma_i\right) = 0.$$

*Proof.* This follows from Theorem 2.1 and [MM, Theorem 3.9(1)]. The proof is a relatively easy application of Whitney’s extension theorem.

*2.3. Remark.* Even if we would assume that the integral in (2.1) is uniformly bounded for  $x \in \mathbf{R}^2$ ,  $E$  is not necessarily contained in a single  $C^1$  curve, that is, the tangent need not vary continuously. For example, let  $C$  be a compact  $s$ -regular set lying on the unit circle  $S^1$  and let  $D$  be an  $s$ -regular Cantor set on  $\{(x, y) : 0 \leq x \leq 1, y = 0\}$  with  $0 \in D$ . Choose a sequence  $I_j$  of complementary intervals of  $D$  with mid-points  $x_j$  and lengths  $l_j$  in such a way that  $x_j \rightarrow 0$ , and  $l_j/x_j \rightarrow 0$  very quickly. Let

$$E = D \cup \bigcup_{j=1}^{\infty} \left(\frac{1}{4}l_j C + x_j\right).$$

Then  $E$  is  $s$ -regular. If  $l_j/x_j \rightarrow 0$  sufficiently quickly, then  $c^{2s}(E, )$  is uniformly bounded, but  $E$  is not contained in any  $C^1$  curve.

We now give an example to show that Theorem 2.1 and Corollary 2.2 fail for  $1/2 < s < 1$ .

**2.4. Example.** Let  $1/2 < s < 1$ . Then there is a compact  $s$ -regular set  $E \subset \mathbf{R}^2$  such that  $c^{2s}(E, x)$  is uniformly bounded for  $x \in E$ , but  $\mathcal{H}^s(E \cap \Gamma) = 0$  for every  $C^1$  curve  $\Gamma$ .

*Proof.* We shall construct  $E$  with a von Koch-type construction similar to that used in [DS, §20]. Define  $r \in (0, 1)$  by

$$2r^s = 1.$$

Let  $J_{0,1}$  be a closed oriented line-segment of length 1 in  $\mathbf{R}^2$ . Let  $J_{1,1}$  and  $J_{1,2}$  be the closed oriented line-segments of length  $r$  in  $\mathbf{R}^2$  such that the initial point of  $J_{1,1}$  is the initial point of  $J_{0,1}$ , the initial point of  $J_{1,2}$  is the mid-point of  $J_{0,1}$ , and the oriented angle from  $J_{0,1}$  to both  $J_{1,1}$  and  $J_{1,2}$  is 1. Suppose we have constructed the closed oriented line-segments  $J_{k,1}, \dots, J_{k,2^k}$  of length  $r^k$ . We apply the above operation to each  $J_{k,i}$  with the angle 1 replaced by  $1/(k + 1)$  to obtain the line-segments  $J_{k+1,1}, \dots, J_{k+1,2^{k+1}}$  of length  $r^{k+1}$ . It is clear that the unions  $\bigcup_{i=1}^{2^k} J_{k,i}$  converge as  $k \rightarrow \infty$  to a compact  $s$ -regular set  $E$ . For each  $k$  and  $j$  we denote by  $E_{k,j}$  the subset of  $E$  generated by  $J_{k,j}$  (in the obvious way). Then for all  $k$ ,

$$E = \bigcup_{j=1}^{2^k} E_{k,j}.$$

Since  $\sum_k k^{-1} = \infty$ , one sees easily that  $E$  has tangent at none of its points. In fact,  $E$  approaches all of its points along all directions in the sense that for any  $x \in E$  and any line  $L$  through  $x$  there is a sequence  $x_i \in E \setminus \{x\}$  such that  $x_i \rightarrow x$  and  $\text{dist}(x_i, L)/|x_i - x| \rightarrow 0$ . This together with the  $s$ -regularity of  $E$  implies that  $\mathcal{H}^s(E \cap f([0, 1])) = 0$  for any regular  $C^1$  mapping  $f : [0, 1] \rightarrow \mathbf{R}^2$ . This can be checked by using the regularity of  $f$  to write  $[0, 1] = \bigcup_{k=1}^{\infty} A_k$  where each  $A_k$  is a Borel set such that for some  $e_k \in S^1$

$$|(f(x) - f(y))/|f(x) - f(y)| - e_k| < 1/2 \quad \text{for } x, y \in A_k.$$

Then  $\mathcal{H}^s(E \cap f(A_k)) = 0$  for all  $k$  by the above scatteredness property of  $E$ . It remains to show that  $c^{2s}(E, \cdot)$  is uniformly bounded.

Fix  $x \in E$ . For  $y \in E, y \neq x$ , let  $k(y)$  be the largest  $k$  such that  $x, y \in E_{k-1,j}$  for some  $j$ . Here  $E_{0,j} = E$ . Let  $y, z \in E \setminus \{x\}$  with  $y \neq z$ . Denote  $k = k(y), l = k(z)$  and assume that  $k \leq l$ .

Suppose first that  $k = l$ . Then for some  $j, y, z \in E_{k,j}$  whereas  $x \notin E_{k,j}$ . Let  $m = m(y, z)$  be the largest  $m$  such that  $y, z \in E_{m-1,j}$  for some  $j$ . Then  $m > k$ . It follows from the construction that there is a positive number  $b$ , depending only on  $r$ , such that

$$\begin{aligned} |x - z| &\geq b^{-1}r^k, & |y - z| &\geq b^{-1}r^m, \\ \text{dist}(z, L_{x,y}) &\leq br^m. \end{aligned}$$

If  $m$  is not much bigger than  $k$ , we can improve the last estimate. Since for  $m \leq 2k$  the angle between the lines  $L_{x,y}$  and  $L_{y,z}$  is at most a constant times

$$\sum_{j=k+1}^m \frac{1}{j} \approx \log \frac{m}{k} \approx \frac{m - k}{k},$$

we can choose  $b$  so that also

$$\text{dist}(z, L_{x,y}) \leq b \frac{m-k}{k} r^m.$$

Consequently by (2.7) we have both

$$\begin{aligned} c(x, y, z) &\leq 2b^3 r^{-k} && \text{and} \\ c(x, y, z) &\leq 2b^3 \frac{m-k}{k} r^{-k}. \end{aligned}$$

If  $k < l$  we get in the same way interchanging  $x$  and  $y$  in the above argument

$$\begin{aligned} c(x, y, z) &\leq 2b^3 r^{-k} && \text{and} \\ c(x, y, z) &\leq 2b^3 \frac{l-k}{k} r^{-k}. \end{aligned}$$

Set

$$\begin{aligned} F_k &= \{y \in E : k(y) = k\} && \text{and} \\ F_m(y) &= \{z \in E : k(z) = k(y), k(y, z) = m\}. \end{aligned}$$

Then

$$\begin{aligned} \mathcal{H}^s(F_k) &\leq C_1 r^{sk} && \text{and} \\ \mathcal{H}^s(F_m(y)) &\leq C_1 r^{sm} \end{aligned}$$

where  $C_1$  depends only on  $r$ . Therefore, changing  $m - k$  to  $n$ ,

$$\begin{aligned} c^{2s}(E, x) &= \int_E \int_E c(x, y, z)^{2s} d\mathcal{H}^s z d\mathcal{H}^s y \\ &\leq 2 \sum_{k=1}^{\infty} \sum_{m=k+1}^{\infty} \int_{F_k} \int_{F_m(y)} c(x, y, z)^{2s} d\mathcal{H}^s z d\mathcal{H}^s y \\ &\quad + 2 \sum_{k=1}^{\infty} \sum_{l=k+1}^{\infty} \int_{F_k} \int_{F_l} c(x, y, z)^{2s} d\mathcal{H}^s z d\mathcal{H}^s y \\ &\leq 4 \cdot 2^{2s} b^{6s} C_1^2 \sum_{k=1}^{\infty} \sum_{m=k+1}^{2k} \left(\frac{m-k}{k}\right)^{2s} r^{-2sk} r^{sk} r^{sm} \\ &\quad + 4 \cdot 2^{2s} b^{6s} C_1^2 \sum_{k=1}^{\infty} \sum_{m=2k+1}^{\infty} r^{-2sk} r^{sk} r^{sm} \\ &= 4 \cdot 2^{2s} b^{6s} C_1^2 \sum_{n=1}^{\infty} r^{ns} n^{2s} \sum_{k=n}^{\infty} k^{-2s} \\ &\quad + 4 \cdot 2^{2s} b^{6s} C_1^2 \sum_{k=1}^{\infty} \sum_{n=k+1}^{\infty} r^{sn} < \infty \end{aligned}$$

since  $2s > 1$ . Thus  $c^{2s}(E, \cdot)$  is bounded.

We now show that Theorem 2.1 and Corollary 2.2 fail if we replace the regularity assumption (1.1) by  $\mathcal{H}^s(E) < \infty$ .

**2.5. Example.** Given  $0 < s < 1$  there exists a compact set  $E \subset \mathbf{R}^2$  such that  $0 < \mathcal{H}^s(E) < \infty$ ,  $c^{2s}(E, x)$  is uniformly bounded for  $x \in \mathbf{R}^2$  and  $\mathcal{H}^s(E \cap \Gamma) = 0$  for every  $C^1$  curve  $\Gamma$ .

*Proof.* Choose an integer  $n_1 > 1$ , let  $J = \{(x, y) \in \mathbf{R}^2 : 0 \leq x \leq 1, y = 0\}$  and

$$J_i = \{(x, y) \in \mathbf{R}^2 : x = i/n_1, 0 \leq y \leq r_1\}, \quad i = 1, \dots, n_1,$$

where  $r_1$  is determined by

$$n_1 r_1^s = 1.$$

Let  $\mu_1$  be the normalized, i.e.,  $\mu_1\left(\bigcup_{i=1}^{n_1} J_i\right) = 1$ , length measure on  $\bigcup_{i=1}^{n_1} J_i$ . If we choose  $n_1$  very large,  $r_1$  will be very small and the distance  $1/n_1 = r_1^s$  between any  $J_i$  and  $J_{i+1}$  will be much bigger than  $r_1$ . From this we see easily that choosing  $n_1$  large enough, we have

$$c^{2s}(\mu_1, x) = \iint c(x, y, z)^{2s} d\mu_1 y d\mu_1 z \leq C_0$$

for all  $x \in \mathbf{R}^2$ , where  $C_0$  is an absolute constant.

Next we replace each vertical line segment  $J_i$  by  $n_2$  horizontal line segments  $J_{i,j}$  of length  $r_2$  such that  $n_2 r_2^s = r_1^s$  in the same way. Let  $\mu_2$  be the normalized length measure on  $\bigcup_{i,j} J_{i,j}$ . Choosing  $n_2$  sufficiently large we can keep  $c^{2s}(\mu_2, x)$  as close to  $c^{2s}(\mu_1, x)$ , uniformly, as we want. We choose it so that  $c^{2s}(\mu_2, x) \leq C_0 + \frac{1}{2}$  for  $x \in \mathbf{R}^2$ . The point here is that for some small  $\delta > 0$  looking from any  $x \in \mathbf{R}^2$  the part of  $\mu_2$  outside  $B(x, \delta)$  looks very much like  $\mu_1$  and the contribution of  $\mu_2$  in  $B(x, \delta)$  to  $c^{2s}(\mu_2, x)$  is very small.

Continuing this we get unions  $E_k$  of line segments  $J_{i_1, \dots, i_k}$  of length  $r_k$ . Every second time these line segments are horizontal and every second time vertical. We also have uniformly distributed probability measures  $\mu_k$  on  $E_k$  satisfying  $c^{2s}(\mu_k, x) \leq C_0 + \sum_{i=2}^k 2^{-i}$  for all  $k$  and  $x \in \mathbf{R}^2$ . The sets  $E_k$  converge to a compact set  $E$  with  $0 < \mathcal{H}^s(E) < \infty$  and the measures  $\mu_k$  converge weakly to a probability measure  $\mu$  supported on  $E$  such that

$$c^{2s}(\mu, x) \leq 2 \quad \text{for } x \in \mathbf{R}^2.$$

It is easy to check that  $\mu$  is comparable with  $\mathcal{H}^s$  restricted to  $E$ . Thus  $c^{2s}(E, x)$  is uniformly bounded.

Finally that  $\mathcal{H}^s(E \cap \Gamma) = 0$  for every  $C^1$  curve  $\Gamma$  can be checked with the help of decompositions such as in the proof of Example 2.4.

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