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GEODESICS ON THE SPACE OF LAGRANGIAN SUBMANIFOLDS IN COTANGENT BUNDLES

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ABSTRACT. We prove that the space of Hamiltonian deformations of zero section in a cotangent bundle of a compact manifold is locally flat in the Hofer metric and we describe its geodesics.

1. INTRODUCTION

Let M be a compact smooth manifold. Denote by $\mathcal{L}(M)$ the space of Hamiltonian deformations of zero section O_M in its cotangent bundle T^*M. For $L_0 \in \mathcal{L}(M)$ and for a path $L_t := \phi_t^H(L_0)$, where ϕ_t^H is a Hamiltonian isotopy generated by a smooth compactly supported Hamiltonian function $H : [0,1] \times T^*M \to \mathbf{R}$, we define

$$(1) \qquad \mathit{length}(\{L_t\}) := \inf \int_0^1 (\max_x H(t,x) - \min_x H(t,x))dt,$$

where the infimum is taken over all H such that $\phi_t^H(L_0) = L_t$. For $L_1 \in \mathcal{L}(M)$, define distance between L_0 and L_1 as the infimum of lengths over connecting paths. More precisely,

$$(2) \qquad d(L_0, L_1) := \inf_H \{ \int_0^1 (\max_x H(t,x) - \min_x H(t,x))dt \mid \phi_1^H(L_0) = L_1 \}.$$

Note that $\{L_t\}$ depends only on values of $H(t,x)$ for x near $\bigcup_t L_t$. Since the distance defined by (2) is the infimum of lengths over all connecting paths, d does not depend on whether the maximum and the minimum are taken over $x \in \bigcup_t L_t$ or over T^*M. In this paper $\max_x$ and $\min_x$ will denote the maximum and minimum over $x \in \bigcup_t L_t$.

A group of Hamiltonian diffeomorphisms of T^*M acts on $\mathcal{L}(M)$ via $(\psi, L) \mapsto \psi(L)$. It is known that d is an invariant distance on $\mathcal{L}(M)$ with respect to this action. The most delicate fact in the proof of this fact is the non-degeneracy of d (see [2], [8]). A proof given by Oh [8] is based on a study of invariants defined by

$$\rho(H) = \inf\{\lambda \mid HF_*^{(-\infty,\lambda)}(H) \to HF_*(H) \text{ is surjective}\},$$

where HF_* is Floer homology. After a certain normalization, $\rho(H)$ depends only on $L = \phi_1^H(O_M)$ and it is denoted by $\rho(L)$ (see [8] or Section 2 below for more details).

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In this paper we prove that every $L \in \mathcal{L}(M)$ has a flat C^1-neighborhood. More precisely, let $\mathcal{G}$ be the C^1-neighborhood (in the space of Lagrangian embeddings) of O_M such that if $L \in \mathcal{G}$, then $L = Graph(dS)$ for some smooth function S on M. We prove that for $L_i = Graph(dS_i) \in \mathcal{G},\ i \in \{0,1\}$,

$$d(L_0, L_1) = \|S_1 - S_0\| := \max(S_1 - S_0) - \min(S_1 - S_0).$$

As a corollary, we obtain a description of geodesics on $\mathcal{L}(M)$. This generalizes an analogous result by Bialy and Polterovich for Hamiltonian diffeomorphisms in $\mathbf{R}^{2n}$ [1]. A description of geodesics on the group of Hamiltonian diffeomorphisms in general symplectic manifolds is obtained by Lalonde and McDuff [5].

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2. Preliminaries

Let $\{L_t\}_{0\le t\le 1}$ be a smooth regular path in $\mathcal{L}$, i.e. $\frac{d}{dt}L_t \neq 0$ for every $t \in [0,1]$. L_t is called a *minimal geodesic* if $length(\{L_t\}) = d(L_0, L_1)$. It is called *geodesic* if it is a minimal geodesic locally on $[0,1]$ (compare [1]).

A Hamiltonian $H(t,x)$ is called *quasi-autonomous* if there exist $x_+, x_- \in \bigcup_t L_t$ such that $\max_x H(t,x) = H(t,x_+)$ and $\min_x H(t,x) = H(t,x_-)$ for every t. This is equivalent to

$$(3)\qquad \int_0^1 (\max_x H(t,x) - \min_x H(t,x))dt = \max_x \int_0^1 H(t,x)dt - \min_x \int_0^1 H(t,x)dt$$

(see [1]).

Let us recall a construction of symplectic invariants by Oh [8]. For $H \in C_0^\infty([0,1]\times T^*M)$ consider the classical action functional

$$\mathcal{A}_H(\gamma) := \int_0^1 \gamma^* pdq - Hdt,$$

for $\gamma : [0,1] \to T^*M,\ \gamma(0) \in O_M$, where pdq is a canonical Liouville 1-form on T^*M. Floer chain complexes $CF_*(H)$ are defined as free $\mathbf{Z}$-modules over

$$Crit_*(\mathcal{A}_H) := \{\gamma : [0,1] \to T^*M \mid \frac{d\gamma}{dt} = X_H(\gamma),\ \gamma(0), \gamma(1) \in O_M\}$$

where X_H is a Hamiltonian vector field corresponding to H. They are graded by the Maslov index, and filtered by level sets of $\mathcal{A}_H$: $CF_*^\lambda(H)$ denotes a free $\mathbf{Z}$-module over

$$Crit_*(\mathcal{A}_H) := \{\gamma \in CF_*(H) \mid \mathcal{A}_H(\gamma) \le \lambda\}.$$

Recall that $HF_*(H)$ is defined as a homology group of $CF_*(H)$ with respect to the boundary homomorphism

$$\partial : CF_*(H) \to CF_*(H),\ \partial(x) := \sum_{y\in CF(H)} n(x,y)y,$$

where $n(x,y)$ is the number of solutions of

$$\begin{cases} \frac{\partial u}{\partial \tau} + J(\frac{\partial u}{\partial t} - X_H(u)) = 0, \\ u(\tau,0), u(\tau,1) \in O_M, \\ u(-\infty,t) = x(t), u(+\infty,t) = y(t) \end{cases}$$

("negative gradient flow of $\mathcal{A}_H$"). Here J is some almost complex structure compatible with the symplectic form. Since $\mathcal{A}_H$ decreases along its "negative gradient lines" (2), ∂ restricts to $CF_*^\lambda(H)$; the corresponding homology group is denoted by $HF_*^\lambda(H)$. An obvious inclusion $CF_*^\lambda(H) \to CF_*(H)$ induces a homomorphism

$$j_*^\lambda : HF_*^\lambda(H) \to HF_*(H).$$

Oh [8] defined for a generic $H \in C_0^\infty([0,1] \times T^*M)$

$$\rho(H) = \inf\{\lambda \mid j_*^\lambda : HF^\lambda(H)_* \to HF_*(H) \text{ is surjective}\},$$

and proved that it depends only on $L := \phi_1^H(O_M)$, but not on a particular choice of *normalized* $H \in C_0^\infty([0,1] \times T^*M)$ that generates L. More precisely, denote by W_H the wave front of H, i.e.

$$W_H := \{(q,s) \in M \times \mathbf{R} \mid q = \pi(x),\ s = \mathcal{A}_H(\phi_t^H \circ (\phi_1^H)^{-1}(x)),\ x \in L\},$$

where $\pi : T^*M \to M$ is the canonical projection. Then, if $\phi_1^H(O_M) = \phi_1^K(O_M)$ and $W_H = W_K$, $\rho(H) = \rho(K)$ (Theorem 8.1 in [8]).

Let $\mathcal{H}(M)$ be a set of Hamiltonians normalized so that their wave fronts depend only on $L := \phi_1^H(O_M)$:

$$\mathcal{H}(M) := \{H \in C_0^\infty([0,1] \times T^*M) \mid \max_{(x,s)\in W_H} s + \min_{(x,s)\in W_H} s = 0\}.$$

Note that definitions (1) and (2) remain the same if we take the infimum over $H \in \mathcal{H}(M)$ only. Indeed, it is clear that

$$\begin{aligned} &\inf_{H\in C_0^\infty([0,1]\times T^*M)} \{\int_0^1 (\max_x H(t,x) - \min_x H(t,x))dt \mid \phi_t^H(0_M) = L_t\} \\ (4) \qquad &\le \inf_{H\in\mathcal{H}(M)} \{\int_0^1 (\max_x H(t,x) - \min_x H(t,x))dt \mid \phi_t^H(0_M) = L_t\}. \end{aligned}$$

For every $H \in C_0^\infty([0,1] \times T^*M)$ there exist $c_0 \in \mathbf{R}$ and $\chi \in C_0^\infty(T^*M)$ such that $\chi = 1$ in a neighborhood of $\bigcup_t \operatorname{supp} H(t,\cdot)$, $\chi \le 1$, and $H^{c_0,\chi} := (H + c_0)\chi \in \mathcal{H}(M)$. Since $H^{c_0,\chi}$ generates L_t and

$$\max_x H^{c_0,\chi}(t,x) - \min_x H^{c_0,\chi}(t,x) \le \max_x H(t,x) - \min_x H(t,x),$$

it follows that (4) is an equality.

Recall that for generic $H, K \in C_0^\infty([0,1] \times T^*M)$ the homomorphism $h_* : HF_*(H) \to HF_*(K)$ is induced by the homomorphism defined on the chain level as

$$h : CF_*(H) \to CF_*(K),\ h(x) := \sum_{y\in CF(K)} n(x,y)y,$$

where $n(x,y)$ is the number of solutions of

$$\begin{cases} \frac{\partial u}{\partial \tau} + J(\frac{\partial u}{\partial t} - X_{\tilde H}(u)) = 0, \\ u(\tau,0), u(\tau,1) \in O_M, \\ u(-\infty,t) = x(t), u(+\infty,t) = y(t). \end{cases}$$

Here $\tilde H(\tau,t,x)$ is a generic path in $C_0^\infty([0,1] \times T^*M)$, such that $\tilde H(\tau,t,x) = H(t,x)$ for $\tau \le -1$ and $\tilde H(\tau,t,x) = K(t,x)$ for $\tau \ge 1$. Since $\mathcal{A}_H$ decreases along its "negative gradient lines" (2), it follows that j_*^λ commutes with h_*. This fact, together with careful analysis of change of $\mathcal{A}_H$ along the trajectories (2) makes it possible to express effects of the homomorphism h_* on the level sets of the action functional and to prove that ρ is C^0 continuous. This extends the definition of ρ

to *all* (not necessarily generic) $H \in \mathcal{H}(M)$ (we refer the reader to [8] for details). Main properties of ρ are summarized in the following

Proposition 1. *The function $\rho : \mathcal{H}(M) \to \mathbf{R}$ satisfies*

1. *If $L := \phi_1^H(O_M) = \phi_1^K(O_M)$, then $\rho(H) = \rho(K)$; hence we can denote $\rho(H)$ by $\rho(L)$.*
2. $\rho(L) \in \mathrm{Spec}(L) := \{\mathcal{A}_H(\phi_t^H \circ (\phi_1^H)^{-1}(x)) \mid x \in O_M \cap L\}$
3. $$-\int_0^1 \max_{x \in T^*M}(H(t,x) - K(t,x))dt \leq \rho(H) - \rho(K) \leq -\int_0^1 \min_{x \in T^*M}(H(t,x) - K(t,x))dt.$$
   In particular, ρ is C^0-continuous and monotone, i.e. if $K \leq H$, then $\rho(H) \leq \rho(K)$.
4. $\rho(0) = 0$.
5. $\rho(\phi_1^H(O_M)) + \rho((\phi_1^H)^{-1}(O_M)) \leq d(O_M, \phi_1^H(O_M))$.
6. *If $S : M \to \mathbf{R}$ is a smooth function, then $\rho(-\pi^*S) = \max S$, where $\pi : T^*M \to M$ is a canonical projection.*

Proof. 1. is the contents of Theorem 8.1 in [8]. 2.-4. are contained in Theorem II in [8]. From 3. and 4. follows

$$\rho(\phi_1^H(O_M)) \leq -\int_0^1 \min_{x \in T^*M} H(t,x)dt. \tag{5}$$

Since $(\phi_1^H)^{-1} = \phi_1^{\overline{H}}$, where $\overline{H}(t,x) = -H(t, \phi_t^H(x))$, (5) gives

$$\begin{aligned} \rho((\phi_1^H)^{-1}(O_M)) &\leq -\int_0^1 \min_{x \in T^*M} \overline{H}(t,x)dt \\ &= -\int_0^1 \min_{x \in T^*M} \{-H(t,x)\}dt \\ &= \int_0^1 \max_{x \in T^*M} H(t,x)dt. \end{aligned} \tag{6}$$

Adding (5) and (6) we get

$$\rho(\phi_1^H(O_M)) + \rho((\phi_1^H)^{-1}(O_M)) \leq length(\{\phi_t^H(O_M)\}).$$

Taking $\inf_H$ we get 5.

To prove 6., note that $CF_*(\pi^*S)$ consists of constant paths (critical points of S), and thus for $x \in CF_*(-\pi^*S)$

$$\mathcal{A}_{-\pi^*S}(x) = S(x).$$

Let $x_+ \in M$ be such that $S(x_+) = \max S$. By the non-triviality of the cap action (see [3], or [9], [6] for similar arguments) for a generic $x \in M$ there exists a trajectory (2) connecting the generators of $CF_{dim M}(-\pi^*S)$ and $CF_0(-\pi^*S)$ that contribute to the generators of $HF_{dim M}(-\pi^*S)$ and $HF_0(-\pi^*S)$, where the grading comes from Maslov (or Morse) index (see [8]). Since $\mathcal{A}_{-\pi^*S}$ (and thus S) increases along the gradient lines (2), by choosing $x \in S^{-1}((\max S - \epsilon, \max S))$ for ϵ small enough, we see that if μ is a generator of $HF_{dimM}(-\pi^*S)$, then

$$\mu = k[x_+] + \sum_{x_+ \neq x_i \in CF_{dimM}} k_i[x_i]$$

for some integers k, k_i, where $k \neq 0$. Assume that $\lambda < S(x_+)$. Then $x_+ \notin CF^{\lambda}_{dimM}(-\pi^*S)$, so that for every $a \in HF^{\lambda}_{dimM}(-\pi^*S)$

$$j^{\lambda}_*(a) = \sum_{x_+ \neq x_i \in CF_{dimM}} l_i[x_i].$$

Since $HF^{\lambda}_{dimM}(-\pi^*S)$ is a free module with one generator μ, it must be $j^{\lambda}_*(a) = c\mu$ for some integer c, i.e.

$$ck[x_+] + \sum_{x_+ \neq x_i \in CF_{dimM}} (ck_i - l_i)[x_i] = 0. \tag{7}$$

Since

$$CF_{dimM}(-\pi^*S) \ni ckx_+ + \sum_{x_+ \neq x_i \in CF_{dimM}} (ck_i - l_i)x_i \notin Image(\partial)$$

and since $k \neq 0$, (7) is possible only if $c = 0$. Hence, j^{λ}_* is not surjective. *q.e.d.*

3. Flatness and geodesics

Denote by $\mathcal{F}(M)$ the space of normalized smooth functions on M:

$$\mathcal{F}(M) := \{S \in C^{\infty}(M) \mid \int_M S(q)dq = 0\},$$

where dq is the Lebesgue measure induced by the Riemannian metric on M. Note that for every C^1-small Hamiltonian deformation L of a zero section O_M there is a *unique* $S \in \mathcal{F}(M)$ such that $L = Graph(dS)$. Define a norm on $\mathcal{F}(M)$ by

$$\|S\| := \max_{q\in M} S - \min_{q \in M} S.$$

Theorem 2. *There exist C^1-neighborhoods $\mathcal{G}$ of $O_M \in \mathcal{L}$ and $\mathcal{U}$ of $0 \in \mathcal{F}(M)$ such that the mapping*

$$\Phi : \mathcal{U} \to \mathcal{G},\ L \mapsto Graph(dS)$$

is an isometry.

Proof. $L := Graph(dS) \in \mathcal{G}$ is generated by the time-independent Hamiltonian $-\pi^*S$ (to simplify notation we keep the same notation for $-\pi^*S$ multiplied by a cut-off function). Hence $\rho(L) = \rho(\phi_1^{-\pi^*S}(O_M)) = \rho(-\pi^*S)$. It is easy to see that $\rho((\phi_1^{-\pi^*S})^{-1}(O_M)) = \rho(\pi^*S)$. Therefore, by Proposition 1

$$\begin{aligned} \max S - \min S &= \max S + \max(-S) \\ &= \rho(\pi^*S) + \rho(-\pi^*S) \\ &\leq d(O_M, L) \\ &\leq \int_0^1 (\max S - \min S)dt \\ &= \max S - \min S. \end{aligned}$$

Hence, $d(O_M, L) = \|S\|$. The statement of the theorem follows from the invariance of d:

$$\begin{aligned} d(Graph(dS_0), Graph(dS_1)) &= d(O_M, (\phi_1^{-\pi^*S_0})^{-1} \circ \phi^{-\pi^*S_1}(O_M)) \\ &= d(O_M, \phi_1^{\pi^*S_0-\pi^*S_1}(O_M)) \\ &= d(O_M, Graph(d(S_1 - S_0))) \\ &= \|S_1 - S_0\|. \end{aligned}$$

q.e.d.

Remark 3. Theorem 2 complements the conclusion made at the end of Section 7 of [7] that, if $c(\mu, L)$, $c(1, L)$ are Viterbo's invariants of L (see [10]), then $c(\mu, L) - c(1, L) \leq d(O_M, L)$. From Theorem 2 it follows that the strict equality holds at least for L C^1-close to O_M.

Now we prove several statements analogous to the ones in [1] (or Section 5.7 in [4]).

Lemma 4 (Hamilton-Jacobi equation)**.** *Let $H(t, x)$ be a Hamiltonian generating $L_t = Graph(dS_t)$. Then*

$$\frac{\partial S}{\partial t}(q) + H(t, dS_t(q)) = const.$$

Proof. Let $q = \pi(\phi_s^H(y))$ for $y \in M$, $s \in [0, 1]$. Let (q, p) be the canonical coordinates around $\phi_s^H(y)$. Denote $(q_t, p_t) = \phi_t^H(y)$ for t near s. Then

$$(q_t, p_t) = \left(q_t, \frac{\partial S}{\partial q}(q_t)\right).$$

Differentiating with respect to t we get

$$\left(\frac{dq}{dt}, \frac{dp}{dt}\right) = \left(\frac{dq}{dt}, \frac{\partial^2 S}{\partial t \partial q} + \frac{\partial^2 S}{\partial q^2}\frac{dq}{dt}\right). \tag{8}$$

On the other hand, differentiating $H(t, dS(q)) := H(t, (q, \frac{\partial S}{\partial q}))$ with respect to q we obtain

$$\frac{\partial}{\partial q}\left(H(t, q, \frac{\partial S}{\partial q})\right) = \frac{\partial H}{\partial q} + \frac{\partial H}{\partial p}\frac{\partial^2 S}{\partial q^2}. \tag{9}$$

Applying Hamiltonian equations

$$\begin{cases} \dfrac{dq}{dt} = \dfrac{\partial H}{\partial p}, \\ \dfrac{dp}{dt} = -\dfrac{\partial H}{\partial q} \end{cases}$$

to (8) and (9) we obtain

$$\frac{\partial}{\partial q}\left(H(t, dS(q)) + \frac{\partial S}{\partial t}(q)\right) = 0.$$

q.e.d.

Corollary 5. *If $H(t, x)$ is a Hamiltonian generating $L_t = Graph(dS_t)$, then*

$$\left\|\frac{\partial S}{\partial t}\right\| \leq \max_{x \in T^*M} H(t, x) - \min_{x \in T^*M} H(t, x)$$

for every t.

Proof. By Lemma 4

$$
\begin{aligned}
\max_{x\in T^*M} H(t,x) &\geq \max_{q\in M} H(t, dS_t(q)) \\
&= c + \max_{q\in M}\left(-\frac{\partial S}{\partial t}(q)\right) \\
&= c - \min_{q\in M}\left(\frac{\partial S}{\partial t}(q)\right)
\end{aligned}
\tag{10}
$$

and

$$
\begin{aligned}
\min_{x\in T^*M} H(t,x) &\leq \min_{q\in M} H(t, dS_t(q)) \\
&= c + \min_{q\in M}\left(-\frac{\partial S}{\partial t}(q)\right) \\
&= c - \max_{q\in M}\left(\frac{\partial S}{\partial t}(q)\right),
\end{aligned}
$$

i.e.

$$
-\min_{x\in T^*M} H(t,x) \geq -c + \max_{q\in M}\left(\frac{\partial S}{\partial t}(q)\right). \tag{11}
$$

Adding (10) and (11) finishes the proof. *q.e.d.*

Corollary 6 (Compare [1], Proposition 3.3.A)**.** *If $L_t = Graph(dS_t)$, then $\frac{\partial S}{\partial t}$ is quasi-autonomous if and only if $\{L_t\}$ is generated by a quasi-autonomous Hamiltonian.*

Proof. Note that $\{L_t\}$ is completely determined by values of $H(t,x)$ for x near $\bigcup_t L_t$. Therefore, according to the comment after (2), we can assume that

$$
\max_x H(t,x) = \max_{x\in\cup_t L_t} H(t,x), \ \min_x H(t,x) = \min_{x\in\cup_t L_t} H(t,x).
$$

Assume first that $\frac{\partial S}{\partial t}$ is quasi-autonomous, so that $\max_{q\in M}\frac{\partial S}{\partial t} = \frac{\partial S}{\partial t}(q_+)$ and $\min_{q\in M}\frac{\partial S}{\partial t} = \frac{\partial S}{\partial t}(q_-)$. Then

$$
\begin{aligned}
0 &= \frac{\partial}{\partial q}\frac{\partial S}{\partial t}(q_\pm) \\
&= \frac{\partial}{\partial t}\frac{\partial S}{\partial q}(q_\pm)
\end{aligned}
$$

and thus $x_\mp := dS_t(q_\pm)$ does not depend on t. Let H_t be a Hamiltonian generating $\{L_t\}$. By Lemma 4, for $x \in \bigcup_t L_t$

$$
\begin{aligned}
H(t,x) &= c - \frac{\partial S}{\partial t}(\pi(x)) \\
&\leq c - \frac{\partial S}{\partial t}(q_-) \\
&= H(t, x_+).
\end{aligned}
$$

Similarly, $H(t,x) \geq H(t,x_-)$, i.e. H is quasi-autonomous.

Assume now that $H(t,x)$ is a quasi-autonomous Hamiltonian, such that $\phi_t^H(O_M) = L_t$. Let $\max_x H(t,x) = H(t,x_+)$ and $\min_x H(t,x) = H(t,x_-)$. Again, we can assume that $x_\pm \in \bigcup_t L_t$. Let $q_\pm := \pi(x_\mp)$. Then by Lemma 4

$$\begin{aligned}\frac{\partial S}{\partial t}(q) &= c - H(t, dS_t(q)) \\ &\le c - H(t, x_-) \\ &= \frac{\partial S}{\partial t}(q_+).\end{aligned}$$

Similarly, $\frac{\partial S}{\partial t}(q) \ge \frac{\partial S}{\partial t}(q_-)$, hence S is quasi-autonomous. *q.e.d.*

Remark 7. From the proof of Corollary 6 it follows that, if H_t is a quasi-autonomous Hamiltonian generating $Graph(dS_t)$, then (after modifying H away from $\bigcup_t L_t$ if necessary)

$$\max_{x\in T^*M} H(t,x) - \min_{x\in T^*M} H(t,x) = \max_{q\in M} \frac{\partial S}{\partial t}(q) - \min_{q\in M} \frac{\partial S}{\partial t}(q).$$

Theorem 8. *A regular path $\{L_t\} \in \mathcal{L}$ is a geodesic if and only if it is generated by a locally quasi-autonomous Hamiltonian function.*

Proof. Assume, without loss of generality, that $L_0 = O_M$. Choose $\epsilon > 0$ such that $L_t \in \mathcal{G}$ for $t \in (0, \epsilon)$. Let $S_t = \Phi^{-1}(L_t)$, where Φ is as in Theorem 2. $\{L_t\}_{0\le t\le\epsilon}$ is a minimizing geodesic if and only if for every $\delta > 0$ there exists a Hamiltonian H such that $\phi_t^H(O_M) = L_t$ and

$$\int_0^\epsilon (\max_x H(t,x) - \min_x H(t,x))dt - \delta \le d(O_M, L_\epsilon). \tag{12}$$

Since Φ is an isometry (Theorem 2)

$$\begin{aligned} d(O_M, L_\epsilon) &= \|S_\epsilon\| \\ &= \left\| \int_0^\epsilon \frac{\partial S}{\partial t} dt \right\| \\ &\le \int_0^\epsilon \left\| \frac{\partial S}{\partial t} \right\| dt \\ &\le \int_0^\epsilon (\max_x H(t,x) - \min_x H(t,x))dt \end{aligned} \tag{13}$$

(the last inequality follows from Corollary 5). Since δ in (12) is arbitrary, it follows from (12) and (13) that

$$\int_0^\epsilon \left\| \frac{\partial S}{\partial t} \right\| dt = \left\| \int_0^\epsilon \frac{\partial S}{\partial t} dt \right\|,$$

which according to (3) means that $\frac{\partial S}{\partial t}$ is quasi-autonomous. It follows from Corollary 6 that this is equivalent to $H(t,x)$ being quasi-autonomous. Vice versa, if H (and thus S as well) is quasi-autonomous, by (3) and Remark 7 both inequalities in (13) are equalities, and this gives

$$d(O_M, L_\epsilon) = \int_0^\epsilon (\max_x H(t,x) - \min_x H(t,x))dt.$$

q.e.d.

Theorem 8 extends Theorem 1.3.D in [1] which states that a regular path in a group $Ham(\mathbf{R}^{2n})$ of compactly supported Hamiltonian diffeomorphisms of $\mathbf{R}^{2n}$ is a geodesic if and only if it is generated by a locally quasi-autonomous Hamiltonian function. Indeed, a graph of every $\psi \in Ham(\mathbf{R}^{2n})$ is a Hamiltonian deformation of a diagonal $\Delta \subset \mathbf{R}^{2n} \times \mathbf{R}^{2n}$. We can identify $(\mathbf{R}^{2n} \times \mathbf{R}^{2n}, dq \wedge dp - dQ \wedge dP)$ with $(T^*\Delta, -d(pdq))$ through a *symplectic* identification

$$(q, p, Q, P) \mapsto \left(\frac{q+Q}{2}, \frac{p+P}{2}, P - p, q - Q \right).$$

Since ψ is compactly supported, the image of $Graph(\psi)$ coincides with the zero section of T^*M outside a compact set. Thus, after adding the fiber at infinity, $Graph(\psi)$ can be considered as a Hamiltonian deformation of the zero section in $T^*\mathbf{S}^{2n}$. Hence, Theorem 2 and Theorem 8 extend the analogous results in [1].

References

[1] Bialy, M. and Polterovich, L., *Geodesics of Hofer's metric on the group of Hamiltonian diffeomorphisms*, Duke Math. J., 76 (1994), 273–292. MR **96b:**58030

[2] Chekanov, Yu., *Lagrangian intersections, symplectic energy, and areas of holomorphic curves*, Duke Math. J., 95 (1998), 213–226. MR **99k:**58034

[3] Floer, A., *Cup-length estimates for Lagrangian intersections*, Comm. Pure App. Math., 42 (1989), 335–356. MR **90g:**58034

[4] Hofer, H., and Zehnder, E., *Symplectic invariants and Hamiltonian dynamics*, Birkhäuser Verlag, 1994. MR **96g:**58001

[5] Lalonde, F., and McDuff, D., *Hofer's L^∞-geometry: energy and stability of Hamiltonian flows, I, II*, Invent. Math., 122 (1995), 1–69. MR **96h:**58063; errata MR **97b:**58060

[6] Milinković, D., *Morse homology for generating functions of Lagrangian submanifolds*, Trans. AMS, 351 (1999), 3953–3974. MR **99m:**58048

[7] Milinković, D., *On equivalence of two constructions of invariants of Lagrangian submanifolds*, Pacific J. Math., to appear.

[8] Oh, Y.-G., *Symplectic topology as the geometry of action functional I - Relative Floer theory on the cotangent bundle*, J. Diff. Geom., 45 (1997), 499–577. MR **99a:**58032

[9] Oh, Y.-G., *Symplectic topology as the geometry of action functional II - Pants product and cohomological invariants*, Comm. Anal. Geom., 7 (1999), 1–55. MR **98a:**58032

[10] Viterbo, C., *Symplectic topology as the geometry of generating functions*, Math. Ann., 292 (1992), 685–710. MR **93b:**58058

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