

## SUBDIVISION SCHEMES FOR ITERATED FUNCTION SYSTEMS

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ABSTRACT. We identify iterated function systems  $\Phi$  and regular Borel measures  $\mu$  such that the matrix subdivision process relative to a finite family  $\mathcal{A}$  converges if and only if  $\mathcal{A}$  satisfies certain spectral properties.

### 1. INTRODUCTION

Given any finite family of contractive mappings  $\Phi$  on a metric space  $X$  there is a unique invariant subset  $\Omega$  of  $X$  which is the union of all of its images under the contractive mappings. We suppose that there is a regular Borel measure  $\mu$  on  $\Omega$  such that the images of  $\Omega$  under the contractive mappings partition  $\Omega$  relative to  $\mu$ . Repeated iterations of the contractions lead to a sequence of finer partitions of  $\Omega$ . Moreover, the contractions partition any of the cells obtained at iteration level  $r$  into subcells of level  $r + 1$ . This refinement structure on  $\Omega$  can be used to create vector fields on  $\Omega$  with range in  $\mathbb{R}^n$  in the following way. To this end, we require a family  $\mathcal{A}$  of  $n \times n$  matrices of the same cardinality as  $\Phi$ . Each mapping in  $\Phi$  is associated with a unique matrix in  $\mathcal{A}$ . We start with a constant vector field  $\mathbf{h}$  on  $\Omega$  and break it into a piecewise constant function whose value on the cell  $\phi(\Omega)$  is  $\mathbf{A}\mathbf{h}$  where  $\phi \in \Phi$  and  $\mathbf{A} \in \mathcal{A}$  are associated. We iterate this procedure to get a sequence of piecewise constant functions on finer and finer partitions of  $\Omega$ . This process is referred to as *matrix subdivision* and is important in both geometric modeling and wavelet analysis [1, 4, 5, 6, 7, 8]. If it converges, it produces a *refinable function* relative to  $\mathcal{A}$  and  $\Phi$ . Our purpose here is to study it in greater generality than previously available. Our goal is to identify conditions on  $\Phi$  and the measure  $\mu$  such that a useful characterization of the convergence of the subdivision process to a function in  $L_p(\Omega, \mu)$  can be given in terms of the spectral properties of  $\mathcal{A}$ .

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This paper is organized into four sections. Section 2 is devoted to a description of subdivision schemes and the definition of convergence. A few simple observations are also made on convergent subdivision schemes. In Section 3 we define and discuss the concept of  $\mu$ -uniformity of a family of contractions for any  $\mu$ -measurable set. Finally, in Section 4, we establish a characterization of convergent subdivision schemes and prove that the subdivision scheme converges exponentially fast and that its limit function enjoys Hölder regularity.

## 2. CONTRACTIONS AND SUBDIVISION

For any integer  $n > 0$  we denote the set  $\{0, 1, \dots, n-1\}$  by  $\mathbb{Z}_n$  and use  $\mathbb{Z}_\infty = \{0, 1, \dots\}$ . Let  $X$  be a complete metric space, and let  $\Phi = \{\phi_\ell : \ell \in \mathbb{Z}_m\}$  be a finite set of contractions on  $X$ . Recall the following facts from [2]. There exists a unique closed and bounded set  $\Omega$  such that

$$(2.1) \quad \Omega = \bigcup_{\ell \in \mathbb{Z}_m} \phi_\ell(\Omega).$$

To any vector  $\mathbf{j} = (j_k : k \in \mathbb{Z}_r) \in \mathbb{Z}_m^r$ , that is,  $j_k \in \mathbb{Z}_m$ ,  $k \in \mathbb{Z}_r$ , we associate the composite mapping

$$(2.2) \quad \phi_{\mathbf{j}} = \phi_{j_0} \circ \dots \circ \phi_{j_{r-1}}$$

and observe that

$$(2.3) \quad \Omega = \bigcup_{\mathbf{j} \in \mathbb{Z}_m^r} \phi_{\mathbf{j}}(\Omega).$$

Also, for any nonempty set  $\Gamma \subseteq X$  we define the set-valued mapping

$$\mathcal{S}(\Gamma) = \bigcup_{\ell \in \mathbb{Z}_m} \phi_\ell(\Gamma)$$

and, for  $r \in \mathbb{Z}_\infty$ , its iterates  $\mathcal{S}^{r+1}(\Gamma) = \mathcal{S}(\mathcal{S}^r(\Gamma))$ . It is known that

$$(2.4) \quad \lim_{r \rightarrow \infty} \mathcal{S}^r(\Gamma) = \Omega$$

in the Hausdorff metric [2].

Let  $x_{\mathbf{j}} \in X$ ,  $\mathbf{j} \in \mathbb{Z}_m^r$ ,  $r \in \mathbb{Z}_\infty$ , be the fixed point of the contraction  $\phi_{\mathbf{j}}$ . It is proved in [2] that  $\Omega$  is compact and is the closure of the set

$$(2.5) \quad F_\Phi = \{x_{\mathbf{j}} : \mathbf{j} \in \mathbb{Z}_m^r, r \in \mathbb{Z}_\infty\}.$$

For every (infinite) vector  $\mathbf{j} = (j_\ell : \ell \in \mathbb{Z}_\infty) \in \mathbb{Z}_m^\infty$  and  $r \in \mathbb{Z}_\infty$  we form the (finite) vectors  $\mathbf{j}^r = (j_\ell : \ell \in \mathbb{Z}_r)$  and recall that  $\lim_{r \rightarrow \infty} x_{\mathbf{j}^r}$  exists. We shall denote this limit by  $x_{\mathbf{j}}$ . Moreover, for every  $x, y \in \Omega$  there exists a  $\mathbf{j} \in \mathbb{Z}_m^\infty$  such that  $x = \lim_{r \rightarrow \infty} \phi_{\mathbf{j}^r}(y)$ .

Suppose that  $\mu$  is a regular Borel measure on  $\Omega$  with  $0 < \mu(\Omega) < \infty$ ; cf. [9]. For  $\Gamma \subseteq \Omega$ , we denote by  $L_p(\Gamma, \mu)$ ,  $1 \leq p < \infty$ , the spaces of  $p$ -integrable functions  $f : \Gamma \rightarrow \mathbb{R}$  for which

$$\|f\|_{L_p(\Gamma, \mu)} := \left( \int_\Gamma |f(x)|^p d\mu(x) \right)^{1/p}$$

is finite. Mostly, we will write  $\|f\|_{L_p(\Gamma)}$  and in the case that  $\Gamma = \Omega$  use  $\|f\|_p$ . Moreover, we will use  $L_p^{s \times t}(\Gamma, \mu)$  for all  $\mathbb{R}^{s \times t}$  matrix-valued functions on  $\Gamma$ , that

is,  $\mathbf{F} = (f_{\ell k} : \ell \in \mathbb{Z}_s, k \in \mathbb{Z}_t)$ , such that  $f_{\ell k} \in L_p(\Gamma, \mu)$ ,  $\ell \in \mathbb{Z}_s, k \in \mathbb{Z}_t$ , and equip it with the norm

$$\|\mathbf{F}\|_{L_p^{s \times t}(\Gamma, \mu)} = \sum_{\ell \in \mathbb{Z}_s} \sum_{k \in \mathbb{Z}_t} \|f_{\ell k}\|_{L_p(\Gamma, \mu)}.$$

As above, we use the abbreviated notation  $\|\mathbf{F}\|_{L_p(\Gamma)}$ ,  $\|\mathbf{F}\|_p$  and also  $L_p^s(\Gamma, \mu)$  for the space  $L_p^{s \times 1}(\Gamma, \mu)$ . In this way, the norm of a vector field  $\mathbf{f} = (f_j : j \in \mathbb{Z}_s) \in L_p^s(\Gamma, \mu)$  is defined as the norm of  $\mathbf{f}$ , interpreted as an  $s \times 1$  matrix-valued function. For any two matrices  $\mathbf{A} \in \mathbb{R}^{n \times s}$ ,  $\mathbf{B} \in \mathbb{R}^{t \times n}$  it follows from Minkowski's inequality that

$$\|\mathbf{A}\mathbf{F}\mathbf{B}\|_{L_p^{n \times n}(\Gamma)} \leq \|\mathbf{A}^T\| \|\mathbf{B}\| \|\mathbf{F}\|_{L_p^{s \times t}(\Gamma)},$$

where

$$\|\mathbf{A}\| := \max_{j \in \mathbb{Z}_n} \sum_{k \in \mathbb{Z}_s} |A_{jk}|.$$

We assume that the measure  $\mu$  and the family  $\Phi$  of contractive mappings have the following properties. There exist positive constants  $C, D$  such that for all  $\ell \in \mathbb{Z}_m$ ,  $\Gamma \subseteq \Omega$  and  $f \in L_p(\Gamma, \mu)$ ,

$$(2.6) \quad C \|f\|_{L_p(\phi_\ell(\Gamma), \mu)} \leq \|f \circ \phi_\ell\|_{L_p(\Gamma, \mu)} \leq D \|f\|_{L_p(\phi_\ell(\Gamma), \mu)}.$$

Moreover, we require that

$$(2.7) \quad \phi_\ell \text{ is injective,} \quad \ell \in \mathbb{Z}_m,$$

and

$$(2.8) \quad \mu(\phi_\ell(\Omega) \cap \phi_k(\Omega)) = 0, \quad \ell, k \in \mathbb{Z}_m, \ell \neq k.$$

For  $\Gamma \subseteq \Omega$  and  $\ell \in \mathbb{Z}_m$ , equation (2.6) implies that

$$(2.9) \quad C^p \mu(\phi_\ell(\Gamma)) \leq \mu(\Gamma) \leq D^p \mu(\phi_\ell(\Gamma)).$$

In particular, we obtain that  $\mu(\phi_j(\Omega)) > 0$ ,  $\mathbf{j} \in \mathbb{Z}_m^r, r \in \mathbb{Z}_\infty$ . Moreover, from (2.6) and (2.8) it follows that

$$\mu(\phi_j(\Omega) \cap \phi_k(\Omega)) = 0, \quad \mathbf{j}, \mathbf{k} \in \mathbb{Z}_m^r, \mathbf{j} \neq \mathbf{k}, \quad r \in \mathbb{Z}_\infty.$$

We shall make use of the family of *pushforward measures*  $\{\mu_j : \mathbf{j} \in \mathbb{Z}_m^r, r \in \mathbb{Z}_\infty\}$  defined by setting, for any  $\mu$ -measurable set  $\Gamma \subseteq \Omega$ ,

$$\mu_j(\Gamma) = \mu(\phi_j(\Gamma)).$$

Since for all  $f \in L_p(\Omega)$  we have that

$$\|f \circ \phi_j\|_{L_p(\Gamma, \mu_j)} = \|f\|_{L_p(\phi_j(\Gamma), \mu)},$$

condition (2.6) says, for  $\ell \in \mathbb{Z}_m$ , that on the set  $\phi_\ell(\Omega)$  the  $L_p(\Gamma, \mu)$  and  $L_p(\phi_\ell(\Gamma), \mu_\ell)$  norms are equivalent.

Let  $\mathcal{A} = \{\mathbf{A}_\ell : \ell \in \mathbb{Z}_m\}$  be a finite collection of  $n \times n$  matrices. For every  $\mathbf{j} = (j_k : k \in \mathbb{Z}_r) \in \mathbb{Z}_m^r$  and  $r \in \mathbb{Z}_\infty$ , we define the matrices  $\mathbf{A}_\mathbf{j} = \mathbf{A}_{j_{r-1}} \cdots \mathbf{A}_{j_0}$  and the piecewise constant matrix-valued function  $\mathbf{A}^{[r]} \in L_p^{n \times n}(\Omega)$

$$(2.10) \quad \mathbf{A}^{[r]} = \sum_{\mathbf{j} \in \mathbb{Z}_m^r} \mathbf{A}_\mathbf{j} \chi_{\phi_j(\Omega)}, \quad r \in \mathbb{Z}_\infty,$$

where, for  $\Gamma \subseteq \Omega$ , we write  $\chi_\Gamma$  for the characteristic function of  $\Gamma$ . Note that the order of the matrices in the product (2.10) is the *reverse* of the order of the

mappings in the composition (2.2). From the upper bound in equation (2.6) and the injectivity of  $\phi_\ell, \ell \in \mathbb{Z}_m$ , it follows, for  $r \in \mathbb{Z}_\infty$  and  $k \in \mathbb{Z}_m$ , that

$$(2.11) \quad \mathbf{A}^{[r+1]} \circ \phi_k = \mathbf{A}^{[r]} \mathbf{A}_k, \quad \mu\text{-a.e.}$$

Given any two vectors  $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$  we denote by  $\mathbf{a} \otimes \mathbf{b} \in \mathbb{R}^{n \times n}$  their *tensor product*

$$\mathbf{a} \otimes \mathbf{b} = (a_\ell b_k : \ell, k \in \mathbb{Z}_n)$$

and use  $\mathbf{1}$  for the vector in  $\mathbb{R}^n$  whose entries are all 1.

**Definition 2.1.** A finite collection  $\mathcal{A}$  of  $n \times n$  matrices admits a  $L_p(\Omega, \mu)$  convergent subdivision scheme if there exists a nontrivial vector field  $\mathbf{f} \in L_p^n(\Omega, \mu)$  such that

$$(2.12) \quad \lim_{r \rightarrow \infty} \mathbf{A}^{[r]} = \mathbf{1} \otimes \mathbf{f}.$$

Whenever the subdivision scheme converges we call  $\mathbf{f}$  its limit.

Later, we shall use the fact that the set of matrices

$$\mathbf{A}_\ell := \mathbf{H} := \frac{1}{n} \mathbf{1} \otimes \mathbf{1}, \quad \ell \in \mathbb{Z}_m,$$

admit an  $L_p(\Omega, \mu)$  convergent subdivision scheme with limit  $\mathbf{h} := n^{-1} \mathbf{1}$ .

**Proposition 2.2.** If  $\mathcal{A}$  admits a convergent subdivision scheme with limit  $\mathbf{f}$ , then, for every  $\ell \in \mathbb{Z}_m$ , we have that

$$(2.13) \quad \mathbf{f} \circ \phi_\ell = \mathbf{A}_\ell^T \mathbf{f}, \quad \mu\text{-a.e.}$$

*Proof.* We fix  $\ell \in \mathbb{Z}_m$ . The inequalities

$$\begin{aligned} & \left\| \mathbf{1} \otimes \mathbf{f} \circ \phi_\ell - \mathbf{1} \otimes \left( \mathbf{A}^T \mathbf{f} \right) \right\|_p = \left\| \mathbf{1} \otimes \mathbf{f} \circ \phi_\ell - (\mathbf{1} \otimes \mathbf{f}) \mathbf{A}_\ell \right\|_p \\ & \leq \left\| \mathbf{1} \otimes \mathbf{f} \circ \phi_\ell - \mathbf{A}^{[r+1]} \circ \phi_\ell \right\|_p + \left\| (\mathbf{1} \otimes \mathbf{f} - \mathbf{A}^{[r]}) \mathbf{A}_\ell \right\|_p \\ & \leq D \left\| \mathbf{1} \otimes \mathbf{f} - \mathbf{A}^{[r+1]} \right\|_p + \|\mathbf{A}_\ell\| \left\| \mathbf{1} \otimes \mathbf{f} - \mathbf{A}^{[r]} \right\|_p \end{aligned}$$

yield the result. □

We call any solution of equation (2.13) a *refinable vector field* associated to the *refinement matrices*  $\mathcal{A}$ . Therefore, the limit of a convergent subdivision scheme is always a refinable vector field.

We say that the measure  $\mu$  is *strictly positive* on  $\Omega$  if, for all  $x \in \Omega$  and  $\delta > 0$ , the measure of the ball  $B(x; \delta) = \{y : y \in \Omega, d(x, y) \leq \delta\}$  is strictly positive, that is,  $\mu(B(x; \delta)) > 0$ . Observe that a strictly positive measure  $\mu$  satisfies, for any continuous function  $f \in C(\Omega)$  and  $x \in \Omega$ , the inequality

$$\left| \frac{1}{\mu(B(x; \delta))} \int_{B(x; \delta)} f(y) d\mu(y) - f(x) \right| \leq \omega_\infty(f; \delta),$$

where  $\omega_\infty$  denotes the modulus of continuity of  $f$  given by

$$\omega_\infty(f; \delta) = \max \{|f(x) - f(y)| : x, y \in \Omega, d(x, y) < \delta\}.$$

Since  $\lim_{\delta \rightarrow 0} \omega_\infty(f; \delta) = 0$  for any  $f \in C(\Omega)$  we conclude that whenever  $f \in C(\Omega)$  vanishes  $\mu$ -a.e. on  $\Omega$  it must be identically zero on  $\Omega$ . This fact leads us to the next result.

**Corollary 2.3.** *Suppose that  $\mu$  is strictly positive on  $\Omega$  and that  $\mathcal{A}$  admits a convergent subdivision scheme with limit  $\mathbf{f}$ . If  $\mathbf{f}$  is continuous on  $\Omega$ , then, for every  $x \in \Omega$ ,*

$$\mathbf{f}(x) \neq 0.$$

*Proof.* Since  $\mathbf{f}$  and  $\phi_\ell, \ell \in \mathbb{Z}_m$ , are continuous we have, for any  $y \in \Omega$  and  $\mathbf{j} \in \mathbb{Z}_m^r, r \in \mathbb{Z}_\infty$ , that

$$(2.14) \quad \mathbf{f}(\phi_{\mathbf{j}}(y)) = \mathbf{A}_{\mathbf{j}}^T \mathbf{f}(y).$$

Suppose that there exists a  $y \in \Omega$  such that  $\mathbf{f}(y) = 0$ . For any  $x \in \Omega$  we choose a sequence  $\mathbf{j}^r \in \mathbb{Z}_m^r, r \in \mathbb{Z}_\infty$ , such that  $\lim_{r \rightarrow \infty} \phi_{\mathbf{j}^r}(y) = x$ . Since  $\mathbf{f}(y) = 0$  equation (2.14) and the continuity of  $\mathbf{f}$  imply that

$$\mathbf{f}(x) = \lim_{r \rightarrow \infty} \mathbf{f}(\phi_{\mathbf{j}^r}(y)) = \mathbf{A}_{\mathbf{j}^r}^T \mathbf{f}(y) = 0,$$

which violates the assumption that  $\mathcal{A}$  admits a convergent subdivision scheme. This contradiction proves the result. □

**Corollary 2.4.** *Suppose that  $\mathcal{A}$  admits a convergent subdivision scheme with limit  $\mathbf{f}$ . If at least two of the matrices in  $\mathcal{A}$  are nonsingular, then  $\mathbf{f} \neq 0, \mu$ -a.e.*

*Proof.* Let  $\mathbf{A}_\ell$  and  $\mathbf{A}_k, \ell \neq k \in \mathbb{Z}_m$ , be two nonsingular matrices in  $\mathcal{A}$  and set  $\Gamma = \{x \in \Omega : \mathbf{f}(x) = 0\}$ . From the refinement equation (2.13) and the invertibility of  $\mathbf{A}_\ell$  and  $\mathbf{A}_k$  we conclude that

$$\Gamma = \phi_\ell(\Gamma) = \phi_k(\Gamma)$$

and therefore, by (2.8) we obtain

$$\mu(\Gamma) \leq \mu(\phi_\ell(\Gamma) \cap \phi_k(\Gamma)) \leq \mu(\phi_\ell(\Omega) \cap \phi_k(\Omega)) = 0.$$

□

### 3. $\mu$ -UNIFORMITY

For  $\ell \in \mathbb{Z}_m$  we define the set

$$(3.1) \quad \Omega_\ell^r = \bigcup \{ \phi_{\mathbf{j}}(\Omega) : \mathbf{j} = (j_k : k \in \mathbb{Z}_{r+1}) \in \mathbb{Z}_m^{r+1}, j_r = \ell \}$$

and note that

$$\Omega = \bigcup_{\ell \in \mathbb{Z}_m} \Omega_\ell^r.$$

From (2.4) it follows that

$$\lim_{r \rightarrow \infty} \Omega_\ell^r = \Omega, \quad \ell \in \mathbb{Z}_m,$$

in the Hausdorff metric on  $X$ . We require an additional property.

**Definition 3.1.** We say that the family  $\Phi$  of contractions is  $\mu$ -uniform if for any  $\mu$ -measurable set  $\Gamma \subseteq \Omega, \mu(\Gamma) > 0$  and  $\ell \in \mathbb{Z}_m$  the condition

$$(3.2) \quad \liminf_{r \rightarrow \infty} \mu(\Gamma \cap \Omega_\ell^r) > 0$$

is satisfied.

Let us first provide sufficient conditions on the family of contractions  $\Phi$  and the measure  $\mu$  so that  $\Phi$  is  $\mu$ -uniform.

**Lemma 3.2.** *If there exist positive constants  $\rho_\ell$ ,  $\ell \in \mathbb{Z}_m$ , such that  $\mu_\ell = \rho_\ell \mu$ ,  $\ell \in \mathbb{Z}_m$ , then, for any  $f \in L_p(\Omega)$  and  $\ell \in \mathbb{Z}_m$ , we have that*

$$(3.3) \quad \lim_{r \rightarrow \infty} \|f\|_{L_p(\Omega_r^\ell)} = \rho_\ell^{1/p} \|f\|_{L_p(\Omega)}.$$

Choosing any  $\mu$ -measurable  $\Gamma \subseteq \Omega$  and the function  $f = \chi_\Gamma$  in (3.3) yields the following result.

**Corollary 3.3.** *If there exist positive constants  $\rho_\ell$ ,  $\ell \in \mathbb{Z}_m$ , such that  $\mu_\ell = \rho_\ell \mu$ ,  $\ell \in \mathbb{Z}_m$ , then  $\Phi$  is  $\mu$ -uniform.*

A change of variables arguments implies the following result.

**Corollary 3.4.** *If  $X = \mathbb{R}^s$ ,  $\Phi$  consists of affine mappings and  $\mu$  is the Lebesgue measure on  $\mathbb{R}^s$ , then  $\Phi$  is  $\mu$ -uniform.*

To prove Lemma 3.2 we need another ancillary fact.

**Lemma 3.5.** *For any  $f \in C(\Omega)$  there exists a sequence  $\varphi^r$ ,  $r \in \mathbb{Z}_\infty$ , of piecewise constant functions of the form*

$$(3.4) \quad \varphi^r = \sum_{\mathbf{j} \in \mathbb{Z}_m^r} c_{\mathbf{j}} \chi_{\phi_{\mathbf{j}}(\Omega)}, \quad c_{\mathbf{j}} \in \mathbb{R}, \quad \mathbf{j} \in \mathbb{Z}_m^r,$$

such that

$$\lim_{r \rightarrow \infty} \|f - \varphi^r\|_{L_p(\Omega)} = 0.$$

*Proof.* We choose some  $y \in \Omega$  and set

$$\varphi^r(f) = \sum_{\mathbf{j} \in \mathbb{Z}_m^r} (f \circ \phi_{\mathbf{j}})(y) \chi_{\phi_{\mathbf{j}}(\Omega)}.$$

Let  $\delta_r = \max \{d(x, y) : x, y \in \phi_{\mathbf{j}}(\Omega), \mathbf{j} \in \mathbb{Z}_m^r\}$  and note that, since  $\Omega$  is bounded and  $\Phi$  is a family of contractions,  $\lim_{r \rightarrow \infty} \delta_r = 0$ . Note that, for any  $r \in \mathbb{Z}_\infty$ , we can write

$$f(x) = \sum_{\mathbf{j} \in \mathbb{Z}_m^r} f(x) \chi_{\phi_{\mathbf{j}}(\Omega)}(x), \quad x \in \Omega,$$

and therefore

$$\begin{aligned} \|\varphi^r - f\|_{L_p(\Omega)} &= \left( \int_{\Omega} \left| \sum_{\mathbf{j} \in \mathbb{Z}_m^r} ((f \circ \phi_{\mathbf{j}})(y) - f(x)) \chi_{\phi_{\mathbf{j}}(\Omega)}(x) \right|^p d\mu(x) \right)^{1/p} \\ &= \left( \sum_{\mathbf{j} \in \mathbb{Z}_m^r} \int_{\phi_{\mathbf{j}}(\Omega)} |f \circ \phi_{\mathbf{j}}(y) - f(x)|^p d\mu(x) \right)^{1/p} \\ &\leq \omega_\infty(f; \delta_r) (\mu(\Omega))^{1/p}. \end{aligned}$$

Thus, the result follows. □

*Proof of Lemma 3.2.* For  $r \in \mathbb{Z}_\infty$  let  $\varphi^r$  be a function of the form (3.3). Associated to any  $\mathbf{j} = (j_k : k \in \mathbb{Z}_r)$  we introduce the constant  $\rho_{\mathbf{j}} = \prod_{k \in \mathbb{Z}_r} \rho_{j_k}$  and note that  $\rho_{\mathbf{j}} = \frac{\mu_{\mathbf{j}}(\Omega)}{\mu(\Omega)}$ . For  $\ell \in \mathbb{Z}_m$  we have that

$$\|\varphi^r\|_{L_p(\Omega_r^\ell)}^p = \sum_{\mathbf{j} \in \mathbb{Z}_m^r} |c_{\mathbf{j}}|^p \rho_{\mathbf{j}} \rho_\ell \mu(\Omega) = \rho_\ell \sum_{\mathbf{j} \in \mathbb{Z}_m^r} |c_{\mathbf{j}}|^p \rho_{\mathbf{j}} \mu(\Omega) = \rho_\ell \|\varphi^r\|_{L_p(\Omega)}^p.$$

Next, we choose  $g \in C(\Omega)$ . By Lemma 3.5, given any  $\varepsilon > 0$  there is an  $r_0 \in \mathbb{Z}_\infty$  such that for all  $r > r_0$  we have that  $\|\varphi^r(g) - g\|_{L_p(\Omega)} < \varepsilon/2$  and therefore, since  $\rho_\ell \leq 1, \ell \in \mathbb{Z}_m$ ,

$$\begin{aligned} \|g\|_{L_p(\Omega_\ell^r)} &\leq \|\varphi^r(g)\|_{L_p(\Omega_\ell^r)} + \|g - \varphi^r(g)\|_{L_p(\Omega)} \leq \rho_\ell^{1/p} \|\varphi^r(g)\|_{L_p(\Omega)} + \frac{\varepsilon}{2} \\ &\leq \rho_\ell^{1/p} \|g\|_{L_p(\Omega)} + \varepsilon. \end{aligned}$$

In exactly the same way we also obtain

$$\|g\|_{L_p(\Omega_{\ell,r})} \geq \rho_\ell^{1/p} \|g\|_{L_p(\Omega)} - \varepsilon$$

which proves (3.3) for continuous functions. For  $f \in L_p(\Omega)$  we let  $\varepsilon > 0$  and pick  $g \in C(\Omega)$  such that  $\|f - g\|_{L_p(\Omega)} < \varepsilon/3$ . By what we just proved, we can also choose an  $r_0 \in \mathbb{Z}_\infty$  such that, for any  $r > r_0$  and  $\ell \in \mathbb{Z}_m$ ,

$$\left| \|g\|_{L_p(\Omega_\ell^r)} - \rho_\ell^{1/p} \|g\|_{L_p(\Omega)} \right| \leq \varepsilon/3.$$

Hence, for  $r > r_0$  and  $\ell \in \mathbb{Z}_m$  we obtain

$$\|f\|_{L_p(\Omega_\ell^r)} \leq \|g\|_{L_p(\Omega_\ell^r)} + \frac{\varepsilon}{3} \leq \rho_\ell^{1/p} \|g\|_{L_p(\Omega)} + \frac{2}{3}\varepsilon \leq \rho_\ell^{1/p} \|f\|_{L_p(\Omega)} + \varepsilon,$$

and likewise

$$\|f\|_{L_p(\Omega_\ell^r)} \geq \rho_\ell^{1/p} \|g\|_{L_p(\Omega)} - \varepsilon,$$

which proves the lemma. □

Based on property (3.2) we have the following result.

**Lemma 3.6.** *If  $\Phi$  is  $\mu$ -uniform in  $\Omega$ , then, for any  $f \in L_p(\Omega)$  for which there exists  $\ell \in \mathbb{Z}_m$  such that*

$$(3.5) \quad \lim_{r \rightarrow \infty} \|f\|_{L_p(\Omega_\ell^r)} = 0,$$

*it follows that  $f = 0, \mu$ -a.e.*

*Proof.* Let  $f \in L_p(\Omega)$  satisfy (3.5) and suppose that  $f \neq 0$ . Then there exists a constant  $\delta > 0$  such that  $\mu(F_\delta) > 0$ , where

$$F_\delta := \{x : |f(x)| > \delta\}.$$

Therefore, we obtain

$$\|f\|_{L_p(\Omega_\ell^r)} \geq \delta (\mu(F_\delta \cap \Omega_\ell^r))^{1/p}.$$

Since  $\Phi$  is  $\mu$ -uniform, this yields the contradiction

$$0 = \liminf_{r \rightarrow \infty} \|f\|_{L_p(\Omega_\ell^r)} \geq \delta \liminf_{r \rightarrow \infty} (\mu(F_\delta \cap \Omega_\ell^r))^{1/p} > 0.$$

□

4. CONVERGENCE OF SUBDIVISION SCHEMES

Our first result in this section provides a necessary condition for the convergence of the matrix subdivision scheme.

**Proposition 4.1.** *If  $\Phi$  is  $\mu$ -uniform and  $\mathcal{A}$  admits a convergent subdivision scheme, then*

$$(4.1) \quad \mathbf{A}_\ell \mathbf{1} = \mathbf{1}, \quad \ell \in \mathbb{Z}_m.$$

*Proof.* For any  $\mathbf{j} \in \mathbb{Z}_m^{r+1}$  and  $\mu$ -a.e.  $x \in \phi_{\mathbf{j}}(\Omega)$  we conclude from the definition (2.10) that  $\mathbf{A}^{[r+1]}(x) = \mathbf{A}_{\mathbf{j}}$ . Hence, for  $\ell \in \mathbb{Z}_m$  and  $r \in \mathbb{Z}_\infty$  we get that

$$(4.2) \quad \mathbf{A}^{[r+1]} \Big|_{\Omega_\ell^r} = \mathbf{A}_\ell \mathbf{A}^{[r]} \Big|_{\Omega_\ell^r}, \quad \mu\text{-a.e.},$$

and consequently we obtain

$$\begin{aligned} \|(\mathbf{1} - \mathbf{A}_\ell \mathbf{1}) \otimes \mathbf{f}\|_{L_p(\Omega_\ell^r)} &= \|\mathbf{1} \otimes \mathbf{f} - \mathbf{A}_\ell (\mathbf{1} \otimes \mathbf{f})\|_{L_p(\Omega_\ell^r)} \\ &\leq \left\| \mathbf{1} \otimes \mathbf{f} - \mathbf{A}^{[r+1]} \right\|_p + \left\| \mathbf{A}^{[r+1]} - \mathbf{A}_\ell (\mathbf{1} \otimes \mathbf{f}) \right\|_{L_p(\Omega_\ell^r)} \\ &= \left\| \mathbf{1} \otimes \mathbf{f} - \mathbf{A}^{[r+1]} \right\|_p + \left\| \mathbf{A}_\ell (\mathbf{A}^{[r]} - \mathbf{1} \otimes \mathbf{f}) \right\|_{L_p(\Omega_\ell^r)} \\ &\leq \left\| \mathbf{1} \otimes \mathbf{f} - \mathbf{A}^{[r+1]} \right\|_p + \left\| \mathbf{A}_\ell^T \right\| \left\| \mathbf{1} \otimes \mathbf{f} - \mathbf{A}^{[r]} \right\|_p. \end{aligned}$$

Since the upper bound in the last inequality converges to zero for  $r \rightarrow \infty$ , Lemma 3.6 proves the result. □

To state the next result we use  $\langle \cdot, \cdot \rangle$  for the standard inner product on  $\mathbb{R}^n$ .

**Corollary 4.2.** *If the hypothesis of Proposition 4.1 is satisfied and  $\mathbf{f}$  is the limit of the subdivision scheme  $\mathcal{A}$ , then*

$$(4.3) \quad \langle \mathbf{f}, \mathbf{1} \rangle = 1, \quad \mu\text{-a.e.}$$

*Proof.* By Proposition 4.1, for  $r \in \mathbb{Z}_\infty$ ,  $\mathbf{A}^{[r]} \mathbf{1} = \mathbf{1}$ ,  $\mu$ -a.e., and therefore

$$\mathbf{1} = \lim_{r \rightarrow \infty} \mathbf{A}^{[r]} \mathbf{1} = (\mathbf{1} \otimes \mathbf{f}) \mathbf{1},$$

which proves equation (4.3). □

For the next corollary we introduce, for  $\mu$ -a.e.  $y \in \Omega$ ,  $f \in L_p(\Omega)$  and  $r \in \mathbb{Z}_\infty$ , the following function on  $\Omega$  which acts like a modulus of continuity,

$$(4.4) \quad \omega_r(f)(y) = \left( \sum_{\mathbf{j} \in \mathbb{Z}_m^r} \int_{\phi_{\mathbf{j}}(\Omega)} |f(x) - (f \circ \phi_{\mathbf{j}})(y)|^p \, d\mu(x) \right)^{1/p},$$

and extend it to vector fields  $\mathbf{f} = (f_k : k \in \mathbb{Z}_n)$  by setting

$$\omega_r(\mathbf{f}) = \sum_{k \in \mathbb{Z}_n} \omega_r(f_k).$$

**Corollary 4.3.** *If  $\mathcal{A}$  admits a convergent subdivision scheme with limit  $\mathbf{f}$ , then*

$$\lim_{r \rightarrow \infty} \omega_r(\mathbf{f}) = 0, \quad \mu\text{-a.e.}$$

*Proof.* We have  $\mu$ -a.e.  $y \in \Omega$  such that

$$(4.5) \quad \omega_r(\mathbf{f})(y) = \left\| \mathbf{f}^T(y) \left( \mathbf{1} \otimes \mathbf{f} - \mathbf{A}^{[r]} \right) \right\|_p = \left\| \mathbf{f} - \left( \mathbf{A}^{[r]} \mathbf{f} \right)(y) \right\|_p.$$

The first identity follows readily from the fact that  $\mathbf{f}$  is the limit function of a convergent subdivision scheme and thus satisfies (2.13) and (4.3). For the second identity we compute that

$$\begin{aligned} \left\| \mathbf{f} - \left( \mathbf{A}^{[r]} \mathbf{f} \right)(y) \right\|_p &= \sum_{k \in \mathbb{Z}_n} \left( \int_{\Omega} \left| f_k(x) - \left( \mathbf{A}^{[r]} \mathbf{f} \right)_k(y) \right|^p d\mu(x) \right)^{1/p} \\ &= \sum_{k \in \mathbb{Z}_n} \left( \sum_{j \in \mathbb{Z}_n} \int_{\phi_j(\Omega)} \left| f_k(x) - \left( \mathbf{A}_j \mathbf{f} \right)_k(y) \right|^p d\mu(x) \right)^{1/p} \\ &= \omega_r(\mathbf{f})(y). \end{aligned}$$

□

This fact is a consequence of Theorem 4.8, however, we put it here since it has a partial converse which we state next.

**Proposition 4.4.** *Suppose that  $\mathbf{f} \in L_p(\Omega)$  is a nontrivial vector field which is refinable relative to a family  $\mathcal{A}$  of matrices, which satisfies the equation  $\langle \mathbf{1}, \mathbf{f} \rangle = 1$   $\mu$ -a.e. and the conditions*

$$\lim_{r \rightarrow \infty} \omega_r(\mathbf{f}) = 0, \quad \mu\text{-a.e.},$$

and

$$\mathbb{R}^n = \text{span} \{ \mathbf{f}(y) : \mu\text{-a.e. } y \in \Omega \}.$$

Then  $\mathcal{A}$  admits a convergent subdivision scheme with limit  $\mathbf{f}$ .

*Proof.* This fact follows conveniently from equation (4.5). □

Now we will characterize the convergence of matrix subdivision schemes. For this purpose we define the *difference operator*  $\mathbf{D} : \mathbb{R}^n \rightarrow \mathbb{R}^{n-1}$  by setting, for any  $\mathbf{c} = (c_\ell : \ell \in \mathbb{Z}_n)$ ,

$$\mathbf{D}\mathbf{c} = (c_{\ell+1} - c_\ell : \ell \in \mathbb{Z}_{n-1}).$$

Its representation as an  $(n - 1) \times n$  matrix is

$$\mathbf{D} = \begin{bmatrix} -1 & 1 & & & \\ & \ddots & \ddots & & \\ & & -1 & 1 & \end{bmatrix}.$$

Observe the following fact.

**Lemma 4.5.** *Let  $\mathbf{A} \in \mathbb{R}^{n \times n}$ .*

1.  $\mathbf{A}\mathbf{1} = \mathbf{0}$  if and only if there is a matrix  $\mathbf{B} \in \mathbb{R}^{n \times (n-1)}$  such that  $\mathbf{A} = \mathbf{B}\mathbf{D}$ .
2. If  $\mathbf{A}\mathbf{1} = \mathbf{1}$ , then there is a matrix  $\mathbf{B} \in \mathbb{R}^{(n-1) \times (n-1)}$  such that  $\mathbf{D}\mathbf{A} = \mathbf{B}\mathbf{D}$ .

We pattern the following notion after Jia [3].

**Definition 4.6.** Let  $\mathcal{A}$  be a finite collection of  $n \times n$  matrices of cardinality  $m$ . The  $p$ -joint spectral radius of  $\mathcal{A}$  relative to  $\Phi$  and  $\mu$  is defined as

$$(4.6) \quad \rho_p(\mathcal{A}) = \limsup_{r \rightarrow \infty} \left( \sum_{j \in \mathbb{Z}_m^r} \mu_j(\Omega) \|\mathbf{A}_j\|^p \right)^{1/rp}.$$

**Theorem 4.7.** Let  $\Phi$  be  $\mu$ -uniform. Then  $\mathcal{A}$  admits an  $L_p(\Omega, \mu)$  convergent subdivision scheme if and only if the following two conditions hold:

1.

$$(4.7) \quad \mathbf{A}_\ell \mathbf{1} = \mathbf{1}, \quad \ell \in \mathbb{Z}_m.$$

2. The matrices  $\mathbf{B}_\ell \in \mathbb{R}^{n-1 \times n-1}$ ,  $\ell \in \mathbb{Z}_m$ , defined by  $D\mathbf{A}_\ell = \mathbf{B}_\ell D$ , satisfy

$$(4.8) \quad \rho_p(\mathcal{B}) < 1.$$

*Proof.* Let us assume that (4.7) and (4.8) are satisfied and show that  $\mathcal{A}$  admits a convergent subdivision scheme. To this end, we consider a sequence of vector fields  $\mathbf{h}_r$ ,  $r \in \mathbb{Z}_\infty$ , on  $\Omega$  defined as

$$(4.9) \quad \mathbf{h}_r^T = \frac{1}{n} \mathbf{1}^T \mathbf{A}^{[r]}$$

with the convention that  $\mathbf{A}^{[0]} = \mathbf{I}$ , the identity matrix. Now, for  $r \in \mathbb{Z}_\infty$  and  $\mathbf{c} \in \mathbb{R}^n$ , using the vector  $\mathbf{j}^r = (j_k : k \in \mathbb{Z}_r)$ , we obtain the equation

$$\begin{aligned} \|\langle \mathbf{c}, \mathbf{h}_{r+1} - \mathbf{h}_r \rangle\|_{L_p(\Omega)}^p &= \sum_{j \in \mathbb{Z}_m^{r+1}} \|\langle \mathbf{c}, \mathbf{h}_{r+1} - \mathbf{h}_r \rangle\|_{L_p(\phi_j(\Omega), \mu)}^p \\ &= \sum_{j \in \mathbb{Z}_m^{r+1}} \|\langle \mathbf{c}, \mathbf{h}_{r+1} \circ \phi_j - \mathbf{h}_r \circ \phi_j \rangle\|_{L_p(\Omega, \mu_j)}^p \\ &= \sum_{j \in \mathbb{Z}_m^{r+1}} \left\| \left\langle \mathbf{c}, \mathbf{A}_{j_r}^T (\mathbf{A}_{j_r} - \mathbf{H})^T \mathbf{1} \right\rangle \right\|_{L_p(\Omega, \mu_j)}^p \\ &= \sum_{j \in \mathbb{Z}_m^{r+1}} \|\langle (\mathbf{A}_{j_r} - \mathbf{H}) \mathbf{A}_{j_r} \mathbf{c}, \mathbf{1} \rangle\|_{L_p(\Omega, \mu_j)}^p \\ &= \sum_{j \in \mathbb{Z}_m^{r+1}} n^{-p} |\langle (\mathbf{A}_{j_r} - \mathbf{H}) \mathbf{A}_{j_r} \mathbf{c}, \mathbf{1} \rangle|^p \mu_j(\Omega). \end{aligned}$$

Observe for any  $k \in \mathbb{Z}_m$  that  $(\mathbf{A}_k - \mathbf{H}) \mathbf{1} = \mathbf{0}$ . Thus, there exists a matrix  $\mathbf{C}_k \in \mathbb{R}^{n \times n-1}$  such that  $\mathbf{A}_k - \mathbf{H} = \mathbf{C}_k D$  and we conclude, for  $\mathbf{j} \in \mathbb{Z}_m^r$  and  $k \in \mathbb{Z}_m$ , that

$$(\mathbf{A}_k - \mathbf{H}) \mathbf{A}_j = \mathbf{C}_k D \mathbf{A}_j = \mathbf{C}_k \mathbf{B}_j D.$$

Consequently there exists a positive constant  $\alpha$  such that

$$\|\langle \mathbf{c}, \mathbf{h}_{r+1} - \mathbf{h}_r \rangle\|_p^p \leq \alpha \|\mathbf{c}\|_p^p \sum_{j \in \mathbb{Z}_m^r} \mu_j(\Omega) \|\mathbf{B}_j\|^p.$$

Thus, for any  $\sigma$  such that  $\rho_p(\mathcal{B}) < \sigma < 1$  there exists an  $r_0 \in \mathbb{Z}_\infty$  and a positive constant  $\beta$  such that

$$(4.10) \quad \|\mathbf{h}_{r+1} - \mathbf{h}_r\|_p \leq \beta \sigma^r, \quad r \geq r_0.$$

Therefore,  $\mathbf{h}_r$ ,  $r \in \mathbb{Z}_\infty$ , is a Cauchy sequence in  $L_p^n(\Omega)$  which necessarily converges to some function  $\mathbf{f} \in L_p^n(\Omega)$ .

It remains to verify that  $\mathbf{f}$  is indeed the limit of the subdivision scheme. For this purpose, we observe that, for  $r \in \mathbb{Z}_\infty$ ,

$$(4.11) \quad \left\| \mathbf{D}\mathbf{A}^{[r]} \right\|_p = \left\| \mathbf{B}^{[r]}\mathbf{D} \right\|_p \leq \|\mathbf{D}\| \left\| \mathbf{B}^{[r]} \right\|_p = \|\mathbf{D}\| \left( \sum_{\mathbf{j} \in \mathbb{Z}_m^r} \mu_{\mathbf{j}}(\Omega) \|\mathbf{B}_{\mathbf{j}}\|^p \right)^{1/p}.$$

Hence, we conclude that

$$(4.12) \quad \lim_{r \rightarrow \infty} \left\| \mathbf{D}\mathbf{A}^{[r]} \right\|_p = 0.$$

In other words, we have shown, for any  $\mathbf{y} \in \mathbb{R}^n$  such that  $\langle \mathbf{y}, \mathbf{1} \rangle = 0$ , that

$$(4.13) \quad \lim_{r \rightarrow \infty} \mathbf{y}^T \mathbf{A}^{[r]} = \mathbf{y}^T (\mathbf{1} \otimes \mathbf{f}).$$

Moreover, (4.9) yields for all  $r \in \mathbb{Z}_\infty$  that equation (4.13) holds with  $\mathbf{y}$  replaced by  $\mathbf{1}$  and therefore we have confirmed that  $\mathbf{f}$  is the limit of the subdivision scheme  $\mathcal{A}$ .

For the converse, we assume that  $\mathcal{A}$  admits a convergent subdivision scheme with limit  $\mathbf{f}$ . Then, for any  $\mathbf{j} \in \mathbb{Z}_m^r$  and  $k, \ell \in \mathbb{Z}_n$ , we have that

$$\begin{aligned} \mu_{\mathbf{j}}(\Omega) \left| (\mathbf{D}\mathbf{A}_{\mathbf{j}})_{k,\ell} \right|^p &= \left\| (\mathbf{A}_{\mathbf{j}})_{k+1,\ell} - (\mathbf{A}_{\mathbf{j}})_{k,\ell} \right\|_{L_p(\phi_{\mathbf{j}}(\Omega), \mu)}^p \\ &\leq 2^p \left( \left\| (\mathbf{A}_{\mathbf{j}})_{k+1,\ell} - f_{\ell} \right\|_{L_p(\phi_{\mathbf{j}}(\Omega))}^p + \left\| (\mathbf{A}_{\mathbf{j}})_{k,\ell} - f_{\ell} \right\|_{L_p(\phi_{\mathbf{j}}(\Omega))}^p \right) \\ &= 2^p \left( \left\| (\mathbf{A}^{[r]})_{k+1,\ell} - f_{\ell} \right\|_{L_p(\phi_{\mathbf{j}}(\Omega))}^p + \left\| (\mathbf{A}^{[r]})_{k,\ell} - f_{\ell} \right\|_{L_p(\phi_{\mathbf{j}}(\Omega))}^p \right). \end{aligned}$$

Therefore, there exists a positive constant  $\gamma$  such that

$$\mu_{\mathbf{j}}(\Omega) \|\mathbf{B}_{\mathbf{j}}\mathbf{D}\|^p \leq \gamma \left\| \mathbf{A}^{[r]} - \mathbf{1} \otimes \mathbf{f} \right\|_{L_p(\phi_{\mathbf{j}}(\Omega))}^p.$$

Consequently, summing both sides of this inequality over  $\mathbf{j}$  and taking into account the fact that  $\mathbf{D}$  is surjective, we conclude that there is a positive constant  $\eta$  such that

$$(4.14) \quad \sum_{\mathbf{j} \in \mathbb{Z}_m^r} \mu_{\mathbf{j}}(\Omega) \|\mathbf{B}_{\mathbf{j}}\mathbf{D}\|^p \leq \eta \left\| \mathbf{A}^{[r]} - \mathbf{1} \otimes \mathbf{f} \right\|_p^p$$

which proves the claim. □

We finally show that a convergent subdivision scheme converges exponentially fast and that its limit enjoys Hölder regularity.

**Theorem 4.8.** *If  $\mathcal{A}$  admits a convergent subdivision scheme with limit  $\mathbf{f}$ , then*

$$\lim_{r \rightarrow \infty} \omega_r^{1/r}(\mathbf{f}) \leq \rho_p(\mathcal{B}), \quad \mu\text{-a.e.}$$

*Proof.* For any  $\sigma$  with  $\rho_p(\mathcal{B}) < \sigma < 1$  there exist, by (4.10), numbers  $r_0 \in \mathbb{Z}_\infty$  and  $\alpha > 0$  such that, for  $r > r_0$ ,

$$(4.15) \quad \left\| \mathbf{f} - \mathbf{h}_r \right\|_p = \alpha \sigma^r.$$

Let  $\mathbf{P} := \mathbf{H} - \mathbf{I}$  and observe that  $\mathbf{P}\mathbf{1} = 0$ . Hence, there exists a matrix  $\mathbf{Q} \in \mathbb{R}^{n \times (n-1)}$  such that  $\mathbf{P} = \mathbf{Q}\mathbf{D}$ . Using (4.9) and (4.11) we find an  $r_1 \in \mathbb{Z}_\infty$  and  $\beta > 0$  such that, for  $r \geq r_1$ ,

$$(4.16) \quad \left\| \mathbf{1} \otimes \mathbf{h}_r - \mathbf{A}^{[r]} \right\|_p = \left\| \mathbf{P}\mathbf{A}^{[r]} \right\|_p = \left\| \mathbf{Q}\mathbf{D}\mathbf{A}^{[r]} \right\|_p \leq \beta \sigma^r(\mathcal{B}).$$

Thus, there is a constant  $\gamma > 0$  such that, for  $r \geq \max\{r_0, r_1\}$ ,

$$\left\| \mathbf{1} \otimes \mathbf{f} - \mathbf{A}^{[r]} \right\|_p \leq \left\| \mathbf{1} \otimes \mathbf{f} - \mathbf{1} \otimes \mathbf{h}_r \right\|_p + \left\| \mathbf{1} \otimes \mathbf{h}_r - \mathbf{A}^{[r]} \right\|_p \leq \gamma \sigma^r.$$

Hence, we have, for  $r \geq \max\{r_0, r_1\}$ , that

$$\omega_r(\mathbf{f})(y) = \left\| \mathbf{f}^T(y) \left( \mathbf{1} \otimes \mathbf{f} - \mathbf{A}^{[r]} \right) \right\|_p \leq \gamma \sum_{k \in \mathbb{Z}_n} |f_k(y)| \sigma^r.$$

Since  $\mathbf{f} \in L_p(\Omega)$ , it is finite  $\mu$ -a.e. and hence, for  $\mu$ -a.e.  $y \in \Omega$  we conclude that

$$\lim_{r \rightarrow \infty} \omega_r^{1/r}(\mathbf{f})(y) \leq \sigma.$$

However,  $\sigma$  was chosen arbitrarily greater than  $\rho_p(\mathcal{B})$ . This proves the result.  $\square$

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