

A NONCOMMUTATIVE MOMENT PROBLEM

DON HADWIN

(Communicated by David R. Larson)

Dedicated to the memory of my brother, Gary, whose cheerful spirit still fills my heart

ABSTRACT. We prove a noncommutative moment theorem and relate it to Connes' problem of embedding finite factor von Neumann algebras into an ultraproduct of the hyperfinite II_1 factor. We include a linear-algebraic equivalent of Connes' problem, which asks for a characterization of all noncommutative polynomials which have positive trace when the variables are replaced by contractive hermitian $n \times n$ matrices.

1. NONCOMMUTATIVE MOMENTS

The theory of moments has played a fundamental role in probability theory and many areas of analysis. More recently noncommutative versions of classical results have been obtained and applied to C^* -algebras, Lie algebras, Lie groups, algebraic quantum field theory, and other areas [1], [4], [5], [12], [13], [14]. In this paper we solve a special noncommutative moment problem and show how it relates to a problem of A. Connes [3] concerning the possibility of embedding every von Neumann algebra with a faithful trace into an ultrapower of the hyperfinite II_1 factor. We then present a tantalizing equivalent of Connes' embedding problem that involves characterizing noncommutative polynomials with the property that, whenever its variables are replaced with hermitian contractive matrices, the resulting matrix has nonnegative trace.

Classical moment problems characterize when a given sequence of numbers is the moment sequence of a measure on \mathbb{R} . The solution of the classical moment problems of Hamburger, Stieltjes, and Hausdorff is contained in the following *classical moment theorem*.

Theorem 1.1. *Suppose $\{\kappa_m\}_{m \geq 0}$ is a sequence of real numbers. Then*

- (1) *there is a positive Borel measure μ on \mathbb{R} such that*

$$\kappa_m = \int_{\mathbb{R}} t^m d\mu(t)$$

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for $m = 0, 1, 2, \dots$ if and only if, for all finite sequences $\{\alpha_m\}_{m=0}^N$ of complex numbers,

$$\sum_{i,j=1}^N \overline{\alpha_i} \alpha_j \kappa_{i+j} \geq 0;$$

(2) the measure μ in (1) is supported on $[0, \infty)$ if and only if, in addition to the condition in (1), we have

$$\sum_{i,j=1}^N \overline{\alpha_i} \alpha_j \kappa_{i+j+1} \geq 0;$$

(3) the measure μ in (1) has compact support if and only if, in addition to the condition in (1), we have

$$\liminf_{m \rightarrow \infty} |\kappa_{2m}|^{\frac{1}{2m}} < \infty;$$

(4) the measure μ in (1) is a probability measure with compact support contained in $[-1, 1]$ if and only if, in addition to the condition in (1), we have $\kappa_0 = 1$ and

$$\liminf_{m \rightarrow \infty} \kappa_{2m} < \infty.$$

Note that the fourth statement follows immediately from the first and the monotone convergence theorem, and it is clear that we actually have $|\kappa_m| \leq 1$ for $m \geq 0$ whenever the condition in (4) holds.

Since the monomials $\{1, t, t^2, \dots\}$ form a basis for the vector space $\mathbb{C}[t]$ of complex polynomials, there is a natural bijection between complex sequences $\{\kappa_m\}_{m \geq 0}$ and linear functionals ϕ on $\mathbb{C}[t]$ (with $\phi(t^m) = \kappa_m$ for $m \geq 0$). If we consider $\mathbb{C}[t] \subset C[-1, 1]$, then statement (4) in the preceding theorem tells when a linear functional on $\mathbb{C}[t]$ can be extended to a state (i.e., positive unital linear functional) on $C[-1, 1]$.

Corollary 1.2. *A complex linear functional ϕ on $\mathbb{C}[t]$ can be extended to a state on $C[-1, 1]$ if and only if*

- (1) $\phi(1) = 1$,
- (2) $\liminf_{m \rightarrow \infty} |\phi(t^{2m})| < \infty$, and
- (3) for every $p \in \mathbb{C}[t]$,

$$\phi(p^*p) \geq 0.$$

The preceding corollary is the statement that we want to generalize to the non-commutative setting. First note that $C[-1, 1]$ is the universal unital C^* -algebra generated by an element t subject to the relations $t = t^*$ and $-1 \leq t \leq 1$. Suppose n is a positive integer. Let \mathcal{A}_n be the universal unital C^* -algebra generated by t_1, \dots, t_n subject to the relations $t_j = t_j^*$ and $-1 \leq t_j \leq 1$ for $1 \leq j \leq n$. This means that if \mathcal{B} is any unital C^* -algebra with contractive hermitian elements $b_1, \dots, b_n \in \mathcal{B}$, then there is a unique unital $*$ -homomorphism $\pi: \mathcal{A}_n \rightarrow \mathcal{B}$ such that $\pi(t_j) = b_j$ for $1 \leq j \leq n$. In different terminology \mathcal{A}_n is the C^* -algebraic free product of n copies of $C[-1, 1]$. Let \mathbb{P}_n denote the vector space of all non-commutative polynomials in t_1, \dots, t_n , and let \mathbb{M}_n denote all of the monomials in t_1, \dots, t_n . When $n = 1$, letting $t = t_1$, we have $\mathbb{P}_n = \mathbb{C}[t]$, $\mathbb{M}_n = \{1, t, t^2, \dots\}$, and $\mathcal{A}_n = C[-1, 1]$. It is clear that \mathbb{M}_n is a linear basis for the vector space \mathbb{P}_n , so there

is a bijection between the functions on \mathbb{M}_n and the linear functionals on \mathbb{P}_n . We are going to restrict ourselves to *tracial* linear functionals ϕ on \mathbb{P}_n , i.e., functionals satisfying

$$\phi(pq) = \phi(qp)$$

for all $p, q \in \mathcal{P}_n$. The corresponding restriction on maps $\kappa: \mathbb{M}_n \rightarrow \mathbb{C}$ is $\kappa(ab) = \kappa(ba)$ for all $a, b \in \mathbb{M}_n$. Our noncommutative moment theorem can now be stated.

Theorem 1.3. *A tracial linear functional ϕ on \mathbb{P}_n can be extended to a (tracial) state on \mathcal{A}_n if and only if*

- (1) $\phi(1) = 1$,
- (2) $\liminf_{m \rightarrow \infty} |\phi(t_j^{2m})| < \infty$ for $1 \leq j \leq n$, and
- (3) for every $p \in \mathbb{P}_n[t]$,

$$\phi(p^*p) \geq 0.$$

Proof. The “only if” part is obvious. Suppose (1)–(3) are true. Since $\phi(a) = \frac{1}{2}[\phi((1+a)^2) - \phi(a^2)]$, it is clear from (3) that $\phi(a) \in \mathbb{R}$ whenever $a = a^*$, so $\phi(t_j^m) \in \mathbb{R}$ for $1 \leq j \leq n$, $0 \leq m < \infty$. It follows from the classical moment theorem above that $|\phi(t_j^m)| \leq 1$ for $1 \leq j \leq n$, $0 \leq m < \infty$. A simple induction argument yields a special case of a generalized Hölder’s inequality [10], namely, if w_1, \dots, w_m are selfadjoint elements in \mathbb{P}_n , then

$$|\phi(w_1 \cdots w_m)| \leq \prod_{k=1}^m \phi(w_k^{2^m})^{\frac{1}{2^m}}.$$

The case $m = 1$ follows from the Cauchy-Schwarz inequality and the fact that $\phi(1) = 1$. Exactly the same argument shows for positive integers m, v that $\phi(w_1^{2^m})^{\frac{1}{2^v}} \leq \phi(w_1^{2^{m+v}})^{\frac{1}{2^{v+1}}}$. For the induction argument let k be the greatest integer function of $(m+1)/2$, and let $r = m+1-k$. Then

$$\begin{aligned} |\phi(w_1 \cdots w_m)| &\leq \phi((w_1 \cdots w_k)(w_1 \cdots w_k)^*)^{\frac{1}{2}} \phi((w_{k+1} \cdots w_{k+r})(w_1 \cdots w_k)^*)^{\frac{1}{2}}. \end{aligned}$$

However, since ϕ is tracial,

$$\phi((w_1 \cdots w_k)(w_1 \cdots w_k)^*) = \phi(w_1^2 w_2 \cdots w_{k-1} w_k^2 w_{k-1} \cdots w_2),$$

and we can apply the induction assumption. A similar argument for the other factor yields the desired conclusion. It follows from this generalized Hölder’s inequality that

$$|\phi(a)| \leq 1$$

for every $a \in \mathbb{M}_n$.

As in the GNS construction define a semi-inner product (\cdot, \cdot) on \mathbb{P}_n by $(p, q) = \phi(q^*p) = \phi(pq^*)$ and, for $1 \leq j \leq n$, define a linear transformation T_j on \mathbb{P}_n by $T_j p = t_j p$. We want to show that each T_j induces a contractive linear mapping on the semi-inner product space, i.e.,

$$(T_j p, T_j p) = \phi(t_j^2 p p^*) \leq \phi(p p^*) = (p, p)$$

for every $p \in \mathbb{P}_n$. Clearly, we can assume that $\phi(p p^*) > 0$. Define a functional ψ on $\mathbb{C}[t_j]$ by $\psi(q) = \phi(q p p^*) / \phi(p p^*)$. Clearly, $\psi(1) = 1$ and $\psi(q^* q) = \phi((q p)^* q p) / \phi(p p^*) \geq 0$. Since $|\phi(a)| \leq 1$ for every $a \in \mathbb{M}_n$, it follows that

$\sup_{m \geq 0} |\phi(t_j^m p p^*)| < \infty$. Hence, by Corollary 1.2, we have $\psi(t_j^2) \leq 1$, which is the desired inequality $\phi(t_j^2 p p^*) \leq \phi(p p^*)$.

Hence each T_j defines a selfadjoint contraction operator on the Hilbert-space completion of our semi-inner product space \mathbb{P}_n . It follows from the definition of \mathcal{A}_n that there is a unital $*$ -homomorphism π on \mathcal{A}_n such that $\pi(t_j) = T_j$ for $1 \leq j \leq n$. Clearly, for each $p \in \mathbb{P}_n$,

$$\phi(p) = (\pi(p)[1], [1]),$$

which completes the proof. □

Remark 1.1. The arguments in the preceding proof can be used to show that any positive unital tracial linear functional on the algebraic free product of C^* -algebras extends to a tracial state on the C^* -algebraic free product completion. The key additional idea is that the restriction to each of the C^* -algebras in the product must be a state and that the algebraic free product is spanned by finite products of selfadjoint contractions from these algebras.

2. CONNES' ULTRAPOWER EMBEDDING PROBLEM

There is an open problem, raised by A. Connes in his pioneering paper [3], that asks whether every finite von Neumann algebra with a faithful trace can be embedded in an ultrapower of the hyperfinite II_1 factor. D. Voiculescu has discussed how this embedding problem is related to free entropy, and in [6] L. Ge and the author discuss how this problem is related to tracial approximation in matrix algebras. For basic facts about ultraproducts the reader can consult [6]. Suppose \mathcal{R} is a finite von Neumann algebra with faithful tracial state τ . The embeddability of \mathcal{R} in an ultrapower of the hyperfinite II_1 factor is equivalent to the following:

Given positive integers n and N , selfadjoint contractions $A_1, \dots, A_n \in \mathcal{R}$, and $\varepsilon > 0$, there are a positive integer k and selfadjoint contractive $k \times k$ matrices T_1, \dots, T_n such that for every monomial $m \in \mathbb{M}_n$ with degree at most N we have

$$|\tau(m(A_1, \dots, A_n)) - \tau_k(m(T_1, \dots, T_n))| < \varepsilon,$$

where τ_k denotes the normalized trace on the algebra \mathcal{M}_k of $k \times k$ matrices.

To translate the preceding condition into one about tracial states on \mathcal{A}_n , note that conditions that the A_j 's and T_j 's are selfadjoint contractions is the same as saying there are unital $*$ -homomorphisms $\pi: \mathcal{A}_n \rightarrow \mathcal{R}$ and $\rho: \mathcal{A}_n \rightarrow \mathcal{M}_k$ such that $\pi(t_j) = A_j$ and $\rho(t_j) = T_j$ for $1 \leq j \leq n$. Then $\phi = \tau \circ \pi$ and $\psi = \tau_k \circ \rho$ are tracial states on \mathcal{A}_n , and the above inequality says that, for every monomial $m \in \mathbb{M}$ with degree at most N , we have

$$|\phi(m(t_1, \dots, t_n)) - \psi(m(t_1, \dots, t_n))| < \varepsilon.$$

For the reverse direction, suppose ϕ is a tracial state on \mathcal{A}_n and the GNS construction for ϕ is represented as $\phi(x) = (\pi(x)e, e)$ where $\pi: \mathcal{A}_n \rightarrow B(H_\pi)$ is a unital representation and e is a unit vector in the Hilbert space H_π . Then $\{A \in \pi(\mathcal{A}_n)'' : \|Ae\|^2 = (A^* A e, e) = 0\}$ is a weak-operator closed two-sided ideal in $\pi(\mathcal{A}_n)''$ and thus equals $P\pi(\mathcal{A}_n)''$ for some central projection P in $\pi(\mathcal{A}_n)''$. Hence $\mathcal{R} = (1 - P)\pi(\mathcal{A}_n)''$ is a von Neumann algebra with a faithful tracial state τ defined by $\tau(A) = (Ae, e)$. This eliminates the dependence on \mathcal{R} .

The tracial states that can be represented in the form $\tau_k \circ \rho$ for some unital representation $\rho: \mathcal{A}_n \rightarrow \mathcal{M}_k$ are more special, but not up to arbitrarily close approximation. A state on a C^* -algebra is a *factor state* if the GNS representation

of the algebra generates a factor von Neumann algebra (i.e., one whose center is trivial). A state is called *finite-dimensional* if the GNS representation acts on a finite-dimensional Hilbert space. Let \mathbb{Q} denote the set of rational numbers. The \mathbb{Q} -convex hull of a set S in a vector space is the set of all convex combinations of elements in S whose coefficients are in \mathbb{Q} . The following lemma is probably well known, but we do not know a reference. The proof of (2) \Rightarrow (3) follows from Milman’s converse to the Krein-Milman theorem or, more simply, from a theorem of P. R. Halmos [8], which states that the irreducible pairs of hermitian operators is norm dense in the set of all hermitian pairs.

Lemma 2.1. *Suppose \mathcal{A} is a unital C^* -algebra.*

- (1) *The extreme point of the (weak*-compact convex) set of tracial states on \mathcal{A} is the set of factor tracial states on \mathcal{A} .*
- (2) *A finite-dimensional factor tracial state τ on \mathcal{A} has the form*

$$\tau = \tau_k \circ \rho$$

where $\rho: \mathcal{A} \rightarrow \mathcal{M}_k$ is an irreducible representation.

- (3) *The \mathbb{Q} -convex hull of the set of finite-dimensional factor tracial states on \mathcal{A} is the set of finite-dimensional states of the form*

$$\tau = \tau_k \circ \rho$$

where $\rho: \mathcal{A} \rightarrow \mathcal{M}_k$ is an arbitrary representation.

- (4) *The convex hull of the set of finite-dimensional factor tracial states on \mathcal{A} is the set of all finite-dimensional states in \mathcal{A} .*

We can now state our reformulation of Connes’ embedding problem. Note that statement (4) gives a truncated type of moment problem where the target states are those in the weak*-closure of the finite-dimensional tracial states.

Theorem 2.2. *The following are equivalent:*

- (1) *Every von Neuman algebra with faithful trace can be embedded in an ultrapower of the hyperfinite II_1 factor.*
- (2) *For each positive integer n , the weak* closure of the finite-dimensional tracial states on \mathcal{A}_n contains the set of all factor tracial states on \mathcal{A}_n .*
- (3) *For each positive integer n , the weak* closure of the set of finite-dimensional factor states on \mathcal{A}_n contains the set of all factor tracial states on \mathcal{A} .*
- (4) *For each positive integer n , each finite subset $F \subset \mathbb{M}_n$, each $\varepsilon > 0$, and each tracial state τ on \mathcal{A}_n there is a finite-dimensional tracial state on \mathcal{A}_n such that, for every $m \in F$,*

$$|\phi(m) - \tau(m)| < \varepsilon.$$

Using the Hahn-Banach separation theorem and the fact that \mathbb{P}_n is norm dense in \mathcal{A}_n , we obtain a tantalizing equivalent of Connes’ problem.

Corollary 2.3. *There is a von Neumann algebra with faithful trace that cannot be embedded in an ultrapower of the hyperfinite II_1 factor if and only if there is a positive integer n , a polynomial $p \in \mathbb{P}_n$, and an n -tuple (A_1, \dots, A_n) of selfadjoint contraction operators such that*

- (1) *$\tau_k(p(T_1, \dots, T_n)) \geq 0$ for every positive integer k and every n -tuple (T_1, \dots, T_n) of selfadjoint contractions in \mathcal{M}_k , and*

(2) $W^*(A_1, \dots, A_n)$ has a faithful tracial state τ and

$$\tau(p(A_1, \dots, A_n)) < 0.$$

Suppose $p \in \mathbb{P}_n$. What does it mean to say $\phi(p(A_1, \dots, A_n)) \geq 0$ for every n -tuple (A_1, \dots, A_n) of selfadjoint contractions in arbitrary von Neuman algebras with faithful tracial state ϕ ?

Lemma 2.4. *Suppose \mathcal{A} is a separable unital C^* -algebra and $a \in \mathcal{A}$. The following are equivalent:*

- (1) $\phi(a) \geq 0$ for every tracial state ϕ on \mathcal{A} ,
- (2) there is a positive integer r and bounded sequences $\{q_k\}, \{u_{j,k}\}, \{v_{j,k}\}$ ($1 \leq j \leq r$) in \mathcal{A} such that

$$q_k q_k^* + \sum_{j=1}^r (u_{j,k} v_{j,k} - v_{j,k} u_{j,k}) \rightarrow a$$

in the weak topology on \mathcal{A} .

Proof. We need only prove (1) \Rightarrow (2). Suppose (1) is true. The second dual $\mathcal{A}^{\#\#}$ of \mathcal{A} is a von Neumann algebra acting on a separable Hilbert space \mathcal{H} . Then we can write $\mathcal{A}^{\#\#} = \mathcal{B} \oplus \mathcal{D}$ where \mathcal{B} is a finite von Neumann algebra and \mathcal{D} is properly infinite. Let $Q = 1 \oplus 0$. On \mathcal{B} there is a faithful center-valued tracial conditional expectation τ , and since \mathcal{H} is separable, \mathcal{B} can be written as a direct integral of factors and τ can be written as a direct integral of tracial states. From (1) we have $\tau(Qa) \geq 0$. Moreover, $\tau(Qa - \tau(Qa)) = 0$. However, it follows from [2], [7], [9] that $Qa - \tau(Qa)$ and $(1 - Q)a$ are each a finite sum of commutators in $\mathcal{A}^{\#\#}$, and thus $a - \tau(Qa)$ is a finite sum of commutators in $\mathcal{A}^{\#\#}$, i.e.,

$$a - \tau(Qa) = \sum_{j=1}^r (U_j V_j - U_j V_j)$$

with $U_1, V_1, \dots, U_r, V_r \in \mathcal{A}^{\#\#}$. Hence

$$a = \tau(Qa) + \sum_{j=1}^r (U_j V_j - U_j V_j).$$

Using the Kaplansky density theorem there are bounded sequences $\{q_k\}, \{u_{j,k}\}, \{v_{j,k}\}$ ($1 \leq j \leq r$) in \mathcal{A} such that

$$q_k q_k^* + \sum_{j=1}^r (u_{j,k} v_{j,k} - v_{j,k} u_{j,k}) \rightarrow a$$

in the weak topology on \mathcal{A} . □

Thus Connes' embedding problem asks whether condition (1) in Corollary 2.3 implies that there are bounded sequences $\{q_k\}, \{u_{j,k}\}, \{v_{j,k}\}$ ($1 \leq j \leq r$) in \mathbb{P}_n such that

$$q_k q_k^* + \sum_{j=1}^r (u_{j,k} v_{j,k} - v_{j,k} u_{j,k}) \rightarrow p$$

in the weak topology on \mathcal{A}_n .

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REFERENCES

- [1] J. Alcantara-Bode and J. Yngvason, *Algebraic quantum field theory and noncommutative moment problems. I*, Ann. Inst. H. Poincaré Phys. Théor. **48** (1988), no. 2, 147–159. MR **90e**:46062
- [2] A. Brown and C. Pearcy, *Commutators in factors of type III*, Canad. J. Math. **18** (1966), 1152–1160. MR **34**:1864
- [3] A. Connes, *Classification of injective factors. Cases $II_1, II_\infty, III_\lambda$, $\lambda \neq 1$* , Ann. of Math. (2) **104** (1976), 73–115. MR **56**:12908
- [4] A. Yu Daletskiui, Yu. S. Samouilenko, *A noncommutative moment problem*, (Russian) Funktsional. Anal. i Prilozhen. **21** (1987), no. 2, 72–73.
- [5] Michel Dubois-Violette, *Topics on a noncommutative moment problem*, Proceedings of the International Conference on Operator Algebras, Ideals, and their Applications in Theoretical Physics (Leipzig, 1977), pp. 241–254, Teubner, Leipzig, 1978. MR **80g**:46056
- [6] L. Ge and D. Hadwin, *Ultraproducts of C^* -algebras*, preprint.
- [7] Th. Fack and P. de la Harpe, *Sommes de commutateurs dans les algèbres de von Neumann finies continues*, (French) Ann. Inst. Fourier (Grenoble) **30** (1980), no. 3, 49–73. MR **81m**:46085
- [8] P. R. Hamos, *Irreducible operators*, Michigan Math. J. **15** (1968), 215–223. MR **37**:6788
- [9] H. Halpern, *Commutators in properly infinite von Neumann algebras*, Trans. Amer. Math. Soc. **139** (1969), 55–73. MR **40**:4773
- [10] R. Kunze, *L_p -Fourier transform on locally compact unimodular groups*, Trans. Amer. Math. Soc. **89** (1958), 519–540. MR **20**:6668
- [11] C. Pearcy and D. Topping, *Commutators and Certain II_1 Factors*, J. Funct. Analysis **3** (1969), 69–78. MR **39**:789
- [12] R. Schmüdgen, *Commutative and noncommutative moment problems*, Operator Algebras and Operator Theory (Craiova, 1989), 184–197, Pitman Res. Notes Math. Ser. 271, Longman Sci. Tech., Harlow, 1992. MR **93i**:47016
- [13] K. Schmüdgen, *Noncommutative moment problems*, Math. Z. **206** (1991), no. 4, 623–649. MR **92f**:22030
- [14] J. Yngvason, *Algebraic quantum field theory and noncommutative moment problems. II*, Ann. Inst. H. Poincaré Phys. Thé. **48** (1988), no. 2, 161–173. MR **90e**:46063

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF NEW HAMPSHIRE, DURHAM, NEW HAMPSHIRE 03824

E-mail address: don@math.unh.edu

URL: <http://www.spicerack.sr.unh.edu/~don>