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FIXED POINTS FOR CONVEX CONTINUOUS MAPPINGS IN TOPOLOGICAL VECTOR SPACES

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Abstract. We prove the following result. Let C be a convex compact subset in a topological vector space, and $T : C \to C$ a convex continuous mapping. (See Definition 1.1.) Then T has a fixed point. Moreover, continuous mappings that can be approximated by convex continuous mappings also have the fixed point property.

1. Introduction

The following result is the well-known Schauder fixed point theorem:

Theorem ([25]). *Let C be a convex compact subset in a normed space, and $T : C \to C$ be a continuous mapping. Then T has a fixed point in C.*

This theorem has been generalized to locally convex spaces by many authors for various types of mappings. See, e.g., Tychonoff [27], Browder [1], Fan [5], Glicksberg [6], Himmelberg [10], Reich [19], [20]. The following is an outstanding problem. See [7].

Schauder's conjecture. *Let C be a convex compact subset in a topological vector space. Then a continuous mapping $T : C \to C$ has a fixed point.*

Many mathematicians have studied this problem and some progress has been made in topological vector spaces with some special structure. See Klee [14], Zima [29], Rzepecki [24], Hadzic [8], [9], Idzik [11], Nguyen [16], [17], Nguyen and Le [18]. However, this problem still remains unsolved. In this paper, by strengthening the continuity condition on the mapping T, we give a partial answer to Schauder's conjecture. To be more precise, we introduce the following concept:

Definition 1.1. Let X be a topological vector space. A mapping $T : X \to X$ is said to be convex continuous at $x_0 \in X$ if for any open neighborhood $N(Tx_0)$ of Tx_0, there exists an open neighborhood $V(x_0)$ of x_0 such that $Conv(TV(x_0)) \subset N(Tx_0)$, where $Conv(TV(x_0))$ represents the convex hull of $TV(x_0)$.

It is easy to see that the constant mapping is convex continuous. One can also show that a mapping with a finite dimensional range is convex continuous. If X is not locally convex, then the identity mapping on X is continuous but not convex

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continuous. Even though convex continuity is stronger, we show in section 3 that some continuous mappings can be approximated by convex continuous mappings. More precisely, we show that continuous mappings in Roberts spaces can be approximated by convex continuous mappings. We prove that Schauder's fixed point theorem is also true for a convex continuous approximatable mapping. When X is a locally convex space, it turns out that these two concepts are equivalent. We have the following:

Proposition 1.1. *Let X be a locally convex space. Then $T : X \to X$ is convex continuous if and only if T is continuous.*

The proof follows from the fact that every open neighborhood of 0 contains an open convex neighborhood of 0.

2. Schauder's fixed point theorem

In the following, we always assume that X is a Hausdorff topological vector space with property (W):

X has a local base $\{W_i\}_{i\in I}$ of 0, where I is an index set with a partial order "$<$", such that $W_i \subset W_j$ if $i < j$ and for any $i_1, i_2, \cdots, i_k$, there is an $l \in \{1, 2, \cdots, k\}$, such that $i_l = max\{i_1, i_2, \cdots, i_k\}$ and $W_{i_j} \subset W_{i_l}$ for $j = 1, 2, \cdots, k, j \neq l$.

For example, if X is first countable (equivalently, metrizable), then such a base exists.

Theorem 2.1. *Let X be a Hausdorff topological vector space with property (W), and $C \subset X$ a convex compact subset. Suppose $T : C \to C$ is a convex continuous mapping. Then T has a fixed point in C.*

Proof. Let $\mathcal{N} = \{W_i\}_{i\in I}$ be the local base of 0 such that property (W) holds. We may assume that $\{W_i\}_{i\in I}$ is a symmetric base. (Otherwise, put $W_i' = W_i \cap (-W_i)$.)

For any open neighborhood V of 0 and $x \in C$, by the convex continuity of T, there exists a $W_{i_x} \in \mathcal{N}$, such that $W_{i_x} \subset V$ and $Conv\{T(x + W_{i_x})\} \subset Tx + V$. By continuity of addition in X, we may choose $W'_{i_x} \in \mathcal{N}$, such that $W'_{i_x} + W'_{i_x} \subset W_{i_x}$.

Now, $\{x + W'_{i_x} : x \in C\}$ is an open covering of C, so it contains a finite open sub-covering of C. We denote it by $\bigcup_{j=1}^n (x_j + W'_{i_{x_j}})$.

Let $\{\psi_j(x)\}_{j=1}^n$ be a continuous partition of unity subordinated to the covering $\{x_j + W'_{i_{x_j}}\}_{j=1}^n$, $\sum_{j=1}^n \psi_j(x) = 1$, $\forall x \in C$.

We define a mapping $T_V : C \to C$ by

$$T_V x = \sum_{j=1}^n \psi_j(x) T x_j, \forall x \in C.$$

Then it follows from the Brouwer fixed point theorem that T_V has a fixed point $x_V \in C$. We claim that there exists $y_V \in C$, such that

$$x_V - y_V \in V \text{ and } x_V - Ty_V \in V. \tag{2.1}$$

Let $i_{x_{j_0}} = max\{i_{x_1}, i_{x_2}, \cdots, i_{x_n}\}$. We may assume that $\psi_j(x_V) \neq 0$, $j = 1, 2, \cdots, n$. (Otherwise, we exclude that term.)

Since $x_V \in x_j + W'_{i_{x_j}}$, $j = 1, 2, \cdots, n$, it follows that

$$\begin{aligned} x_j \in x_V + W'_{i_{x_j}} &\subset x_{j_0} + W'_{i_{x_{j_0}}} + W'_{i_{x_j}} \subset x_{j_0} + W'_{i_{x_{j_0}}} + W'_{i_{x_{j_0}}} \\ &\subset x_{j_0} + W_{i_{x_{j_0}}}, j \neq j_0. \end{aligned}$$

Hence we have

$$T_V x_V = \sum_{j=1}^{n} \psi_j(x) T x_j \in Conv\{T(x_{j_0} + W_{i_{x_{j_0}}})\} \subset T x_{j_0} + V.$$

Let $y_V = x_{j_0}$. Then $x_V - y_V \in V$, $x_V - T y_V \in V$, as desired.

By the compactness of C, we may assume that $\{x_V, V \in \mathcal{N}\}$ has a subnet $\{x_{V'}\}$ such that $x_{V'}$ converges to $x_0 \in C$.

By (2.1), we know that $T x_0 = x_0$, i.e. x_0 is a fixed point of T in C.

This completes the proof.

Theorem 2.1 gives a positive answer to the Schauder conjecture under the convex continuity condition. The following (almost trivial) result shows that the convex continuity is also necessary in some sense.

Theorem 2.2. *Let X be a Hausdorff topological vector space with property (W), and $C \subset X$ a convex compact subset, $T : C \to C$ a continuous mapping. Then T has a fixed point in C if and only if there exists a convex compact subset C' of C such that $T : C' \to C'$ and T is convex continuous on C'.*

Proof. The sufficiency follows from Theorem 2.1.

For the necessity, suppose that T has a fixed point $x_0 \in C$. Set $C' = \{x_0\}$. Then $T : C' \to C'$ is convex continuous.

3. Convex continuous approximatable mappings

In the following, we assume that X is a topological vector space unless specified otherwise. In section 2, we proved Schauder's fixed point theorem under the convex continuity condition. However, we know that the identity mapping is not convex continuous, and more generally, the continuous open mappings are not convex continuous, so a natural question is: How large is this class of mappings? It is difficult to answer this question completely, but we can give a partial answer to some extent. We shall approximate a continuous mapping by convex continuous mappings. To be precise, we introduce the following:

Definition 3.1. Let $T : D \subseteq X \to D$ be a continuous mapping, where D is a closed convex subset. If for any open neighborhood V of 0 there exist a closed convex subset $D' \subset D$ and a convex continuous mapping $T_V : D' \to D'$ such that $Tx - T_V x \in V$ for all $x \in D'$, then we say that T is a convex continuous approximatable mapping.

Proposition 3.1. *The identity mapping $I : D \to D$ of a closed convex subset D in X is convex continuous approximatable.*

Proof. For any open neighborhood V of 0, take a finite dimensional subspace F of X such that $F \cap D = D' \neq \varnothing$. Let $I' = I|_{D'}$. Then $I' : D' \to D'$ is convex continuous and satisfies $Ix - I'x = 0 \in V$ for all $x \in D'$. By Definition 3.1, I is convex continuous approximatable.

The following result shows that Schauder's fixed point theorem still holds for convex continuous approximatable mappings.

Theorem 3.1. *Let $C \subset X$ be a compact convex subset, and assume that X has property (W). (See section 2.) Suppose $T : C \to C$ is a convex continuous approximatable mapping. Then T has a fixed point in C.*

Proof. For each open neighborhood V of X, there exist a closed convex subset C' of C and a convex continuous mapping $T' : C' \to C'$ such that $Tx - T'x \in V$ for all $x \in C'$.

By Theorem 2.1, T' has a fixed point $x_V \in C'$. Hence we have

$$Tx_V - x_V \in V. \tag{3.1}$$

By the compactness of C, we know that $\{x_V : V \in \mathcal{N}\}$ has a subnet $\{x_{V'}\}$ such that $x_{V'}$ converges to x_0, where $\mathcal{N}$ is a local base of 0.

In view of (3.1), we know that $Tx_0 = x_0$. Therefore T has a fixed point in C.

This completes the proof.

In the following, we show that a continuous mapping in a so-called Roberts space is convex continuous approximatable. (See [13], [18], [21], [22].) First, we recall the definition of a Roberts space as follows:

Let X be a metrizable topological vector space. Then there exists a pseudo-norm $\|\cdot\|$ on X which is monotone, i.e. $\|sx\| \leq \|x\|$ if $s \leq 1$ (see [14], [23]), and induces an invariant metric on X. Let $A \subset X$ be a subset. We denote by: $A^+ = conv(A \cup \{0\}), A^* = conv(A^+ \cup (-A^+))$.

Following [21], a non-zero point x_0 of X is called a needle point if for every $\epsilon > 0$ there exists a finite set $A(x_0, \epsilon) = \{x_1, x_2, \cdots, x_n\}$ satisfying the following conditions:

1. $\|x_i\| < \epsilon$, $i = 1, 2, \cdots, n$;
2. for each $x \in A^+(x_0, \epsilon)$, there exists an $\alpha \in [0, 1]$ such that $\|x - \alpha x_0\| < \epsilon$;
3. $x_0 = \frac{1}{n}\sum_{i=1}^{n} x_i$.

X is called a needle point space if X is complete metrizable and every non-zero point in X is a needle point. See [13], [21] for examples.

Now assume that X is a needle point space. Following Roberts' construction, take $x_0 \neq 0$, and put $A_0 = \{x_0\}$. Define a sequence $\{A_n\}$ of finite subsets of X by induction as follows:

(4) $\|x\| < \epsilon_n$ for $x \in A_n$, where $\epsilon_n = (m_{n-1})^{-1}2^{-n}$, $m_n = cardA_n$, $n \geq 1$;
(5) if $a_n = \{x_1^n, x_2^n, \cdots, x_{m_n}^n\}$, then

$$A_{n+1} = \bigcup_{i=1}^{m_n} A(x_i^n, \epsilon_{n+1}), \text{ where } A(x_i^n, \epsilon_{n+1}) \text{ satisfies (1)-(3) for } i = 1, 2, \cdots, m_n.$$

Finally, we put $C = \overline{\bigcup_{i=1}^{\infty} A_n^*}$. Then C is a compact convex set with no extreme point. See [21]. Following [18], we call C a Roberts space.

Theorem 3.2. *Let $T : C \to C$ be a continuous mapping, where C is a Roberts space. Then T is a convex continuous approximatable mapping.*

Proof. Since X is metrizable, we only need to show that for each $\epsilon > 0$, there exists a closed convex subset $C_\epsilon \subset C$, and a convex continuous mapping $T' : C_\epsilon \to C_\epsilon$ such that $\|Tx - T'x\| < \epsilon$.

We use an auxiliary mapping defined in [18]. Take n sufficiently large such that $2^{-n+8} < \epsilon$. Then there exists a continuous mapping $g : A_n^* \to A_n^*$ such that $\|Tx - g(x)\| < 2^{-1}\epsilon$ for all $x \in A_n^*$, where g is defined in the proof of Theorem 3 in [18].

Let $C_\epsilon = A_n^*$, $T' = g$. By the construction of a Roberts space, A_n^* is closed convex and contained in a finite dimensional space, so T' is convex continuous.

The proof is complete.

In the following, we assume that X is metrizable, $\Omega \subset X$ is an open nonempty subset, $T : \overline{\Omega} \to X$ is a convex continuous mapping and $T\overline{\Omega}$ is relatively compact in X. We will try to approximate T by continuous mappings with finite dimensional range. Such an approach was used in [14], and also in [12] for the construction of the Leray-Schauder degree. See also [2].

Theorem 3.3. *Suppose $T\overline{\Omega} \cap \overline{\Omega} \neq \varnothing$. Then for each $V \in \mathcal{N}$, where $\mathcal{N} = \{W_i\}_{i \in I}$, I-countable, is a local base of 0 with property (W), there exist an open subset $\Omega_V \subseteq \Omega$, $\overline{T\overline{\Omega}} \cap \overline{\Omega} \subseteq \Omega_V$, a finite dimensional subspace F_V of X, and a continuous mapping $T_V : \Omega_V \to F_V$, such that for each $x \in \Omega_V$, there exists a $y^x \in \Omega_V$, such that*

$$x - y^x \in V, T_V x - T y^x \in V. \tag{3.2}$$

Proof. For each $V \in \mathcal{N}$, $x \in \overline{\Omega}$, by the convex continuity of T, there exists a $W_{i_x} \in \mathcal{N}$ such that $Conv\{T(x + W_{i_x})\} \subset Tx + V$. Let $W'_{i_x} \in \mathcal{N}$ be such that $W'_{i_x} + W'_{i_x} \subset W_{i_x}$.

Since $\{x + W'_{i_x}\}_{x \in \overline{\Omega}}$ is an open covering of $\overline{T\overline{\Omega}} \cap \overline{\Omega}$, it has a finite sub-covering $\{x_j + W'_{i_{x_j}} : j = 1, 2, \cdots, n\}$.

Now, put $\Omega_V = \bigcup_{j=1}^n (x_j + W'_{i_{x_j}})$. Let $\{\psi_j(x), j = 1, 2, \cdots, n\}$ be a continuous partition of unity subordinated to the covering $\{x_j + W'_{i_{x_j}}\}_{j=1}^n$, such that $\sum_{j=1}^n \psi_j(x) = 1$. (Remark that X is metrizable, so that such a partition of unity exists.)

We define the mapping $T_V : \overline{\Omega_V} \to F_V = span\{Tx_j, j = 1, 2, \cdots, n\}$ as follows:

$$T_V x = \sum_{j=1}^n \psi_j(x) T x_j, x \in \overline{\Omega_V}.$$

Following the proof of (2.1), we conclude that (3.2) is true.

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