

GLOBAL ANALYTIC AND GEVREY HYPOELLIPTICITY OF SUBLAPLACIANS UNDER DIOPHANTINE CONDITIONS

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ABSTRACT. In this paper we consider the problem of global Gevrey and analytic regularity for a class of partial differential operators on a torus in the form of a sum of squares of vector fields, which may not satisfy the bracket condition. We show that these operators are globally Gevrey or analytic hypoelliptic on the torus if and only if the coefficients satisfy certain Diophantine approximation properties.

1. INTRODUCTION AND RESULT

On a torus and more generally on an analytic manifold the characterization of the analytic operators, written as a sum of squares of vector fields, *sublaplacians*, which are globally analytic, or Gevrey hypoelliptic, is an open problem. The motivation of this work is to show that some of the operators considered in [CH] can be analytic or Gevrey hypoelliptic under a condition weaker than the bracket condition. For stating this condition and our result, we need the following definitions.

Let U be an open subset of \mathbb{R}^N , and $s \geq 1$. A distribution $u \in \mathcal{D}'(U)$ is said to belong to the Gevrey class G^s at $x_0 \in U$ if there exists a neighborhood $V \subset U$ of x_0 such that $u \in C^\infty(V)$, and a constant $C > 0$ such that

$$|\partial^\alpha u(x)| \leq C^{|\alpha|+1} |\alpha|^{s\alpha}, \quad \forall x \in V, \quad \forall \alpha \in \mathbb{N}^N.$$

If u is in G^s at every point $x_0 \in U$, then u is said to belong in $G^s(U)$. It can be shown (see for example Rodino [R]) that $u \in \mathcal{D}'(U)$ is in G^s at $x_0 \in U$ if, and only if, there is a distribution v with compact support agreeing with u near x_0 and such that its Fourier transform satisfies the estimate

$$|\hat{v}(\xi)| \leq C e^{-\varepsilon|\xi|^{\frac{1}{s}}}, \quad \forall \xi \in \mathbb{R}^N,$$

for some $\varepsilon > 0$ and $C > 0$.

Let \mathbb{T}^N denote the N -dimensional torus. Then a distribution $u \in \mathcal{D}'(\mathbb{T}^N)$ is in $G^s(\mathbb{T}^N)$ if and only if

$$|\hat{u}(\xi)| \leq C e^{-\varepsilon|\xi|^{\frac{1}{s}}}, \quad \forall \xi \in \mathbb{Z}^N,$$

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for some $\varepsilon > 0$ and $C > 0$. A linear partial differential operator P defined on \mathbb{T}^N is said to be globally G^s hypoelliptic on \mathbb{T}^N if for any $u \in \mathcal{D}'(\mathbb{T}^N)$ the condition $Pu \in G^s(\mathbb{T}^N)$ implies that $u \in G^s(\mathbb{T}^N)$. If P is defined on an open set U of \mathbb{R}^N , then P is said to be locally G^s hypoelliptic if for any V open subset of U and any $u \in \mathcal{D}'(V)$ the condition $Pu \in G^s(V)$ implies that u is in $G^s(V)$. Note that local G^s hypoellipticity implies global. Similar are the definitions for C^∞ hypoellipticity.

Also we shall need the following number theory definitions.

Definition 1.1. Let $s \geq 1$. A collection of vectors v_1, \dots, v_ℓ in \mathbb{R}^d are not simultaneously approximable with exponent s if for any $\varepsilon > 0$ there is $C_\varepsilon > 0$ such that for each $\eta = (\eta_1, \dots, \eta_\ell) \in \mathbb{Z}^\ell$ and each $\xi \in \mathbb{Z}^d - 0$ we have

$$|\eta_j - v_j \cdot \xi| \geq C_\varepsilon e^{-\varepsilon|\xi|^{\frac{1}{s}}}, \text{ for some } j = 1, \dots, \ell.$$

If $\ell = d = s = 1$, then this is the definition of a not exponentially Liouville number. It follows that v_1, \dots, v_ℓ are simultaneously approximable with exponent s if and only if there is $\varepsilon > 0$ for which there exist two sequences $\{\eta_k\} = \{(\eta_{k,1}, \dots, \eta_{k,\ell})\} \subset \mathbb{Z}^\ell$ and $\{\xi_k\} = \{(\xi_{k,1}, \dots, \xi_{k,d})\} \subset \mathbb{Z}^d$ such that

$$|\eta_{k,j} - v_j \cdot \xi_k| \leq e^{-\varepsilon|\xi_k|^{\frac{1}{s}}}, \forall j = 1, \dots, \ell.$$

Definition 1.2. Let $(b_1(t), \dots, b_\ell(t))$ be a vector of real-valued functions which are linearly independent over \mathbb{R} . A vector of functions $(f_1(t), \dots, f_d(t))$ is said to be not simultaneously approximable with exponent s for the basis $(b_1(t), \dots, b_\ell(t))$, and use the notation $(f_1, \dots, f_d) \in (SA)_s^c(b_1, \dots, b_\ell)$, if the following two conditions hold:

- (1) $\{f_1, \dots, f_d\}$ is contained in the linear span of $\{b_1, \dots, b_\ell\}$;
- (2) the ℓ columns of the matrix (λ_{jk}) in the expression

$$(f_1, \dots, f_d)^t = (\lambda_{jk})(b_1, \dots, b_\ell)^t$$

are not simultaneously approximable with exponent s .

Definitions similar to the two above have been used in [HP] for the study of C^∞ hypoellipticity. The simplest such condition was first used by Greenfield and Wallach in [GW], where they showed that $\partial_t - \alpha \partial_x$ is globally hypoelliptic on \mathbb{T}^2 if and only if α is not a Liouville number. Also, the notion of a vector satisfying a Diophantine condition was used by Herman [H] and Yoccoz [Y].

Next we state our result.

Theorem 1. In \mathbb{T}^{m+n} with variables $(t, x) = (t_1, \dots, t_m, x_1, \dots, x_m)$ let the operator

$$P = -\Delta_t - \left(a_1(t) \frac{\partial}{\partial x_1} + \dots + a_n(t) \frac{\partial}{\partial x_n} \right)^2,$$

where $a_j, j = 1, \dots, n$, are real-valued and real-analytic functions defined on \mathbb{T}^m . Then the operator P is globally G^s hypoelliptic on \mathbb{T}^{m+n} if and only if, after a possible renaming of the variables x_1, \dots, x_n and the corresponding coefficients a_1, \dots, a_n , the following Diophantine condition $(DC)_j$ holds for some $j \in \{0, 1, \dots, n - 1\}$.

$$(DC)_j \quad \begin{aligned} & a_1, \dots, a_{n-j} \text{ are independent over } \mathbb{R} \text{ and} \\ & (a_{n-j+1}, \dots, a_n) \in (SA)_s^c(a_1, \dots, a_{n-j}). \end{aligned}$$

For results on the interesting open problems of local and global, analytic and Gevrey hypoellipticity of sublaplacians we refer the reader to the following works as well as the references listed therein: Baouendi and Goulaouic [BG], Bove and Tartakoff [BT1], [BT2], Christ [C1], [C2], [C3], Hanges and Himonas [HH1], [HH2], [HH3], Helffer [H], Metivier [M1], [M2], Pham The Lai and Robert [PR], Sjöstrand [S], Tartakoff [T1], [T2], [T3], and Treves [Tr1], [Tr2]. While, for some results on hypoellipticity we refer to Hörmander [Ho], Kohn [K], Oleinik and Radkevic [OR], and Rothschild and Stein [RS].

2. PROOF OF THE THEOREM

We first prove the necessity of the condition in the theorem. Assume that the Diophantine condition $(DC)_j$ does not hold for any $j \in \{0, 1, \dots, n - 1\}$. Then, after a possible renaming of the variables x_1, \dots, x_n , we must have that either a_1 is identically equal to zero, or for some $j \in \{0, 1, \dots, n - 1\}$ the coefficients a_1, \dots, a_{n-j} are linearly independent, $\{a_{n-j+1}, \dots, a_n\} \subset \text{span} \{a_1, \dots, a_{n-j}\}$, and the $(n - j)$ column vectors of the matrix $(\lambda_{\ell k})$ in the expression

$$(a_{n-j+1}, \dots, a_n)^t = (\lambda_{\ell k})(a_1, \dots, a_{n-j})^t$$

are simultaneously approximable with exponent s . In the case that $a_1 = 0$, any function $u = u(x_1)$ not belonging to $G^s(\mathbb{T})$ is a solution to $Pu = 0$, and therefore P is not globally G^s hypoelliptic in \mathbb{T}^{m+n} . In the second case we have

$$a_\ell = \sum_{k=1}^{n-j} \lambda_{\ell,k} a_k, \quad \ell = n - j + 1, \dots, n,$$

where the vectors $\lambda_k = (\lambda_{n-j+1,k}, \dots, \lambda_{n,k}), k = 1, \dots, n - j$, are simultaneously approximable with exponent s . Then, the operator P takes the form

$$P = -\Delta_t - \left(\sum_{k=1}^{n-j} a_k(t) \left[\frac{\partial}{\partial x_k} + \sum_{\ell=n-j+1}^n \lambda_{\ell,k} \frac{\partial}{\partial x_\ell} \right] \right)^2.$$

Since $\lambda_1, \dots, \lambda_{n-j}$ are simultaneously approximable with exponent s there exist $\varepsilon > 0$, a sequence $\{\xi'_\nu\} = \{(\xi_{\nu,1}, \dots, \xi_{\nu,n-j})\} \subset \mathbb{Z}^{n-j}$, and another sequence $\{\xi''_\nu\} = \{(\xi''_{\nu,n-j+1}, \dots, \xi''_{\nu,n})\} \subset \mathbb{Z}^j - \{0\}$ such that $|\xi''_\nu| \rightarrow \infty$ as $\nu \rightarrow \infty$ and

$$(2.1) \quad |\xi'_{\nu,k} + \lambda_k \cdot \xi''_\nu| \leq e^{-\varepsilon |\xi''_\nu|^{\frac{1}{s}}}, \quad k = 1, \dots, n - j, \quad \text{and } \nu = 1, 2, \dots$$

Choosing

$$u(t, x) = \sum_{\nu=1}^{\infty} e^{i(\xi'_\nu \cdot x' - \xi''_\nu \cdot x'')},$$

where $x' = (x_1, \dots, x_{n-j})$ and $x'' = (x_{n-j+1}, \dots, x_n)$, we see that $u \in \mathcal{D}'(\mathbb{T}^{m+n}) - G^s(\mathbb{T}^{m+n})$, while

$$Pu = \sum_{\nu=1}^{\infty} \left[\sum_{k=1}^{n-j} a_k(t) (\xi'_{\nu,k} - \lambda_k \cdot \xi''_\nu) \right]^2 e^{i(\xi'_\nu \cdot x' - \xi''_\nu \cdot x'')}$$

is in $G^s(\mathbb{T}^{m+n})$ because of inequality (2.1).

To prove the sufficiency of the condition, let $u \in \mathcal{D}'(\mathbb{T}^{m+n})$ with

$$(2.2) \quad Pu = f, \quad f \in G^s(\mathbb{T}^{m+n}).$$

We will show that $u \in G^s(\mathbb{T}^{m+n})$ if condition $(DC)_j$ holds for some $j \in \{0, 1, \dots, n-1\}$. Taking partial Fourier transform with respect to x in (2.2) gives

$$(2.3) \quad \left(-\Delta_t + [a_1(t)\xi_1 + \dots + a_n(t)\xi_n]^2\right) \hat{u}(t, \xi) = \hat{f}(t, \xi).$$

For simplicity, we shall assume that we have only one t , i.e. $\Delta_t = \partial_t^2$. Multiplying (2.3) by \bar{u} and integrating by parts gives

$$(2.4) \quad \|\hat{u}(\cdot, \xi)\|_w^2 = \int_{\mathbb{T}} \hat{f}(t, \xi) \bar{u}(t, \xi) dt,$$

where for $\varphi \in C^\infty(\mathbb{T})$

$$\|\varphi\|_w^2 = \|\varphi_t\|_{L^2(\mathbb{T})}^2 + \int_{\mathbb{T}} w^2(t, \xi) |\varphi(t)|^2 dt$$

and w is given by

$$w(t, \xi) = a_1(t)\xi_1 + \dots + a_n(t)\xi_n.$$

We shall need the following:

Lemma 2. *If for some $j \in \{0, 1, 2, \dots, n-1\}$ the Diophantine condition $(DC)_j$ holds, then there exist constants $\alpha = \alpha(a_1, \dots, a_n) > 0$ and $\delta = \delta(a_1, \dots, a_n) > 0$ such that for any $\varepsilon > 0$ and any $\xi \in \mathbb{Z}^n - 0$ we can find an open interval $I_\xi \subset \mathbb{T}$ for which*

$$(2.5) \quad w^2(t, \xi) \geq \alpha C_\varepsilon e^{-\varepsilon|\xi|^{\frac{1}{s}}}, \quad \forall t \in I_\xi, |I_\xi| > \delta,$$

where C_ε is a positive constant depending only on ε .

Before proving this lemma we shall use it to complete the proof of the theorem. Let $\xi \in \mathbb{Z}^n - \{0\}$ be fixed. Then for $\varphi \in C^\infty(\mathbb{T})$, $s \in I_\xi$ and $t \in \mathbb{T}$, using the fundamental theorem of calculus we obtain

$$|\varphi(t)|^2 \lesssim |\varphi(s)|^2 + \int_{\mathbb{T}} |\varphi_t(t)|^2 dt.$$

Integrating first $t \in \mathbb{T}$ and then $s \in I_\xi$ gives

$$(2.6) \quad |I_\xi| \|\varphi\|_{L^2(\mathbb{T})}^2 \lesssim \int_{I_\xi} |\varphi(s)|^2 ds + |I_\xi| \|\varphi_t\|_{L^2(\mathbb{T})}^2.$$

Using (2.5), for any $\varepsilon > 0$ we have

$$\begin{aligned} \int_{I_\xi} |\varphi(s)|^2 ds &\lesssim \alpha^{-1} C_\varepsilon^{-1} e^{\varepsilon|\xi|^{\frac{1}{s}}} \int_{I_\xi} w^2(s, \xi) |\varphi(s)|^2 ds \\ &\lesssim \alpha^{-1} C_\varepsilon^{-1} e^{\varepsilon|\xi|^{\frac{1}{s}}} \int_{\mathbb{T}} w^2(t, \xi) |\varphi(t)|^2 dt. \end{aligned}$$

This together with (2.6) shows that for any $\varepsilon > 0$ there is a constant $C_\varepsilon > 0$ such that

$$(2.7) \quad \|\varphi\|_{L^2(\mathbb{T})}^2 \leq C_\alpha C_\varepsilon^{-1} e^{\varepsilon|\xi|^{\frac{1}{s}}} \|\varphi\|_w^2, \quad \forall \varphi \in C^\infty(\mathbb{T}), \quad \forall \xi \neq 0,$$

where C_α is a positive constant depending only on a_1, \dots, a_n .

Applying (2.7) with φ replaced by the C^∞ function $\hat{u}(\cdot, \xi)$ gives

$$\begin{aligned} \|\hat{u}(\cdot, \xi)\|_{L^2(\mathbb{T})}^2 &\leq C_\alpha C_\varepsilon^{-1} e^{\varepsilon|\xi|^{\frac{1}{s}}} \|\hat{u}(\cdot, \xi)\|_w^2 \\ &\leq C_\alpha C_\varepsilon^{-1} e^{\varepsilon|\xi|^{\frac{1}{s}}} \int_{\mathbb{T}} \hat{f}(t, \xi) \bar{u}(t, \xi) dt. \end{aligned}$$

Then applying Cauchy-Schwarz's inequality gives

$$(2.8) \quad \|\hat{u}(\cdot, \xi)\|_{L^2(\mathbb{T})} \leq C_\alpha C_\varepsilon^{-1} e^{\varepsilon|\xi|^{\frac{1}{s}}} \|\hat{f}(\cdot, \xi)\|_{L^2(\mathbb{T})}.$$

Since $f \in G^s(\mathbb{T}^{m+n})$ there is $\varepsilon_0 > 0$ such that

$$(2.9) \quad \|\hat{f}(\cdot, \xi)\| \leq c_0 e^{-\varepsilon_0|\xi|^{\frac{1}{s}}}.$$

Choosing in (2.8) $\varepsilon = \frac{\varepsilon_0}{2} = \varepsilon_1$ and using (2.9) we obtain

$$\|\hat{u}(\cdot, \xi)\|_{L^2(\mathbb{T})} \leq C_1 e^{-\varepsilon_1|\xi|^{\frac{1}{s}}}, \xi \neq 0.$$

This and the Cauchy-Schwarz inequality give

$$(2.10) \quad |\hat{u}(\tau, \xi)| \leq C_2 e^{-\varepsilon_1|\xi|^{\frac{1}{s}}}, \xi \neq 0.$$

Inequality (2.10) together with similar microlocal estimates near each elliptic point $(\tau, 0)$ give the estimate

$$|\hat{u}(\tau, \xi)| \leq C e^{-\varepsilon_2|(\tau, \xi)|^{\frac{1}{s}}}, \forall (\tau, \xi) \in \mathbb{Z}^{m+n},$$

which shows that $u \in G^s(\mathbb{T}^{m+n})$. The proof of the theorem will be completed once we prove Lemma 2.

Proof of Lemma 2. If for some $j \in \{0, 1, \dots, n-1\}$ the Diophantine condition $(DC)_j$ holds, then

$$w(t, \xi) = \sum_{k=1}^{n-j} a_k(t) (\xi_k + \lambda_k \cdot \xi''),$$

where a_1, \dots, a_n are independent functions over \mathbb{R} , and $\lambda_k = (\lambda_{n-j+1, k}, \dots, \lambda_{n, k})$, $k = 1, \dots, n-j$, are not simultaneously approximable. If $\xi'' = 0$, then the independence of a_1, \dots, a_{n-j} and the compactness of the unit sphere S^{n-j-1} give

$$w^2(t, \xi) = \left(\sum_{k=1}^{n-j} a_k(t) \xi_k \right)^2 = |\xi'|^2 w^2 \left(t, \frac{\xi'}{|\xi'|} \right) \geq |\xi'|^2 \alpha,$$

for all t in an open interval I_ξ with $|I_\xi| \geq \delta$ for some $\delta > 0$ and independent of ξ . If $\xi'' \neq 0$, then by letting $\gamma_k = \xi_k + \lambda_k \cdot \xi''$ we write $w^2(t, \xi)$ in the form

$$w^2(t, \xi) = \left(\sum_{k=1}^{n-j} a_k(t) \gamma_k \right)^2 = |\gamma|^2 w^2 \left(t, \frac{\gamma}{|\gamma|} \right),$$

where $\gamma = (\gamma_1, \dots, \gamma_{n-j})$. Again, using the independence of a_1, \dots, a_{n-j} and the compactness of the unit sphere S^{n-j-1} we obtain

$$(2.11) \quad w^2(t, \xi) \geq |\gamma|^2 \alpha, \forall t \in I_\gamma, |I_\gamma| \geq \delta.$$

Since the vectors $\lambda_1, \dots, \lambda_{n-j}$ are not simultaneously approximable with exponent s for any $\varepsilon > 0$ there is $C_\varepsilon > 0$ such that for some $k_0 \in \{1, \dots, n-j\}$ we have

$$(2.12) \quad |\gamma_{k_0}| \geq C_\varepsilon e^{-\varepsilon|\xi''|^{\frac{1}{s}}}.$$

Inequalities (2.11) and (2.12) give

$$w^2(t, \xi) \geq \alpha |\gamma_{k_0}|^2 \geq \alpha C_\varepsilon e^{-\varepsilon|\xi''|^{\frac{1}{s}}} \geq \alpha C_\varepsilon e^{-\varepsilon|\xi|^{\frac{1}{s}}},$$

which completes the proof of the lemma.

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