

SEPARATION FOR KERNELS OF HANKEL OPERATORS

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ABSTRACT. We prove that for two Hankel operators H_{a_1} and H_{a_2} on the Hardy space of the unit disk either the kernel of $H_{a_1}^* H_{a_2}$ equals the kernel of H_{a_2} or the kernel of $H_{a_2}^* H_{a_1}$ equals the kernel of H_{a_1} . In fact we prove a version of the above result for products of an arbitrary finite number of Hankel operators. Some immediate corollaries are generalizations of the result of Brown and Halmos on zero products of two Hankel operators and the result of Axler, Chang and Sarason on finite rank products of two Hankel operators. Simple examples show our results are sharp.

1. INTRODUCTION

Let L^2 be the space of Lebesgue square integrable functions on the unit circle and L^∞ be the space of essentially bounded functions on the unit circle. The Hardy space H^2 is the closed linear span of analytic polynomials in L^2 . Let P be the projection of L^2 onto H^2 . For $f \in L^\infty$, the Toeplitz operator $T_f: H^2 \rightarrow H^2$ with symbol f is defined by the rule $T_f h = P(fh)$. The Hankel operator $H_f: H^2 \rightarrow L^2 \ominus H^2$ with symbol f is defined by $H_f h = (I - P)(fh)$. Let $g \in L^\infty$. The Toeplitz operators and Hankel operators are connected by the following important relation:

$$T_{\bar{f}g} - T_{\bar{f}}T_g = H_f^*H_g.$$

The Hankel operator H_f depends only on $(I - P)f$. If $(I - P)f = \sum_{k=1}^{\infty} f_k \bar{z}^k$, then H_f has a matrix representation (called a Hankel matrix) $(f_{k+l})_{l \geq 1, k \geq 0}$ with respect to bases $\{z^k\}_{k=0}^{\infty}$ of H^2 and $\{\bar{z}^l\}_{l=1}^{\infty}$ of $L^2 \ominus H^2$.

The present work is motivated by a problem on finite rank products of Hankel operators. The classical Kronecker's result [4] on finite Hankel rank matrices is that H_f is of finite rank if and only if $(I - P)f$ is a rational function, and in this case the rank is equal to the degree of $(I - P)f$. Here we recall that if $r = p/q$ for polynomials p and q , the degree of r is the maximum of the degree of p and the degree of q . Axler, Chang and Sarason [1] proved, among other things, that

$$\text{rank}(H_f^*H_g) = \min\{\text{deg}((I - P)f), \text{deg}((I - P)g)\}.$$

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In other words $H_f^*H_g$ is of finite rank if and only if either H_f or H_g is of finite rank, and in this case the rank of $H_f^*H_g$ is equal to the minimum of the rank of H_f and the rank of H_g . The proof of the above result is elegant and uses Beurling's invariant subspace theorem. A purely algebraic proof was given by Richman [6]. He does not assume that H_f and H_g are bounded, so his result is slightly stronger. As long as $H_f^*H_g$ is a bounded operator, the same result holds. He showed that the analogue of his result for products of three Hankel matrices does not hold by displaying three Hankel matrices (not all are bounded) whose product is zero. He raised the question of whether or not the rank of the product of three *bounded* Hankel operators is equal to the minimum of the ranks of the individual ones.

Xia and Zheng [7] discussed zero or compact products of three Hankel operators. Let $\theta_i, i = 1, 2, 3$, be three inner functions. They proved that $H_{\theta_1}H_{\theta_2}^*H_{\theta_3}$ is zero if and only if $\theta_1\theta_3$ divides $\theta_2 - \lambda$ for some constant λ . For example, $H_{\bar{z}^2}H_{\bar{z}^3}H_{\bar{z}} = 0$. This answers negatively Richman's question.

Let $a_i \in L^\infty$ for $i = 1, \dots, n$ throughout the paper. In [3] we characterize when a product of four Hankel operators is of finite rank and use this characterization to prove that

$$\max_{\sigma} \{ \text{rank}(H_{a_{\sigma(1)}}^* H_{a_{\sigma(2)}} H_{a_{\sigma(3)}}^* H_{a_{\sigma(4)}}) \} = \min_{1 \leq i \leq 4} \text{rank}(H_{a_i})$$

where the maximum is taken over all permutations σ of $\{1, 2, 3, 4\}$. A similar rank formula holds for products of three Hankel operators.

In this paper we extend the above formula to products of an arbitrary finite number of Hankel operators by proving a separation principle for kernels of products of Hankel operators, which is interesting in itself. Let S_n be the permutation group of n integers $\{1, \dots, n\}$. For notational convenience, let

$$H(a_{\sigma(1)}, \dots, a_{\sigma(n)}) = H_{a_{\sigma(1)}}^* H_{a_{\sigma(2)}} \cdots H_{a_{\sigma(2k-1)}}^* H_{a_{\sigma(2k)}}$$

if $n = 2k$ is even and

$$H(a_{\sigma(1)}, \dots, a_{\sigma(n)}) = H_{a_{\sigma(1)}} H_{a_{\sigma(2)}}^* \cdots H_{a_{\sigma(2k-1)}} H_{a_{\sigma(2k)}}^* H_{a_{\sigma(2k+1)}}$$

if $n = 2k + 1$ is odd.

Theorem 1.1. *There exists a permutation $\sigma \in S_n$ such that the kernel of $H(a_{\sigma(1)}, \dots, a_{\sigma(n)})$ is equal to the kernel of $H_{a_{\sigma(n)}}$.*

For an operator A , A is of finite rank if and only if the kernel of A is of finite codimension, and in this case the rank of A is equal to the codimension of the kernel of A . Thus we have the following result.

Corollary 1.2. *Using the notation as above, we have*

$$\max_{\sigma \in S_{n-1}} \text{rank}[H(a_{\sigma(1)}, \dots, a_{\sigma(n-1)}, a_n)] = \min_{1 \leq i \leq n} \text{rank}(H_{a_i}).$$

Corollary 1.3. *If $H(a_{\sigma(1)}, \dots, a_{\sigma(n-1)}, a_n) = 0$ for all permutations $\sigma \in S_{n-1}$, then one of H_{a_i} is zero.*

For $n = 2$ in Corollary 1.3, this is the result of Brown and Halmos [2]. For $n = 3$ in Corollary 1.3, this result is slightly stronger than Theorem 3.3 in the paper [7] by Xia and Zheng.

2. THE MAIN RESULT

For an operator A , $Ker(A)$, $Range(A)$ and $rank(A)$ denote the kernel, range and rank of A , respectively ($rank(A) = \infty$ if A is not a finite rank operator). For an inner function $\theta \in H^2$, let $\mathcal{H}(\theta) = H^2 \ominus \theta H^2$.

The following lemma is well known (see [5] for example).

Lemma 2.1. *$Ker(H_f) \neq \{0\}$ if and only if f is of the form $\overline{\theta b}$ where θ is some inner function and $b \in H^\infty$ has the property that the inner part of b and θ are coprime. Furthermore we have:*

1. $Ker(H_{\overline{\theta b}}) = \theta H^2, \quad Ker(H_{\overline{\theta b}}^*) = \overline{z\theta H^2},$
2. $\text{closure}\{Range(H_{\overline{\theta b}}^*)\} = \{Ker(H_{\overline{\theta b}})\}^\perp = H^2 \ominus \theta H^2 = \mathcal{H}(\theta),$
3. $\text{closure}\{Range(H_{\overline{\theta b}})\} = \overline{z\mathcal{H}(\theta)}.$

Lemma 2.2. *Let $\sigma \in S_n$ be the permutation such that $\sigma(1) = n$ and $\sigma(i) = i - 1$ for $2 \leq i \leq n$. Let C be any bounded linear operator with range in H^2 . There exists an i ($1 \leq i \leq n$) such that*

$$(2.1) \quad Ker(H_{a_{\sigma(i)}}^* H_{a_i} C) = Ker(H_{a_i} C).$$

Similarly, there exists an i ($1 \leq i \leq n$) such that

$$Ker(H_{a_{\sigma(i)}} H_{a_i}^* D) = Ker(H_{a_i}^* D),$$

where D is any bounded linear operator with range in $L^2 \ominus H^2$.

Proof. It is clear that if $Ker(H_{a_i}^*) = 0$ for some i , we are done. Therefore by Lemma 2.1 we can assume that for $i = 1, \dots, n$, $a_i = \overline{\theta_i b_i}$, where θ_i is inner and $b_i \in H^\infty$ has the property that the inner part of b_i and θ_i are coprime. Let C be a bounded linear operator from a complex Hilbert space K into H^2 . We prove the lemma by contradiction. If (2.1) does not hold, let p_i be non-zero such that $p_i = C d_i$ for some $d_i \in K$ and

$$p_i \in Ker(H_{a_{\sigma(i)}}^* H_{a_i}) \ominus \theta_i H^2, \quad i = 1, \dots, n.$$

Thus $H_{a_i} p_i \in Ker(H_{a_{\sigma(i)}}^*)$. By Lemma 2.1

$$\overline{\theta_i b_i p_i} = \overline{z\theta_{\sigma(i)} h_i} + k_i, \quad i = 1, \dots, n,$$

for some analytic functions $h_i, k_i \in H^2$. Rewrite the above equations as

$$b_i p_i - \theta_i k_i = \theta_i \overline{\theta_{\sigma(i)} z h_i}$$

for $i = 2, \dots, n$. Taking products of both sides of above n equations gives

$$\begin{aligned} \prod_{i=1}^n (b_i p_i - \theta_i k_i) &= \prod_{i=1}^n (\theta_i \overline{\theta_{\sigma(i)} z h_i}) \\ &= \left[\prod_{i=1}^n (\theta_i \overline{\theta_{\sigma(i)})} \right] \left[\overline{z^n \prod_{i=1}^n h_i} \right]. \end{aligned}$$

Note that $\prod_{i=1}^n (\theta_i \overline{\theta_{\sigma(i)})} = 1$. We have

$$\prod_{i=1}^n (b_i p_i - \theta_i k_i) = \overline{z^n \prod_{i=1}^n h_i}.$$

Note that the left side of the above equation belongs to zH^2 and the right side of the above equation belongs to $\overline{zH^2}$. Therefore

$$\prod_{i=1}^n (b_i p_i - \theta_i k_i) = 0.$$

That is, $b_i p_i - \theta_i k_i = 0$ for some $i \geq 1$. Since the inner part of b_i and θ_i are coprime, we have $p_i \in \theta_i H^2$ which is a contradiction. This completes the proof. \square

Let

$$J(a_1, \dots, a_n) = H_{a_{\sigma(1)}} H_{a_{\sigma(2)}}^* \cdots H_{a_{\sigma(2k-1)}} H_{a_{\sigma(2k)}}^*$$

if $n = 2k$ is even and

$$J(a_1, \dots, a_n) = H_{a_{\sigma(1)}}^* H_{a_{\sigma(2)}} \cdots H_{a_{\sigma(2k)}} H_{a_{\sigma(2k+1)}}^*$$

if $n = 2k + 1$ is odd.

The following is the proof of Theorem 1.1.

Proof. We will prove a slightly stronger version of Theorem 1.1. Let C be any bounded linear operator with range in H^2 and D be any bounded linear operator with range in $L^2 \ominus H^2$. We will show that there exists a permutation $\sigma \in S_n$ such that the kernel of $H(a_{\sigma(1)}, \dots, a_{\sigma(n)})C$ is equal to the kernel of $H_{a_{\sigma(n)}}C$, and similarly there exists a permutation $\sigma \in S_n$ such that the kernel of $J(a_{\sigma(1)}, \dots, a_{\sigma(n)})D$ is equal to the kernel of $H_{a_{\sigma(n)}}^*D$. We prove the result by induction on n . By Lemma 2.2, it is true for $n = 2$. Assume it is true for $n = 2k$. For each $i = 1, \dots, 2k$, let σ_i be a permutation of $2k$ integers $\{1, 2, \dots, i - 1, i + 1, \dots, 2k + 1\}$, such that

$$\begin{aligned} & Ker[J(a_{\sigma_i(1)}, a_{\sigma_i(2)}, \dots, a_{\sigma_i(i-1)}, a_{\sigma_i(i+1)}, \dots, a_{\sigma_i(2k)}, a_{\sigma_i(2k+1)})H_{a_i}C] \\ &= Ker[H_{a_{\sigma_i(2k+1)}}^* H_{a_i}C], \end{aligned}$$

and similarly let σ_{2k+1} be a permutation of $2k$ integers $\{1, 2, \dots, 2k\}$ such that

$$\begin{aligned} & Ker[J(a_{\sigma_{2k+1}(1)}, a_{\sigma_{2k+1}(2)}, \dots, a_{\sigma_{2k+1}(2k)})H_{a_{2k+1}}C] \\ &= Ker[H_{a_{\sigma_{2k+1}(2k)}}^* H_{a_{2k+1}}C]. \end{aligned}$$

Let $\delta_i = \sigma_i(2k + 1)$ for $i = 1, \dots, 2k$ and $\delta_{2k+1} = \sigma_{2k+1}(2k)$. We claim that there exists an $i, 1 \leq i \leq 2k + 1$, such that

$$Ker[H_{a_{\delta_i}}^* H_{a_i}C] = Ker[H_{a_i}C].$$

We first show that there exists a subset $S = \{\alpha_i, i = 1, 2, \dots, m\}$ of $\{1, \dots, 2k + 1\}$ such that the set of all pairs of integers of the form $\{(\delta_i, i), i = 1, \dots, 2k + 1\}$ contains a subset of pairs of integers of the form $\{(\alpha_i, \beta_i) : \alpha_i, \beta_i \in S, i = 1, \dots, m\}$ and $\alpha_i \neq \alpha_j, \beta_i \neq \beta_j$ for $i \neq j$.

Let S_1 be the set of all distinct members of $\{\delta_i, i = 1, \dots, 2k + 1\}$. Since $\delta_i \neq i, S_1$ contains at least two integers. Without loss of generality, let $S_1 = \{\delta_i, i = 1, \dots, m_1\}$. It is easy to see that the set of all pairs of integers of the form $\{(\delta_i, i), i = 1, \dots, 2k + 1\}$ contains a subset of pairs of integers of the form $\{(\alpha_i, \delta_i) : \alpha_i \in S_1, i = 1, \dots, m_1\}$. If $\alpha_i \neq \alpha_j$ for all $i \neq j$, we are done. Otherwise let S_2 be the set of all distinct members of $\{\alpha_i, i = 1, \dots, m_1\}$ and repeat the argument as above. It is easy to see that continuing this process will yield the desired set S .

Let β denote the permutation of m integers in S which maps α_i to β_i for $i = 1, \dots, m$. Without loss of generality, assume β is a cycle (otherwise take any cycle of β). That is, $\beta(\alpha_i) = \beta_i = \alpha_{i+1}$ for $i = 1, \dots, m-1$ and $\beta(\alpha_m) = \beta_m = \alpha_1$. Now again by Lemma 2.2, there exists an i such that

$$\text{Ker}[H_{a_{\alpha_i}}^* H_{a_{\beta(\alpha_i)}} C] = \text{Ker}[H_{a_{\beta(\alpha_i)}} C].$$

This proves that there exists a permutation $\sigma \in S_{2k+1}$ such that the kernel of $H(a_{\sigma(1)}, \dots, a_{\sigma(2k+1)})C$ is equal to the kernel of $H_{a_{\sigma(2k+1)}} C$. Similarly we can show that there exists a permutation $\sigma \in S_{2k+1}$ such that the kernel of

$$J(a_{\sigma(1)}, \dots, a_{\sigma(2k+1)})D$$

is equal to the kernel of $H_{a_{\sigma(2k+1)}}^* D$. This completes the proof. \square

Next we give the proof of Corollary 1.2.

Proof. First we prove the corollary for the case $n = 2$. By Theorem 1.1 with $n = 2$, either the kernel of $H_{a_1}^* H_{a_2}$ is equal to the kernel of H_{a_2} or the kernel of $H_{a_2}^* H_{a_1}$ is equal to the kernel of H_{a_1} . By the fact that an operator is of finite rank if and only if its kernel is of finite codimension, and in this case the rank is equal to the codimension of its kernel, we have that either the rank of $H_{a_1}^* H_{a_2}$ is equal to the rank of H_{a_2} , or the rank of $H_{a_2}^* H_{a_1}$ is equal to the rank of H_{a_1} . But $H_{a_1}^* H_{a_2}$ is the adjoint of $H_{a_2}^* H_{a_1}$. Thus the rank of $H_{a_1}^* H_{a_2}$ is equal to the minimum of the rank of H_{a_1} and the rank of H_{a_2} .

For the general case, by the proof of Theorem 1.1, there exists a permutation $\sigma \in S_{n-1}$ such that the rank of $H(a_{\sigma(1)}, \dots, a_{\sigma(n-1)}, a_n)$ is equal to the rank of $H_{a_{\sigma(n-1)}}^* H_{a_n}$. By the previous analysis, the rank of $H_{a_{\sigma(n-1)}}^* H_{a_n}$ is equal to the minimum of the rank of $H_{a_{\sigma(n-1)}}$ and the rank of H_{a_n} . This proves that the left side of the rank formula in Corollary 1.2 is larger than or equal to the right side. It is obvious that the left side is less than or equal to the right side. This completes the proof. \square

3. EXAMPLES

The following simple examples show that our results are the best possible.

Example 3.1. Let $\{k_i\}_{i \geq 1}$ be the Fibonacci sequence, i.e., $k_1 = 1, k_2 = 2$ and $k_i = k_{i-1} + k_{i-2}$ for $i \geq 3$. Let $a_i = \bar{z}^{k_i}$ for $i = 1, \dots, n$. Note that $H_{\bar{z}^i} H_{\bar{z}^k}^* H_{\bar{z}^j} = 0$ for all i, j and k such that $i + j \leq k$. Therefore $H(a_{\sigma(n)}, a_{\sigma(n-1)}, \dots, a_{\sigma(2)}, a_1)$ is zero for all permutations σ of $\{n, n-1, \dots, 2\}$ except for the identity permutation (i.e., $\sigma(i) = i$ for $i = n, n-1, \dots, 2$). This shows that the results in Corollary 1.2 and Corollary 1.3 are sharp. That is, if we replace the set S_{n-1} by any proper subset of S_{n-1} , both Corollary 1.2 and Corollary 1.3 are false.

Example 3.2. Notation is the same as in Example 3.1. It is clear that the kernel of $H(a_n, a_{n-1}, \dots, a_1)$ is equal to the kernel of H_{a_1} . We claim that the kernel of $H(a_{\sigma(n)}, a_{\sigma(n-1)}, \dots, a_{\sigma(1)})$ is not equal to the kernel of $H_{a_{\sigma(1)}}$ for all non-trivial permutations of $\{n, n-1, \dots, 2, 1\}$. This shows that the result in Theorem 1.1 is also sharp. To see this, note that $H(a_{\sigma(n)}, a_{\sigma(n-1)}, \dots, a_{\sigma(1)})$ is zero unless $\sigma(n) = n$ or $\sigma(1) = n$.

If $\sigma(1) = n$, it is easy to see that the kernel of $H(a_{\sigma(n)}, a_{\sigma(n-1)}, \dots, a_{\sigma(1)})$ is not equal to the kernel of H_{a_n} , since

$$H(a_{\sigma(n)}, a_{\sigma(n-1)}, \dots, a_{\sigma(2)}, a_n) 1 = 0, \quad H_{a_n} 1 = \bar{z}^{k_n} \neq 0.$$

Therefore $\sigma(n) = n$. Similar arguments show that in order for the kernel of $H(a_{\sigma(n)}, a_{\sigma(n-1)}, \dots, a_{\sigma(1)})$ to equal the kernel of $H_{a_{\sigma(1)}}$, we must have $\sigma(i) = i$ for $i = n, n-1, \dots, 2, 1$.

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