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# ON THE NUMBER OF MAXIMAL SUM-FREE SETS

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ABSTRACT. It is shown that the set $\{1, 2, \ldots, n\}$ contains at most $2^{n/2-2^{-28}n}$ maximal sum-free subsets, provided $n$ is large enough.

A set $A \subseteq [n] = \{1, 2, \ldots, n\}$ is sum-free if for any two elements $a, b \in A$ we have $a + b \notin A$. A sum-free set $A \subseteq [n]$ is maximal if it is not contained in any other sum-free subset of $[n]$. Let $s(n)$ and $s_{\max}(n)$ denote the number of sum-free and maximal sum-free subsets of $[n]$, respectively. Since the set of odd numbers is sum-free, and so is each of its subsets, $s(n) \geq 2^{\lceil n/2 \rceil}$. It is conjectured that, in fact, we have $s(n) \leq c2^{n/2}$ for some constant $c > 0$ but at this moment we know only (see Calkin [2] and Alon [1]) that the value of the exponent is close to $n/2$, i.e. the following holds.

**Theorem 1.** $s(n) = 2^{n/2+o(n)}$. □

In this note we study the behaviour of $s_{\max}(n)$. Cameron and Erdős [4] observed that $s_{\max}(n) \geq 2^{\lfloor n/4 \rfloor}$ and asked if $s_{\max}(n) = o(s(n))$, or, maybe, even $s_{\max}(n) \leq 2^{n/2-\epsilon n}$, holds for some constant $\epsilon > 0$. Our main result states that this is indeed the case.

**Theorem 2.** *There exists* $n_0$ *such that for* $n \geq n_0$ *we have* $s_{\max}(n) \leq 2^{n/2-2^{-28}n}$.

For a maximal sum-free set $A \subseteq [n]$ and $B \subseteq A$ let

$$h_A(B) = \big|[n] \setminus \big[(A \setminus B) \cup \big((A \setminus B) + (A \setminus B)\big) \cup \big((A \setminus B) - (A \setminus B)\big) \cup (A \setminus B)/2\big]\big|,$$

i.e. $h_A(B)$ denotes the number of elements $i \in [n]$ one can add to $A \setminus B$ so that $\{i\} \cup (A \setminus B)$ remains sum-free. Our argument relies on the following result of probabilistic flavor.

**Lemma 3.** *Let* $\beta$ *be a constant such that* $0 < \beta < 1/2$. *Then there exists* $n_0$ *such that for* $n \geq n_0$, *every maximal sum-free set* $A \subseteq [n]$ *with* $|A| = m \geq n/11$ *contains at least* $n^{-5\sqrt{n}}\binom{m}{k}$ *subsets* $B$ *of* $k = \lfloor \beta m \rfloor$ *elements for which* $h_A(B) \leq 2\beta n$.

*Proof.* For each $i \in [n] \setminus A$ let us choose a pair $R_i = \{a_i', a_i''\}$ of two, not necessarily distinct, elements of $A$ such that either $i = a_i' + a_i''$, or $i = a_i' - a_i''$, or, maybe, $2i = a_i' = a_i''$ (since $A$ is a maximal sum-free set such a pair always exists). Let

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$A'$ be a set of all $a \in A$ which belong to at most $\sqrt{n}$ from the sets $R_i$, $i \in [n] \setminus A$. Clearly, $m' = |A'| \geq |A| - 2\sqrt{n}$. Let

$$W = \{i : \ R_i \cap A' \neq \emptyset\},$$

and denote by $\mathbf{B}$ a set chosen uniformly at random among all subsets of $A'$ with $k$ elements. We shall study the behaviour of the random variable $\mathbf{X} = \sum_{i\in W} \mathbf{X}_i$, where for $i \in W$

$$\mathbf{X}_i = \begin{cases} 0 & \text{if } R_i \cap \mathbf{B} = \emptyset, \\ 1 & \text{if } R_i \cap \mathbf{B} \neq \emptyset. \end{cases}$$

For the expectation of $\mathbf{X}$, we get

$$\begin{aligned} \mathrm{E}\,\mathbf{X} = \sum_{i\in W} \mathrm{E}\,\mathbf{X}_i &\leq |W| \left(1 - \min\left\{ \frac{\binom{m'-2}{k}}{\binom{m'}{k}}, \frac{\binom{m'-1}{k}}{\binom{m'}{k}} \right\}\right) \\ &\leq (1+o(1))(n-m)\big(1 - \big(1-(k/m')^2\big)\big) \leq 2\beta n - 2\beta m. \end{aligned}$$

In order to estimate the variance $\mathrm{Var}\,\mathbf{X}$ note that

$$\mathrm{Var}\,\mathbf{X} \leq \sum_{i,j:\ R_i\cap R_j \neq \emptyset} |\operatorname{Cov}(\mathbf{X}_i, \mathbf{X}_j)| + \sum_{i,j:\ R_i \cap R_j = \emptyset} |\operatorname{Cov}(\mathbf{X}_i, \mathbf{X}_j)|.$$

As a crude upper bound for the first sum one can take

$$\sum_{i,j:\ R_i\cap R_j\neq\emptyset} |\operatorname{Cov}(\mathbf{X}_i,\mathbf{X}_j)| \leq \sum_{i,j:\ R_i\cap R_j\neq\emptyset} 1 \ \leq \ 2n\sqrt{n},$$

while the second sum is not larger than

$$\Big(\sum_{i\in W} \mathrm{E}\,\mathbf{X}_i\Big) m' \left[ \left| \frac{\binom{m'-2}{k}}{\binom{m'}{k}} - \frac{\binom{m'-4}{k-2}}{\binom{m'-2}{k-2}} \right| + \left| \frac{\binom{m'-1}{k}}{\binom{m'}{k}} - \frac{\binom{m'-3}{k-1}}{\binom{m'-1}{k-1}} \right| \right] \leq \mathrm{E}\,\mathbf{X}\, m' \,\frac{11}{m'} \leq 11\,\mathrm{E}\,\mathbf{X}.$$

Hence

$$\mathrm{Var}\,\mathbf{X} \leq 11\,\mathrm{E}\,\mathbf{X} + 2n\sqrt{n} \leq 3n\sqrt{n} = o\big(n^2\big),$$

and so Chebyshev's inequality implies that, for $n$ large enough, with probability at least $1/2$ we have $\mathbf{X} \leq 2\beta n - \beta m$. Thus, since $h_A(\mathbf{B}) \leq \mathbf{X} + |\mathbf{B}|$, there exist at least

$$\frac{1}{2}\binom{m'}{k} \geq n^{-5\sqrt{n}}\binom{m}{k}$$

subsets $B$ of $A$ with $k$ elements for which $h_A(B) \leq 2\beta n$. □

*Proof of Theorem 2.* Let us first estimate the number $s'_{\max}(n)$ of all maximal sum-free subsets of $[n]$ with at least $n/11$ elements. Set $\beta = 2^{-23}$ and consider the number $t$ of all pairs $(A, A\setminus B)$, where $A$ is a maximal sum-free set of at least $n/11$ elements and $B \subseteq A$ is such that $|B| = \lfloor \beta|A| \rfloor$ and $h_A(B) \leq 2\beta n$. Then, due to Lemma 3, for $n$ large enough we have

$$t \geq s'_{\max}(n) n^{-5\sqrt{n}} \binom{|A|}{|B|} \geq s'_{\max}(n) n^{-5\sqrt{n}} \binom{\lfloor n/11 \rfloor}{\lfloor \beta n/11 \rfloor} \geq s'_{\max}(n)\beta^{-\beta n/11}.$$

On the other hand, there are at most $2^{2\beta n}$ ways to enlarge a set $A \setminus B$ to a maximal sum-free set. Thus, $t \le s(n)2^{2\beta n}$ and using Theorem 1 we infer that

$$s'_{\max}(n) \le s(n)2^{2\beta n}\beta^{\beta n/11} \le 2^{n/2+o(n)}\left(4^{11}\beta\right)^{\beta n/11} \le 2^{n/2-2^{-27}n}.$$

Hence

$$s_{\max}(n) \le s'_{\max}(n) + \sum_{i=1}^{n/11}\binom{n}{i} \le 2^{n/2-2^{-27}n} + 2^{49n/100} \le 2^{n/2-2^{-28}n}. \qquad \square$$

Note that in the proof of Theorem 2 we used no arithmetic properties of the natural numbers. Thus, our result can be generalized as follows. Let $\oplus : \Omega \times \Omega \to \Omega$ be an operation in $A$. We say that $A \subseteq \Omega$ is $\oplus$-free if $a \oplus b \notin A$ for $a, b \in A$ and define a maximal $\oplus$-free set accordingly. Furthermore, for $A \subseteq \Omega$, let $\sigma(A)$ and $\sigma_{\max}(A)$ denote the number of $\oplus$-free and maximal $\oplus$-free subsets of $A$, respectively. Then one can mimic our argument to show the following result.

**Theorem 4.** *For every $\alpha > 0$ there exist $\epsilon > 0$ and $n_0$ such that the following holds. For every set $\Omega$ with operation $\oplus : \Omega \times \Omega \to \Omega$ and every $A \subseteq \Omega$ for which $|A| = n > n_0$ and $\sigma(A) \ge 2^{\alpha n}$, we have $\sigma_{\max}(A) \le \sigma(A)2^{-\epsilon n}$.* $\square$

Finally, we remark that Cameron [3] studied the set of all subsets of the natural numbers as a metric space with the distance defined as

$$d(A, B) = 2^{-\min\{A \triangle B\}},$$

where $A \triangle B$ denote the symmetric difference of sets $A$ and $B$. Then the set of all sum-free subsets of natural numbers has Hausdorff dimension 1/2 (it follows immediately from Theorem 1 and the fact that all sets which contain only odd numbers are sum-free). Theorem 2 implies that the dimension of the set of all maximal sum-free subsets of the natural numbers is at most $1/2 - 2^{-28} < 1/2$.

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