

## AN INEQUALITY FOR THE NORM OF A POLYNOMIAL FACTOR

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ABSTRACT. Let  $p(z)$  be a monic polynomial of degree  $n$ , with complex coefficients, and let  $q(z)$  be its monic factor. We prove an asymptotically sharp inequality of the form  $\|q\|_E \leq C^n \|p\|_E$ , where  $\|\cdot\|_E$  denotes the sup norm on a compact set  $E$  in the plane. The best constant  $C_E$  in this inequality is found by potential theoretic methods. We also consider applications of the general result to the cases of a disk and a segment.

### 1. INTRODUCTION

Let  $p(z)$  be a monic polynomial of degree  $n$ , with complex coefficients. Suppose that  $p(z)$  has a monic factor  $q(z)$ , so that

$$p(z) = q(z)r(z),$$

where  $r(z)$  is also a monic polynomial. Define the uniform (sup) norm on a compact set  $E$  in the complex plane  $\mathbb{C}$  by

$$(1.1) \quad \|f\|_E := \sup_{z \in E} |f(z)|.$$

We study the inequalities of the following form:

$$(1.2) \quad \|q\|_E \leq C^n \|p\|_E, \quad \deg p = n,$$

where the main problem is to find the best (the smallest) constant  $C_E$ , such that (1.2) is valid for *any* monic polynomial  $p(z)$  and *any* monic factor  $q(z)$ .

In the case  $E = \overline{D}$ , where  $D := \{z : |z| < 1\}$ , the inequality (1.2) was considered in a series of papers by Mignotte [9], Granville [7] and Glesser [6], who obtained a number of improvements on the upper bound for  $C_{\overline{D}}$ . D. W. Boyd [3] made the final step here, by proving that

$$(1.3) \quad \|q\|_{\overline{D}} \leq \beta^n \|p\|_{\overline{D}},$$

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with

$$(1.4) \quad \beta := \exp \left( \frac{1}{\pi} \int_0^{2\pi/3} \log \left( 2 \cos \frac{t}{2} \right) dt \right).$$

The constant  $\beta = C_{\overline{D}}$  is asymptotically sharp, as  $n \rightarrow \infty$ , and it can also be expressed in a different way, using Mahler’s measure. This problem is of importance in designing algorithms for factoring polynomials with integer coefficients over integers. We refer to [5] and [8] for more information on the connection with symbolic computations.

A further development related to (1.2) for  $E = [-a, a]$ ,  $a > 0$ , was suggested by P. B. Borwein in [1] (see Theorems 2 and 5 there or see Section 5.3 in [2]). In particular, Borwein proved that if  $\deg q = m$ , then

$$(1.5) \quad |q(-a)| \leq \|p\|_{[-a,a]} a^{m-n} 2^{n-1} \prod_{k=1}^m \left( 1 + \cos \frac{2k-1}{2n} \pi \right),$$

where the bound is attained for a monic Chebyshev polynomial of degree  $n$  on  $[-a, a]$  and a factor  $q$ . He also showed that, for  $E = [-2, 2]$ , the constant in the above inequality satisfies

$$\begin{aligned} & \limsup_{n \rightarrow \infty} \left( 2^{m-1} \prod_{k=1}^m \left( 1 + \cos \frac{2k-1}{2n} \pi \right) \right)^{1/n} \\ & \leq \lim_{n \rightarrow \infty} \left( 2^{[2n/3]-1} \prod_{k=1}^{[2n/3]} \left( 1 + \cos \frac{2k-1}{2n} \pi \right) \right)^{1/n} \\ & = \exp \left( \int_0^{2/3} \log (2 + 2 \cos \pi x) dx \right) = 1.9081 \dots, \end{aligned}$$

which hints that

$$(1.6) \quad C_{[-2,2]} = \exp \left( \int_0^{2/3} \log (2 + 2 \cos \pi x) dx \right) = 1.9081 \dots$$

We find the asymptotically best constant  $C_E$  in (1.2) for a rather arbitrary compact set  $E$ . The general result is then applied to the cases of a disk and a line segment, so that we recover (1.3)–(1.4) and confirm (1.6).

## 2. RESULTS

Our solution of the above problem is based on certain ideas from the logarithmic potential theory (cf. [12] or [13]). Let  $\text{cap}(E)$  be the *logarithmic capacity* of a compact set  $E \subset \mathbb{C}$ . For  $E$  with  $\text{cap}(E) > 0$ , denote the *equilibrium measure* of  $E$  (in the sense of the logarithmic potential theory) by  $\mu_E$ . We remark that  $\mu_E$  is a positive unit Borel measure supported on  $E$ ,  $\text{supp } \mu_E \subset E$  (see [13, p. 55]).

**Theorem 2.1.** *Let  $E \subset \mathbb{C}$  be a compact set,  $\text{cap}(E) > 0$ . Then the best constant  $C_E$  in (1.2) is given by*

$$(2.1) \quad C_E = \frac{\max_{u \in \partial E} \exp \left( \int_{|z-u| \geq 1} \log |z-u| d\mu_E(z) \right)}{\text{cap}(E)}.$$

Furthermore, if  $E$  is regular, then

$$(2.2) \quad C_E = \max_{u \in \partial E} \exp \left( - \int_{|z-u| \leq 1} \log |z-u| d\mu_E(z) \right).$$

The above notion of regularity is to be understood in the sense of the exterior Dirichlet problem (cf. [13, p. 7]). Note that the condition  $\text{cap}(E) > 0$  is usually satisfied for all applications, as it only fails for very *thin* sets (see [13, pp. 63–66]), e.g., finite sets in the plane. But if  $E$  consists of finitely many points, then the inequality (1.2) cannot be true for a polynomial  $p(z)$  with zeros at every point of  $E$  and for its linear factors  $q(z)$ . On the other hand, Theorem 2.1 is applicable to any compact set with a connected component consisting of more than one point (cf. [13, p. 56]).

One can readily see from (1.2) or (2.1) that the best constant  $C_E$  is invariant under the rigid motions of the set  $E$  in the plane. Therefore we consider applications of Theorem 2.1 to the family of disks  $D_r := \{z : |z| < r\}$ , which are centered at the origin, and to the family of segments  $[-a, a]$ ,  $a > 0$ .

**Corollary 2.2.** *Let  $D_r$  be a disk of radius  $r$ . Then the best constant  $C_{\overline{D_r}}$ , for  $E = \overline{D_r}$ , is given by*

$$(2.3) \quad C_{\overline{D_r}} = \begin{cases} \frac{1}{r}, & 0 < r \leq 1/2, \\ \frac{1}{r} \exp \left( \frac{1}{\pi} \int_0^{\pi-2 \arcsin \frac{1}{2r}} \log \left( 2r \cos \frac{x}{2} \right) dx \right), & r > 1/2. \end{cases}$$

Note that (1.3)–(1.4) immediately follow from (2.3) for  $r = 1$ . The graph of  $C_{\overline{D_r}}$ , as a function of  $r$ , is in Figure 1.

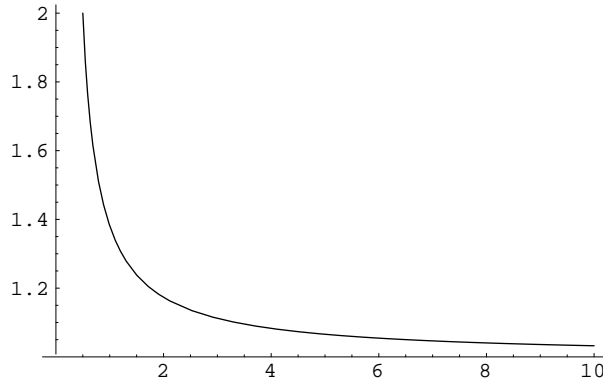


FIGURE 1.  $C_{\overline{D_r}}$  as a function of  $r$ .

**Corollary 2.3.** *If  $E = [-a, a]$ ,  $a > 0$ , then*

$$(2.4) \quad C_{[-a,a]} = \begin{cases} \frac{2}{a}, & 0 < a \leq 1/2, \\ \frac{2}{a} \exp \left( \int_{1-a}^a \frac{\log(t+a)}{\pi \sqrt{a^2-t^2}} dt \right), & a > 1/2. \end{cases}$$

Observe that (2.4), with  $a = 2$ , implies (1.6) by the change of variable  $t = 2 \cos \pi x$ . We include the graph of  $C_{[-a,a]}$ , as a function of  $a$ , in Figure 2.

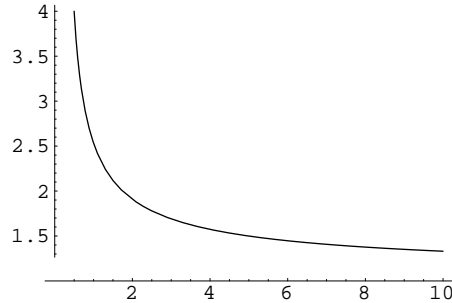


FIGURE 2.  $C_{[-a,a]}$  as a function of  $a$ .

We now state two general consequences of Theorem 2.1. They explain some interesting features of  $C_E$ , which the reader may have noticed in Corollaries 2.2 and 2.3. Let

$$\text{diam}(E) := \max_{z, \zeta \in E} |z - \zeta|$$

be the Euclidean diameter of  $E$ .

**Corollary 2.4.** *Suppose that  $\text{cap}(E) > 0$ . If  $\text{diam}(E) \leq 1$ , then*

$$(2.5) \quad C_E = \frac{1}{\text{cap}(E)}.$$

It is well known that  $\text{cap}(D_r) = r$  and  $\text{cap}([-a, a]) = a/2$  (see [12, p. 135]), which clarifies the first lines of (2.3) and (2.4) by (2.5).

The next corollary shows how the constant  $C_E$  behaves under dilations of the set  $E$ . Let  $\alpha E$  be the dilation of  $E$  with a factor  $\alpha > 0$ .

**Corollary 2.5.** *If  $E$  is regular, then*

$$(2.6) \quad \lim_{\alpha \rightarrow +\infty} C_{\alpha E} = 1.$$

Thus Figures 1 and 2 clearly illustrate (2.6).

We remark that one can deduce inequalities of the type (1.2), for various  $L_p$  norms, from Theorem 2.1, by using relations between  $L_p$  and  $L_\infty$  norms of polynomials on  $E$  (see, e.g., [11]).

### 3. PROOFS

*Proof of Theorem 2.1.* The proof of this result is based on the ideas of [3] and [10]. For  $u \in \mathbb{C}$ , consider a function

$$\rho_u(z) := \max(|z - u|, 1), \quad z \in \mathbb{C}.$$

One can immediately see that  $\log \rho_u(z)$  is a subharmonic function in  $z \in \mathbb{C}$ , which has the following integral representation (see [12, p. 29]):

$$(3.1) \quad \log \rho_u(z) = \int \log |z - t| d\lambda_u(t), \quad z \in \mathbb{C},$$

where  $d\lambda_u(u + e^{i\theta}) = d\theta/(2\pi)$  is the normalized angular measure on  $|t - u| = 1$ . Let  $u \in \partial E$  be such that

$$\|q\|_E = |q(u)|.$$

If  $z_k, k = 1, \dots, n$ , are the zeros of  $p(z)$ , arranged so that the first  $m$  zeros belong to  $q(z)$ , then

$$\begin{aligned} \log \|q\|_E &= \sum_{k=1}^m \log |u - z_k| \leq \sum_{k=1}^m \log \rho_u(z_k) \leq \sum_{k=1}^n \log \rho_u(z_k) \\ (3.2) \quad &= \sum_{k=1}^n \int \log |z_k - t| d\lambda_u(t) = \int \log |p(t)| d\lambda_u(t), \end{aligned}$$

by (3.1).

We use the well known Bernstein-Walsh lemma about the growth of a polynomial outside of the set  $E$  (see [12, p. 156], for example): Let  $E \subset \mathbb{C}$  be a compact set,  $\text{cap}(E) > 0$ , with the unbounded component of  $\mathbb{C} \setminus E$  denoted by  $\Omega$ . Then, for any polynomial  $p(z)$  of degree  $n$ , we have

$$(3.3) \quad |p(z)| \leq \|p\|_E e^{ng_\Omega(z, \infty)}, \quad z \in \mathbb{C},$$

where  $g_\Omega(z, \infty)$  is the Green function of  $\Omega$ , with pole at  $\infty$ . The following representation for  $g_\Omega(z, \infty)$  is found in Theorem III.37 of [13, p. 82]).

$$(3.4) \quad g_\Omega(z, \infty) = \log \frac{1}{\text{cap}(E)} + \int \log |z - t| d\mu_E(t), \quad z \in \mathbb{C}.$$

It follows from (3.1)–(3.4) and Fubini’s theorem that

$$\begin{aligned} \frac{1}{n} \log \frac{\|q\|_E}{\|p\|_E} &\leq \int \log \frac{|p(t)|^{1/n}}{\|p\|_E^{1/n}} d\lambda_u(t) \leq \int g_\Omega(t, \infty) d\lambda_u(t) \\ &= \log \frac{1}{\text{cap}(E)} + \int \int \log |z - t| d\lambda_u(t) d\mu_E(z) \\ &= \log \frac{1}{\text{cap}(E)} + \int \log \rho_u(z) d\mu_E(z). \end{aligned}$$

Using the definition of  $\rho_u(z)$ , we obtain from the above estimate that

$$\begin{aligned} \|q\|_E &\leq \left( \frac{\max_{u \in \partial E} \exp \left( \int \log \rho_u(z) d\mu_E(z) \right)}{\text{cap}(E)} \right)^n \|p\|_E \\ &= \left( \frac{\max_{u \in \partial E} \exp \left( \int_{|z-u| \geq 1} \log |z - u| d\mu_E(z) \right)}{\text{cap}(E)} \right)^n \|p\|_E. \end{aligned}$$

Hence

$$(3.5) \quad C_E \leq \frac{\max_{u \in \partial E} \exp \left( \int_{|z-u| \geq 1} \log |z - u| d\mu_E(z) \right)}{\text{cap}(E)}.$$

In order to prove the inequality opposite to (3.5), we consider the  $n$ -th Fekete points  $\{a_{k,n}\}_{k=1}^n$  for the set  $E$  (cf. [12, p. 152]). Let

$$p_n(z) := \prod_{k=1}^n (z - a_{k,n})$$

be the Fekete polynomial of degree  $n$ . Define the normalized counting measures on the Fekete points by

$$\tau_n := \frac{1}{n} \sum_{k=1}^n \delta_{a_{k,n}}, \quad n \in \mathbb{N}.$$

It is known that (see Theorems 5.5.4 and 5.5.2 in [12, pp. 153–155])

$$(3.6) \quad \lim_{n \rightarrow \infty} \|p_n\|_E^{1/n} = \text{cap}(E).$$

Furthermore, we have the following weak\* convergence of counting measures (cf. [12, p. 159]):

$$(3.7) \quad \tau_n \xrightarrow{*} \mu_E, \quad \text{as } n \rightarrow \infty.$$

Let  $u \in \partial E$  be a point, where the maximum on the right-hand side of (3.5) is attained. Define the factor  $q_n(z)$  for  $p_n(z)$ , with zeros being the  $n$ -th Fekete points satisfying  $|a_{k,n} - u| \geq 1$ . Then we have by (3.7) that

$$\begin{aligned} \lim_{n \rightarrow \infty} \|q_n\|_E^{1/n} &\geq \lim_{n \rightarrow \infty} |q_n(u)|^{1/n} = \lim_{n \rightarrow \infty} \exp \left( \frac{1}{n} \sum_{|a_{k,n}-u| \geq 1} \log |u - a_{k,n}| \right) \\ &= \exp \left( \lim_{n \rightarrow \infty} \int_{|z-u| \geq 1} \log |u - z| d\tau_n(z) \right) \\ &= \exp \left( \int_{|z-u| \geq 1} \log |u - z| d\mu_E(z) \right). \end{aligned}$$

Combining the above inequality with (3.6) and the definition of  $C_E$ , we obtain that

$$C_E \geq \lim_{n \rightarrow \infty} \frac{\|q_n\|_E^{1/n}}{\|p_n\|_E^{1/n}} \geq \frac{\exp \left( \int_{|z-u| \geq 1} \log |z - u| d\mu_E(z) \right)}{\text{cap}(E)}.$$

This shows that (2.1) holds true. Moreover, if  $u \in \partial E$  is a regular point for  $\Omega$ , then we obtain by Theorem III.36 of [13, p. 82]) and (3.4) that

$$\log \frac{1}{\text{cap}(E)} + \int \log |u - t| d\mu_E(t) = g_\Omega(u, \infty) = 0.$$

Hence

$$\log \frac{1}{\text{cap}(E)} + \int_{|z-u| \geq 1} \log |u - t| d\mu_E(t) = - \int_{|z-u| \leq 1} \log |u - t| d\mu_E(t),$$

which implies (2.2) by (2.1). □

*Proof of Corollary 2.2.* It is well known [13, p. 84] that  $\text{cap}(\overline{D_r}) = r$  and  $d\mu_{\overline{D_r}}(re^{i\theta}) = d\theta/(2\pi)$ , where  $d\theta$  is the angular measure on  $\partial D_r$ . If  $r \in (0, 1/2]$ , then the numerator of (2.1) is equal to 1, so that

$$C_{\overline{D_r}} = \frac{1}{r}, \quad 0 < r \leq 1/2.$$

Assume that  $r > 1/2$ . We set  $z = re^{i\theta}$  and let  $u_0 = re^{i\theta_0}$  be a point where the maximum in (2.1) is attained. On writing

$$|z - u_0| = 2r \left| \sin \frac{\theta - \theta_0}{2} \right|,$$

we obtain that

$$\begin{aligned} C_{\overline{D_r}} &= \frac{1}{r} \exp \left( \frac{1}{2\pi} \int_{\theta_0 + 2 \arcsin \frac{1}{2r}}^{2\pi + \theta_0 - 2 \arcsin \frac{1}{2r}} \log \left| 2r \sin \frac{\theta - \theta_0}{2} \right| d\theta \right) \\ &= \frac{1}{r} \exp \left( \frac{1}{2\pi} \int_{2 \arcsin \frac{1}{2r} - \pi}^{\pi - 2 \arcsin \frac{1}{2r}} \log \left( 2r \cos \frac{x}{2} \right) dx \right) \\ &= \frac{1}{r} \exp \left( \frac{1}{\pi} \int_0^{\pi - 2 \arcsin \frac{1}{2r}} \log \left( 2r \cos \frac{x}{2} \right) dx \right), \end{aligned}$$

by the change of variable  $\theta - \theta_0 = \pi - x$ . □

*Proof of Corollary 2.3.* Recall that  $\text{cap}([-a, a]) = a/2$  (see [13, p. 84]) and

$$d\mu_{[-a, a]}(t) = \frac{dt}{\pi\sqrt{a^2 - t^2}}, \quad t \in [-a, a].$$

It follows from (2.1) that

$$(3.8) \quad C_{[-a, a]} = \frac{2}{a} \exp \left( \max_{u \in [-a, a]} \int_{[-a, a] \setminus (u-1, u+1)} \frac{\log |t - u|}{\pi\sqrt{a^2 - t^2}} dt \right).$$

If  $a \in (0, 1/2]$ , then the integral in (3.8) obviously vanishes, so that  $C_{[-a, a]} = 2/a$ . For  $a > 1/2$ , let

$$(3.9) \quad f(u) := \int_{[-a, a] \setminus (u-1, u+1)} \frac{\log |t - u|}{\pi\sqrt{a^2 - t^2}} dt.$$

One can easily see from (3.9) that

$$f'(u) = \int_{u+1}^a \frac{dt}{\pi(u-t)\sqrt{a^2 - t^2}} < 0, \quad u \in [-a, 1 - a],$$

and

$$f'(u) = \int_{-a}^{u-1} \frac{dt}{\pi(u-t)\sqrt{a^2 - t^2}} > 0, \quad u \in [a - 1, a].$$

However, if  $u \in (1 - a, a - 1)$ , then

$$f'(u) = \int_{u+1}^a \frac{dt}{\pi(u-t)\sqrt{a^2 - t^2}} + \int_{-a}^{u-1} \frac{dt}{\pi(u-t)\sqrt{a^2 - t^2}}.$$

It is not difficult to verify directly that

$$\int \frac{dt}{\pi(u-t)\sqrt{a^2-t^2}} = \frac{1}{\pi\sqrt{a^2-u^2}} \log \left| \frac{a^2-ut + \sqrt{a^2-t^2}\sqrt{a^2-u^2}}{t-u} \right| + C,$$

which implies that

$$f'(u) = \frac{1}{\pi\sqrt{a^2-u^2}} \log \left( \frac{a^2-u^2+u + \sqrt{a^2-(u-1)^2}\sqrt{a^2-u^2}}{a^2-u^2-u + \sqrt{a^2-(u+1)^2}\sqrt{a^2-u^2}} \right),$$

for  $u \in (1-a, a-1)$ . Hence

$$f'(u) < 0, \quad u \in (1-a, 0), \quad \text{and} \quad f'(u) > 0, \quad u \in (0, a-1).$$

Collecting all facts, we obtain that the maximum for  $f(u)$  on  $[-a, a]$  is attained at the endpoints  $u = a$  and  $u = -a$ , and it is equal to

$$\max_{u \in [-a, a]} f(u) = \int_{1-a}^a \frac{\log(t+a)}{\pi\sqrt{a^2-t^2}} dt.$$

Thus (2.3) follows from (3.8) and the above equation. □

*Proof of Corollary 2.4.* Note that the numerator of (2.1) is equal to 1, because  $|z-u| \leq 1, \forall z \in E, \forall u \in \partial E$ . Thus (2.5) follows immediately. □

*Proof of Corollary 2.5.* Observe that  $C_E \geq 1$  for any  $E \in \mathbb{C}$ , so that  $C_{\alpha E} \geq 1$ . Since  $E$  is regular, we use the representation for  $C_E$  in (2.2). Let  $T : E \rightarrow \alpha E$  be the dilation mapping. Then  $|Tz-Tu| = \alpha|z-u|, z, u \in E$ , and  $d\mu_{\alpha E}(Tz) = d\mu_E(z)$ . This gives that

$$\begin{aligned} C_{\alpha E} &= \max_{Tu \in \partial(\alpha E)} \exp \left( - \int_{|Tz-Tu| \leq 1} \log |Tz-Tu| d\mu_{\alpha E}(Tz) \right) \\ &= \max_{u \in \partial E} \exp \left( - \int_{|z-u| \leq 1/\alpha} \log(\alpha|z-u|) d\mu_E(z) \right) \\ &= \max_{u \in \partial E} \exp \left( -\mu_E(\overline{D_{1/\alpha}(u)}) \log \alpha - \int_{|z-u| \leq 1/\alpha} \log |z-u| d\mu_E(z) \right) \\ &< \max_{u \in \partial E} \exp \left( - \int_{|z-u| \leq 1/\alpha} \log |z-u| d\mu_E(z) \right), \end{aligned}$$

where  $\alpha \geq 1$ . Using the absolute continuity of the integral, we have that

$$\lim_{\alpha \rightarrow +\infty} \int_{|z-u| \leq 1/\alpha} \log |z-u| d\mu_E(z) = 0,$$

which implies (2.6). □

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