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HYPOELLIPTIC RANDOM HEAT KERNELS: A CASE STUDY

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ABSTRACT. We consider the fundamental solution of a simple hypoelliptic stochastic partial differential equation in which the first-order term is modulated by white noise. We derive some short-time asymptotic formulæ. We discover that the form of the dominant short-time asymptotics depends nontrivially upon the interplay between the geometry of the noisy first-order term and the geometry defined by the hypoelliptic operator.

1. INTRODUCTION

Consider $\mathbb{R}^2$ with its standard Euclidean atlas. Define two vector fields

$$\begin{aligned} A_1(x,y) &\stackrel{\text{def}}{=} \frac{\partial}{\partial x}, \\ A_2(x,y) &\stackrel{\text{def}}{=} x\frac{\partial}{\partial y}, \end{aligned} \qquad (x,y)\in\mathbb{R}^2.$$

Note that

$$A_3(x,y) \stackrel{\text{def}}{=} [A_1,A_2](x,y) = \frac{\partial}{\partial y}, \qquad (x,y)\in\mathbb{R}^2,$$

so that

$$\text{Span}\left\{\text{Lie}\{A_1,A_2\}\right\}(x,y) = T_{(x,y)}\mathbb{R}^2, \qquad (x,y)\in\mathbb{R}^2.$$

Thus the second-order operator

$$\mathcal{L} \stackrel{\text{def}}{=} \frac{1}{2}A_1^2 + \frac{1}{2}A_2^2 = \frac{1}{2}\frac{\partial^2}{\partial x^2} + \frac{x^2}{2}\frac{\partial^2}{\partial y^2}$$

is hypoelliptic (in the literature, this operator is often attributed to Grushin). Now let $(\Omega,\mathcal{F},\mathbb{P})$ be a probability triple on which is defined a Wiener process W. We are interested in the stochastic partial differential equations (SPDE's)

$$\begin{aligned} dp^i &= \mathcal{L}p^i dt + A_i p^i \circ dW_t, \\ p^i(0,\cdot) &= \delta_{(0,0)}, \end{aligned} \qquad t\geq 0, \tag{1}$$

where $\circ$ denotes Stratonovich integration. Specifically, we are interested in the behavior of these three SPDE's as t tends to zero.

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The background of our interest is the general study of SPDE's of the form

$$\begin{aligned} du &= \mathcal{A}u dt + \mathcal{M}u \circ dW_t, \\ u(0,\cdot) &= \delta_y \end{aligned} \tag{2}$$

on d-dimensional differentiable manifolds, where $\mathcal{A}$ is a second-order partial-differential operator and $\mathcal{M}$ is a first-order partial-differential operator, and y is some fixed point in M. Of course if $\mathcal{M} \equiv 0$ and $\mathcal{A}$ is elliptic, the results are classical; for "nice" x,

$$u(t,x) = (2\pi t)^{-d/2} \exp\left[-\frac{d^2(x,y)}{2t} + \mathfrak{W}_y(t,x)\right]\left\{a_0(x) + a_1(x)t + a_2(x)t^2 \dots\right\}$$

as t tends to zero, where d is the Riemannian distance, $\mathfrak{W}_y(t,x)$ is the work done by the non-self-adjoint part of $\mathcal{A}$ along the geodesic from y to x (see [10]), a_0 is related to the Ruse Invariant (see [3]), and the a_i's are functions which can be iteratively derived according to calculations of Minakshisundaram and Pleijel (see [2] and also [4]). If $\mathcal{M} \equiv 0$ and $\mathcal{A}$ is hypoelliptic, Ben Arous in [1] and Leandre in [6] and [7] showed that a similar expansion holds if we interpret d as the Carnot-Carathéodory distance (see also [8]). Since SPDE's where $\mathcal{M}$ is zero-order play a central role in filtering theory, it is natural to ask for similar expansions in the stochastic case. The first result was by Zhang [13], where the case of $\mathcal{A}$ elliptic and $\mathcal{M}$ zero-order was studied; one should replace the deterministic a_i's by a collection of iterated stochastic integrals. The case of $\mathcal{A}$ hypoelliptic and $\mathcal{M}$ zero-order was considered by Mesnager [9].

It turns out that if $\mathcal{M}$ is first-order and $\mathcal{A}$ is elliptic, a new phenomenon occurs. The simplest incarnation of this occurs if the manifold is $\mathbb{R}$, $\mathcal{A} = \frac{1}{2}\frac{d^2}{dx^2}$, and $\mathcal{M} = \frac{d}{dx}$; then it is easily seen that the solution of (2) is

$$\begin{aligned} u(t,x) &= \exp\left[-\frac{|x-y+W_t|^2}{2t}\right]\Big/\sqrt{2\pi t} \\ &= \exp\left[-\frac{|x-y|^2}{2t} - \left(\frac{x-y}{t}\right)W_t - \frac{W_t^2}{2t}\right]\Big/\sqrt{2\pi t}, \end{aligned} \qquad t \geq 0,\ x \in \mathbb{R}.$$

Here the dominant exponential term is the standard distance function, the second term is the work done by $\circ dW_t$ along the line joining x and y (i.e., the geodesic), and the third exponential term, which cannot be neglected, grows like $\log\log t^{-1}$ (due to the law of the iterated logarithm) but its statistics are bounded (i.e., it has a chi-square distribution). In general, the third term involves a randomly-forced Jacobi equation (see [11] and [12]). This brings us to the present work, which is a case study in the final situation, where $\mathcal{A}$ is hypoelliptic and $\mathcal{M}$ is first-order. We have chosen the simplest possible form of this problem to indicate what phenomena are at work. The operator $\mathcal{A}$ is of the Grushin type. This operator is degenerate only along the line $\{0\} \times \mathbb{R}$, so we choose our initial Dirac mass to be at $(0,0)$. By analogy with [12], we expect to see interesting phenomena when $\mathcal{M}$ has a first-order part; i.e., when $\mathcal{M}$ is a vector field. Since a general vector field can be written as a linear combination of A_1, A_2, and A_3, we consider these three cases separately. By understanding the behavior of this simple case, we should be able to glimpse a general theory for (2); we will develop this elsewhere.

The basic idea of our proof is to start along the road of [12]. We shall rescale and then write a variational problem similar to one for a subRiemannian-type distance function. This variational problem is that of finding the control of least cost which drives a certain stochastic differential equation (SDE) from $(0,0)$ to (x,y). We then must be careful with the variational problem. In essence, we must keep track of the directions in which the noise can drive the SDE versus the directions in which controls can drive the SDE, and the rates of each in terms of iterated Lie brackets. Finally, we note that our geometry is not locally constant since the dimension of the span of A_1 and A_2 is not constant (i.e., it is 1 along $\{0\}\times\mathbb{R}$ and 2 elsewhere).

2. A rescaling heuristic

We will first rescale the SPDE's (1); set

$$q^{i,\varepsilon}(t,x,y) \stackrel{\text{def}}{=} p^i(t\varepsilon,x,y), \qquad t\ge 0,\ (x,y)\in\mathbb{R}^2,$$

for all $\varepsilon>0$; thus

$$p^i(t,x,y)=q^{i,t}(1,x,y).$$

We note that $q^{i,\varepsilon}$ satisfies the SPDE

$$\left.\begin{aligned} dq^{i,\varepsilon} &= \varepsilon\mathcal{L}q^{i,\varepsilon}dt+\varepsilon^{1/2}A_i q^{i,\varepsilon}\circ dW^\varepsilon_t,\\ q^{i,\varepsilon}(0,\cdot) &= \delta_{(0,0)}, \end{aligned}\right. \qquad t\ge 0,\ x\in\mathbb{R}^2,$$

where

$$W^\varepsilon_t \stackrel{\text{def}}{=} \varepsilon^{-1/2}W_{t\varepsilon}, \qquad t\ge 0.$$

Next we will replace W^ε by a smooth function b; this leads to the PDE

$$\left.\begin{aligned} dq^{i,\varepsilon,b} &= \varepsilon\mathcal{L}q^{i,\varepsilon,b}dt+\varepsilon^{1/2}A_i q^{i,\varepsilon,b}\dot b(t)dt,\\ q^{i,\varepsilon,b}(0,\cdot) &= \delta_{(0,0)}, \end{aligned}\right. \qquad t\ge 0.$$

A reasonable guess is thus that

$$q^{i,\varepsilon,b}(t,x,y)\asymp \exp\left[-\varepsilon^{-1}\tilde J^{\varepsilon,b}_i(t,x,y)\right]$$

where

$$\tilde J^{\varepsilon,b}_i(t,x,y) \stackrel{\text{def}}{=} \inf\left\{\frac12\int_0^t \sum_{j\in\{1,2\}} p_j^2(s)ds : \right.$$

$$\dot\zeta(s)=\sum_{j\in\{1,2\}}A_j(\zeta(s))p_j(s)+\varepsilon^{1/2}A_i(\zeta(s))\dot b(s), \tag{3}$$

$$\left.\zeta(0)=(0,0),\ \zeta(t)=(x,y)\right\}.$$

We should now retrace our steps and get a formula for $\tilde J^{t,W^t}_i(1,x,y)$.

The main idea of [12] is that, in the elliptic case, one can differentiate the variational formula (3) with respect to ε. This gives the correct exponential expansion in that case. Things are not so simple here, however. In the simplest case, when $i=3$, we see that the b term (which, being a substitute for the Wiener process W^ε, should be thought of as a generalized function) can drive ζ in the A_3 direction; if $y=0$, then the controls p_1 and p_2 need to compensate for this *through the bracket of their respective vector fields*, incurring a much greater cost than if $y\neq 0$. On the other hand, in the case $i=1$, $y=0$, the control p may be directly used, which isn't so costly.

To sort these things out, let's define a semiflow $\{\tilde{\psi}_t^{i,b,\varepsilon};\, t \ge 0\}$ of diffeomorphisms of $\mathbb{R}^2$ by

$$\begin{aligned} \dot{\tilde{\psi}}_s^{i,b,\varepsilon}(x,y) &= \varepsilon^{1/2} A_i(\tilde{\psi}_s^{i,b,\varepsilon}(x,y))\dot{b}(s), \\ \tilde{\psi}_0^{i,b,\varepsilon}(x,y) &= (x,y), \end{aligned} \qquad t \ge 0,\ (x,y) \in \mathbb{R}^2.$$

Recall that if Z is a vector field on $\mathbb{R}^2$ and ψ is a diffeomorphism from $\mathbb{R}^2$ to itself, the pullback of Z through ψ is the vector field $\psi_* Z$ on $\mathbb{R}^2$ defined by

$$(\psi_* Z)(x,y) \stackrel{\text{def}}{=} D\psi Z(\psi^{-1}(x,y)), \qquad (x,y) \in \mathbb{R}^2.$$

We can then rewrite (3) by making the transformation $\zeta(s) = \tilde{\psi}_s^{i,b,\varepsilon}(\xi(s))$; we get that

$$\tilde{J}_i^{\varepsilon,b}(t,x,y) \stackrel{\text{def}}{=} \inf\left\{ \frac{1}{2}\int_0^t \sum_{j\in\{1,2\}} p_j^2(s)\,ds : \right. \tag{8}$$
$$\dot{\xi}(s) = \sum_{j\in\{1,2\}} (\tilde{\psi}_s^{i,\varepsilon,b,-1})_* A_j(\xi(s)) p_j(s),$$
$$\left. \xi(0) = (0,0),\ \tilde{\psi}_t^{i,\varepsilon,b}(\xi(t)) = (x,y) \right\}.$$

Note that with this representation, we can rigorously retrace our steps and get a quantity which should contain the dominant asymptotics of p^i. Let's first replace ε by t and b by W^t. We define a stochastic semiflow $\{\psi_t^i;\, t \ge 0\}$ of diffeomorphisms of $\mathbb{R}^2$ by

$$\begin{aligned} d\psi_t^i(x,y) &= A_i(\psi_t^i(x,y)) \circ dW_t, \\ \psi_0^i(x,y) &= (x,y), \end{aligned} \qquad t \ge 0,\ (x,y) \in \mathbb{R}^2.$$

Note that if b is close in some sense to W^t, then $\tilde{\psi}_s^{i,t,b}$ is close to ψ_{st}^i (this can be made precise, but we will not need to do so). Then we define

$$\begin{aligned} J_i(t,x,y) &\stackrel{\text{def}}{=} \inf\left\{ \frac{1}{2}\int_0^1 \sum_{j\in\{1,2\}} p_j^2(s)\,ds : \right. \\ &\qquad \dot{\xi}(s) = \sum_{j\in\{1,2\}} (\psi_{st}^{i,-1})_* A_j(\xi(s)) p_j(s), \\ &\qquad \left. \xi(0) = (0,0),\ \psi_t^i(\xi(1)) = (x,y) \right\} \\ &= t \inf\left\{ \frac{1}{2}\int_0^t \sum_{j\in\{1,2\}} p_j^2(s) : \right. \\ &\qquad \dot{\xi}(s) = \sum_{j\in\{1,2\}} (\psi_s^{i,-1})_* A_j(\xi(s)) p_j(s), \\ &\qquad \left. \xi(0) = (0,0),\ \psi_t^i(\xi(t)) = (x,y) \right\}. \end{aligned} \tag{4}$$

We can now claim

Proposition 2.1. *We have that* $\mathbb{P}$-*a.s.,*

$$\lim_{t\to 0} t \ln p^i(t,x,y) = -\lim_{t\to 0} J_i(t,x,y). \tag{5}$$

Proof. The proof is essentially that of [6] and [7]. We can write $p^i(t,x,y)$ as a weighted density (with respect to Lebesgue measure on $\mathbb{R}^2$) of an SDE with random coefficients (using standard techniques of decompositions of stochastic flows; see, for example, [5, Chapter 4]). Then [6] and [7] can be used. $\square$

Our goal is to simplify the right-hand side of (5). Not surprisingly, we should look at the subRiemannian distance between (x,y) and the origin of $\mathbb{R}^2$; define

$$d^2_{0,0}(x,y) = t \inf \left\{ \int_0^t \sum_{j\in\{1,2\}} \dot\gamma_j^2(s)ds : \gamma_1,\gamma_2 \in C_0^1([0,t];\mathbb{R}), \right.$$
$$\left. \gamma_1(t) = x,\ \int_0^t \gamma_1(s)\dot\gamma_2(s)ds = y \right\}$$

for all $(x,y)\in\mathbb{R}^2$, where $C_0^1([0,t];\mathbb{R})$ is the collection of differentiable functions γ such that $\gamma(0)=0$ (we note that this definition does not in fact depend upon t). We note that for all (x,y), there are γ_1 and γ_2 which achieve the minimum. Our main result is

Theorem 2.2. *In all cases* (*i.e., for all* $i\in\{1,2,3\}$), *and for all* $(x,y)\in\mathbb{R}^2$,

$$\lim_{t\to 0} t\ln p^i(t,x,y) = -\frac{d^2_{0,0}(x,y)}{2}, \qquad \mathbb{P}\text{-}a.s.$$

Proof. Lemmas 3.1, 4.3, and 5.3. $\square$

The motivation of our work is that the dependence of $J_i(t,x,y)$ of (4) upon W is fairly explicit, so the proof of Theorem 2.2 should involve a minimum of technicalities and should thus allow us to focus on relevant qualitative phenomena. First, note that $\{\psi_t^{i,-1}; t\ge 0\}$ describes a stochastic flow of diffeomorphisms of $\mathbb{R}^2$ given by

$$\begin{aligned} d\psi_s^{i,-1}(x,y) &= -(\psi_t^{i,-1})_* A_i(\psi_s^{i,-1}(x,y)) \circ dW_t, \\ \psi_0^{i,-1}(x,y) &= (x,y), \end{aligned} \qquad t\ge 0,\ (x,y)\in\mathbb{R}^2. \tag{6}$$

Second, note that for any $(x,y)\in\mathbb{R}^2$ and any vector field V on $\mathbb{R}^2$, $\{(\psi_s^{i,-1})_*V(x,y); t\ge 0\}$ satisfies the $T_{(x,y)}\mathbb{R}^2$-valued stochastic differential equation

$$\begin{aligned} d((\psi_t^{i,-1})_*V)(x,y) &= \left((\psi_t^{i,-1})_*[A_i,V]\right)(x,y)\circ dW_t, \\ (\psi_0^{i,-1}V)(x,y) &= V(x,y), \end{aligned} \qquad t\ge 0,\ (x,y)\in\mathbb{R}^2. \tag{7}$$

In the next three sections we will use this last fact to write a stochastic Taylor series for the terms in the minimization problem in (5); this will give us explicit dependencies on W. Since A_1, A_2, and A_3 are linear, the ψ_t^i's can be explicitly solved for. Along the same lines, it can be seen that $[A_1,A_3]=[A_2,A_3]=0$, so $(\psi_t^{i,-1})_*A_1$ and $(\psi_t^{i,-1})_*A_2$ can be explicitly represented.

Before starting our case-by-case analysis, let's make some definitions. For $t > 0$ and $\kappa \in (0, 1/2)$, define

$$[W]_{t,\kappa} \stackrel{\text{def}}{=} \sup_{0<s\le t} \frac{|W_s|}{s^\kappa}.$$

Lemma 2.3. *For any $t > 0$, $\kappa \in (0, 1/2)$, and $\zeta \in L^2([0, t])$,*

$$\left|\int_0^t W_s \zeta(s) ds\right| \le [W]_{t,\kappa} t^{\kappa+1/2} \left\{\int_0^t \zeta^2(s) ds\right\}^{1/2}.$$

Proof. A simple application of Hölder's inequality. □

Also note that for any $\alpha \in \mathbb{R}$ and $\varepsilon > 0$,

$$(1 + \alpha)^2 \le (1 + \varepsilon^2) + (1 + \varepsilon^{-2})\alpha^2,$$

as one can see from Young's inequality. Thirdly for each $\varepsilon \in \mathbb{R}$ and $y \in \mathbb{R}$, we define

$$y_\varepsilon \stackrel{\text{def}}{=} \begin{cases} y & \text{if } y \ne 0, \\ \varepsilon & \text{if } y = 0. \end{cases}$$

3. Case 3

First, we consider the case $i = 3$. This is the easiest case. We have

$$\psi_t^i(x, y) = (x, y + W_t), \qquad t \ge 0,\ (x, y) \in \mathbb{R}^2,$$

and due to (6) and (7), or direct calculations, we know that

$$(\psi_t^{i,-1})_* A_1 = A_1 \quad \text{and} \quad (\psi_t^{i,-1})_* A_2 = A_2.$$

Thus the minimization problem is

$$\begin{aligned} J_3(t, x, y) = t \inf \Bigg\{ & \frac{1}{2} \int_0^t \sum_{j\in\{1,2\}} p_j^2(s) ds : \\ & \dot{\xi}(s) = \sum_{j\in\{1,2\}} A_j(\xi(s)) p_j(s), \\ & \xi(0) = (0, 0),\ \xi(t) = (x, y - W_t) \Bigg\} \\ = & \frac{d_{0,0}^2(x, y - W_t)}{2}. \end{aligned}$$

Thus we have

Proposition 3.1. *For any $(x, y) \in \mathbb{R}^2$,*

$$\lim_{t\to 0} t \ln p^3(t, x, y) = -\frac{d_{0,0}^2(x, y)}{2}, \qquad \mathbb{P}\text{-}a.s.$$

4. Case 2

Next we consider the case $i = 2$. This is the next-to-easiest case. We explicitly have that

$$\psi_t^i(x, y) = (x, y + xW_t), \qquad t \ge 0,\ (x, y) \in \mathbb{R}^2,$$

and due to (6) and (7), or direct calculations, we know that

$$(\psi_t^{i,-1})_* A_1 = A_1 - A_3 W_t \quad \text{and} \quad (\psi_t^{i,-1})_* A_2 = A_2.$$

Thus we can rewrite the minimization problem (4) as

$$\begin{aligned} J_2(t, x, y) = t \inf \Bigg\{ & \frac{1}{2} \int_0^t \dot\gamma_1^2(s) + \dot\gamma_2^2(s) ds : \ \gamma_1, \gamma_2 \in C_0^1([0, t]; \mathbb{R}) \\ & \gamma_1(t) = x,\ \int_0^t \gamma_1(s)\dot\gamma_2(s) ds = y + \int_0^t \{W_s - W_t\}\dot\gamma_1(s) ds \Bigg\}. \end{aligned} \tag{8}$$

An upper bound is

Lemma 4.1. *We have that*

$$J_2(t, x, y) \le \frac{d_{0,0}^2(x, y_\varepsilon)}{2} \left\{ (1 + \varepsilon^2) + 4[W]_{t,\kappa}^2 y_\varepsilon^{-2} t^{2\kappa} d_{0,0}^2(x, y_\varepsilon)(1 + \varepsilon^{-2}) \right\}. \tag{9}$$

Proof. Let γ_1 and γ_2 in $C_0^1([0, t]; \mathbb{R})$ be such that $\gamma_1(t) = x$, $\int_0^t \gamma_1(s)\dot\gamma_2(s) ds = y_\varepsilon$, and

$$\int_0^t \dot\gamma_1^2(s) + \dot\gamma_2^2(s) = \frac{d_{0,0}^2(x, y_\varepsilon)}{t}.$$

Now set $\zeta_1 \overset{\text{def}}{=} \alpha_\varepsilon \gamma_1$ and $\zeta_2(s) \overset{\text{def}}{=} \beta_\varepsilon \gamma_2$. We want to choose α_ε and β_ε such that ζ_1 and ζ_2 are an admissible pair for the variational problem for $J_2(t, x, y)$. Thus we need $\alpha_\varepsilon = 1$. We also need that

$$\beta_\varepsilon y_\varepsilon = y + \int_0^t \dot\gamma_1(s)\{W_s - W_t\} ds.$$

Thus there are two possibilities. If $y \ne 0$, then we need that

$$\beta_\varepsilon = 1 + y^{-1} \int_0^t \dot\gamma_1(s)\{W_s - W_t\} ds$$

and if $y = 0$, then we need that

$$\beta_\varepsilon = \varepsilon^{-1} \int_0^t \dot\gamma_1(s)\{W_s - W_t\} ds.$$

Thus

$$\beta_\varepsilon = \chi_{\{y \ne 0\}} + y_\varepsilon^{-1} \int_0^t \dot\gamma_1(s)\{W_s - W_t\} ds.$$

From Lemma 2.3, we get that

$$\left| \int_0^t \dot\gamma_1(s)\{W_s - W_t\} ds \right| \le 2[W]_{t,\kappa} t^\kappa d_{0,0}(x, y_\varepsilon).$$

Thus

$$\beta_\varepsilon^2 \le (1 + \varepsilon^2) + 4[W]_{t,\kappa}^2 y_\varepsilon^{-2} t^{2\kappa} d_{0,0}^2(x, y_\varepsilon)(1 + \varepsilon^{-2}).$$

This gives us (9). $\square$

From this we get a lower bound.

Lemma 4.2. *We have that*

$$J_2(t,x,y) \geq \frac{\inf_{|y'-y|\leq 4[W]_{t,\kappa}t^{\kappa+1/2}J_2^{1/2}(t,x,y)} d^2_{0,0}(x,y')}{2}.$$

Proof. If γ_1 and γ_2 are sufficiently close to being a minimizer of (8), we must have that

$$\begin{aligned}\left|\int_0^t \dot\gamma_1(s)\{W_s - W_t\}ds\right| &\leq 2[W]_{t,\kappa}t^{\kappa+1/2}\left\{\int_0^t \dot\gamma_1^2(s)ds\right\}^{1/2} \\ &\leq 2^{3/2}[W]_{t,\kappa}t^{\kappa+1/2}\left\{\frac{1}{2}\int_0^t \dot\gamma_1^2(s)ds\right\}^{1/2} \leq 4[W]_{t,\kappa}t^{\kappa+1/2}J_2^{1/2}(t,x,y).\end{aligned}$$

Thus

$$\begin{aligned}J_2(t,x,y) \geq t\inf\Bigg\{&\frac{1}{2}\int_0^t \dot\gamma_1^2(s)+\dot\gamma_2^2(s)ds :\ \gamma_1,\gamma_2 \in C_0^1([0,t];\mathbb{R}) \\ &\gamma_1(t)=x,\ \left|\int_0^t \gamma_1(s)\dot\gamma_2(s)ds - y\right| \leq 4[W]_{t,\kappa}t^{\kappa+1/2}J_2^{1/2}(t,x,y)\Bigg\}.\end{aligned}$$

This gives the result. □

The combination of these gives

Proposition 4.3. *For any* $(x,y)\in\mathbb{R}^2$,

$$\lim_{t\to 0} t\ln p^2(t,x,y) = -\frac{d^2_{0,0}(x,y)}{2}, \qquad \mathbb{P}\text{-a.s.}$$

Proof. First take the limit in t, then in ε. □

5. Case 1

We finally consider the case $i=1$. We explicitly have that

$$\phi_t^i(x,y) = (x+W_t, y), \qquad t\geq 0,\ (x,y)\in\mathbb{R}^2,$$

and due to (6) and (7), or direct calculations, we know that

$$(\psi_t^{i,-1})_* A_1 = A_1 \quad \text{and} \quad (\psi_t^{i,-1})_* A_2 = A_2 + A_3 W_t.$$

Thus the minimization problem is

$$\begin{aligned}J_1(t,x,y) = t\inf\Bigg\{&\frac{1}{2}\int_0^t \dot\gamma_1^2(s)+\dot\gamma_2^2(s)ds :\ \gamma_1,\gamma_2 \in C_0^1([0,t];\mathbb{R}^2) \\ &\gamma_1(t) = x - W_t,\ \int_0^t \{\gamma_1(s)+W_s\}\dot\gamma_2(s)ds = y\Bigg\}.\end{aligned} \tag{10}$$

The upper bound is

Lemma 5.1. *We have that*

(11)

$$J_1(t,x,y) \leq \frac{d^2_{0,0}(x-W_t,y_\varepsilon)}{2}\left\{(1+\varepsilon^2) + 2[W]^2_{t,\kappa}y_\varepsilon^{-2}t^{2\kappa}d^2_{0,0}(x-W_t,y_\varepsilon)(1+\varepsilon^{-2})\right\}.$$

Proof. Let γ_1 and γ_2 in $C_0^1([0,t];\mathbb{R})$ be such that $\gamma_1(t) = x - W_t$, $\int_0^t \gamma_1(s)\dot\gamma_2(s)ds = y_\varepsilon$, and

$$\int_0^t \dot\gamma_1^2(s) + \dot\gamma_2^2(s) = \frac{d_{0,0}^2(x - W_t, y_\varepsilon)}{t}.$$

Now set $\zeta_1 \stackrel{\text{def}}{=} \alpha_\varepsilon \gamma_1$ and $\zeta_2(s) \stackrel{\text{def}}{=} \beta_\varepsilon \gamma_2$. We want to choose α_ε and β_ε such that ζ_1 and ζ_2 are an admissible pair for the variational problem for $J_1(t,x,y)$. Thus we need $\alpha_\varepsilon = 1$. We also need that

$$\beta_\varepsilon y_\varepsilon = y + \int_0^t \dot\gamma_1(s)\{W_s - W_t\}ds.$$

Thus there are two possibilities. If $y \neq 0$, then we need that

$$\beta_\varepsilon = 1 - y^{-1}\int_0^t \dot\gamma_2(s)\{W_s - W_t\}ds$$

and if $y = 0$, then we need that

$$\beta_\varepsilon = -\varepsilon^{-1}\int_0^t \dot\gamma_2(s)\{W_s - W_t\}ds.$$

Thus

$$\beta_\varepsilon = \chi_{\{y\neq 0\}} - y_\varepsilon^{-1}\int_0^t \dot\gamma_2(s) W_s ds.$$

From Lemma 2.3, we get that

$$\left|\int_0^t \dot\gamma_2(s) W_s ds\right| \le [W]_{t,\kappa} t^{2\kappa} d_{0,0}^2(x - W_t, y_\varepsilon).$$

Thus

$$\beta_\varepsilon^2 \le (1+\varepsilon^2) + [W]_{t,\kappa}^2 y_\varepsilon^{-2} t^{2\kappa} d_{0,0}^2(x - W_t, y_\varepsilon)(1+\varepsilon^{-2}).$$

This gives us (11). □

From this we get a lower bound.

Lemma 5.2. *We have that*

$$J_1(t,x,y) \ge \frac{\inf_{|y'-y|\le 2^{3/2}[W]_{t,\kappa} t^{\kappa+1/2} J_1^{1/2}(t,x,y)} d_{0,0}^2(x - W_t, y')}{2}.$$

Proof. If γ_1 and γ_2 are sufficiently close to being a minimizer of (10), we must have that

$$\left|\int_0^t \dot\gamma_2(s) W_s ds\right| \le 2^{1/2}[W]_{t,\kappa} J_1^{1/2}(t,x,y).$$

Thus

$$\begin{aligned} J_1(t,x,y) \ge t \inf \Bigg\{ &\frac{1}{2}\int_0^t \dot\gamma_1^2(s) + \dot\gamma_2^2(s)ds : \gamma_1, \gamma_2 \in C_0^1([0,t];\mathbb{R}) \\ &\gamma_1(t) = x, \left|\int_0^t \gamma_1(s)\dot\gamma_2(s)ds - y\right| \le 2^{1/2}[W]_{t,\kappa} t^{\kappa+1/2} J_1^{1/2}(t,x,y)\Bigg\}. \end{aligned}$$

This gives the result. □

The combination of these gives

Proposition 5.3. *For any* $(x, y) \in \mathbb{R}^2$,

$$\lim_{t \to 0} t \ln p_1(t, x, y) = -\frac{d_{0,0}^2(x, y)}{2}, \qquad \mathbb{P}\text{-}a.s.$$

Proof. First take the limit in t, then in ε. $\square$

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