PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 129, Number 8, Pages 2343–2351
S 0002-9939(00)05844-5
Article electronically published on December 28, 2000

A FORMULA FOR k-HYPONORMALITY OF BACKSTEP EXTENSIONS OF SUBNORMAL WEIGHTED SHIFTS

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(Communicated by David R. Larson)

ABSTRACT. Let $\alpha : \alpha_0, \alpha_1, \cdots$ be a weight sequence of positive real numbers and let W_α be a subnormal weighted shift with a weight sequence α. Consider an extended weight sequence $\alpha(x) : x, \alpha_0, \alpha_1, \cdots$ with $0 < x \leq \alpha_0$ and let $HE(\alpha, k) := \{x > 0 : W_{\alpha(x)}$ is k-hyponormal$\}$ for $k \in \mathbb{N} \cup \{\infty\}$, where $\mathbb{N}$ is the set of natural numbers. We obtain a formula to find the interval $HE(\alpha, k) \setminus HE(\alpha, k+1)$, which provides several examples to distinguish the classes of k-hyponormal operators from one another.

1. INTRODUCTION AND PRELIMINARIES

Let $\mathcal{H}$ be a separable, infinite dimensional, complex Hilbert space and let $\mathcal{L}(\mathcal{H})$ denote the algebra of all bounded linear operators on $\mathcal{H}$. For $A, B \in \mathcal{L}(\mathcal{H})$, let $[A, B] = AB - BA$. We say that an n-tuple $T = (T_1, \cdots, T_n)$ of operators in $\mathcal{L}(\mathcal{H})$ is *hyponormal* if the operator matrix $([T_j^*, T_i])_{i,j=1}^n$ is positive on the direct sum of n copies of $\mathcal{H}$. For arbitrary positive integer k, $T \in \mathcal{L}(\mathcal{H})$ is *k-hyponormal* if $(I, T, \cdots, T^k)$ is hyponormal. It is well known that T is subnormal if and only if T is ∞-hyponormal (cf. [Br], [Hal]).

Let $\alpha : \alpha_0, \alpha_1, \cdots$ be a sequence of positive real numbers. Let $x > 0$ and let $\alpha(x) : x, \alpha_0, \alpha_1, \cdots$ be an augmented weight sequence. For $k \in \mathbb{N} \cup \{\infty\}$, we write $HE(\alpha, k)$ for the set of all positive real variable x such that $W_{\alpha(x)}$ is k-hyponormal (cf. [CuF2]). It follows from [Cu1] that if $\alpha(x) : x, \sqrt{\frac{2}{3}}, \sqrt{\frac{3}{4}}, \sqrt{\frac{4}{5}}, \cdots$, then there exists a sequence $\{\lambda_k\}_{k=1}^\infty$ of positive numbers with $\lim_{k\to\infty} \lambda_k = \sqrt{\frac{1}{2}}$ such that $\lambda_k > \lambda_{k+1}$ $(k \geq 1)$ and $HE(\alpha, k) = (0, \lambda_k]$, where $\lambda_1 = \sqrt{\frac{2}{3}}, \lambda_2 = \frac{3}{4}, \lambda_3 = \sqrt{\frac{8}{15}}, \lambda_4 = \sqrt{\frac{25}{48}}, \cdots$ and $HE(\alpha, \infty) = (0, \sqrt{\frac{1}{2}}\ ]$, which gives an example that distinguishes the classes of k-hyponormal operators from one another. In this paper, we obtain a formula that captures such examples.

Received by the editors January 22, 1999 and, in revised form, December 7, 1999.
2000 *Mathematics Subject Classification.* Primary 47B37.
Key words and phrases. Subnormal weighted shifts, k-hyponormal weighted shifts.
The first author was partially supported by KOSEF grant 971-0102-006-2 and the Korea Research Foundation made in the program year of 1998, 1998-015-D00019. The second author was partially supported by TGRC-KOSEF.

Note that for a unilateral weighted shift W_α with $\alpha_n = \alpha_{n+1}$ for some $n \in \mathbb{N} \cup \{0\}$, 2-hyponormality immediately forces the weight sequence α to be flat, that is, $\alpha_1 = \alpha_2 = \cdots$ (cf. [Cu1]). In [Sta], J. Stampfli had previously established this for subnormal shifts, so if the subnormal weighted shift is not flat, its weight sequence α satisfies $\alpha_0 < \alpha_1 < \alpha_2 < \cdots < \alpha_n < \cdots$. Throughout this paper we may assume that the subnormal weighted shift W_α satisfies $\alpha_0 < \alpha_1 < \alpha_2 < \cdots < \alpha_n < \cdots$ to escape the trivial case.

A weighted shift W_α is said to be *recursively generated* if there exist an integer $r \geq 1$ and a vector $\psi = (\psi_0, \cdots, \psi_{r-1}) \in \mathbb{C}^r$ such that $\gamma_n = \psi_{r-1}\gamma_{n-1} + \cdots + \psi_0\gamma_{n-r}$ $(n \geq r)$, where γ_n $(n \geq 0)$ is the moment of W_α, i.e., $\gamma_0:=1$, $\gamma_n:=\alpha_0^2 \cdots \alpha_{n-1}^2$ $(n \geq 1)$, equivalently,

$$\alpha_n^2 = \psi_{r-1} + \frac{\psi_{r-2}}{\alpha_{n-1}^2} + \cdots + \frac{\psi_0}{\alpha_{n-1}^2 \cdots \alpha_{n-r+1}^2} \qquad (n \geq r). \tag{1.1}$$

The smallest such integer r is called the *rank* of γ. A weighted shift W_α is *non-recursively generated* if it is not recursively generated. Note that a subnormal weighted shift is recursively generated if and only if the corresponding probability measure has finite support (cf. [ShT, p. 6] or [CuF1]).

For the moment sequence $\{\gamma_n\}_{n=0}^{\infty}$ of W_α, we denote

$$A(i,j) := \begin{bmatrix} \gamma_i & \gamma_{i+1} & \cdots & \gamma_{i+j} \\ \gamma_{i+1} & \gamma_{i+2} & \cdots & \gamma_{i+j+1} \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_{i+j} & \gamma_{i+j+1} & \cdots & \gamma_{i+2j} \end{bmatrix}.$$

If a subnormal weighted shift W_α is recursively generated and rank$\gamma = r$, then $\det A(i, r-1) \neq 0$ and $\det A(i,j) = 0$ for any $i \geq 1$, $j \geq r$. Note that if a subnormal weighted shift W_α is non-recursively generated, then $\det A(i,j) > 0$ for any positive integers i and j (cf. [CuF3]).

2. A formula for k-hyponormality

2.1. Non-recursively generated type. Let $\alpha : \alpha_0 < \alpha_1 < \alpha_2 < \cdots < \alpha_n < \cdots$ be a sequence of positive real numbers. Let $x > 0$ and let $\alpha(x) : x, \alpha_0, \alpha_1, \cdots$ be an augmented weight sequence. Assume that W_α is a non-recursively generated subnormal weighted shift. For brevity, let us put $t := \frac{1}{x^2}$. Then it follows from [Cu1, Theorem 4] that $W_{\alpha(x)}$ is k-hyponormal if and only if

$$D_k(t) := \begin{bmatrix} t & \gamma_0 & \gamma_1 & \cdots & \gamma_{k-1} \\ \gamma_0 & \gamma_1 & \gamma_2 & \cdots & \gamma_k \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \gamma_{k-1} & \gamma_k & \gamma_{k+1} & \cdots & \gamma_{2k-1} \end{bmatrix}$$

is non-negative, where $\gamma_0 := 1, \gamma_{n+1} := \alpha_n^2\gamma_n$ $(n \geq 0)$. Note that $d_k(t) := \det D_k(t)$ is a polynomial in t of degree 1. Since W_α is non-recursively generated subnormal, the coefficient of t in $d_k(t)$, $\det A(1, k-1)$, is positive. Hence $d_k(t)$ has a unique zero. We write $t_k := t_k(\alpha)$ for the unique zero of $d_k(t)$.

Theorem 2.1. *Let $\alpha : \alpha_0 < \alpha_1 < \alpha_2 < \cdots < \alpha_n < \cdots$ be a sequence of positive real numbers. Assume that W_α is a non-recursively generated subnormal weighted shift. Let $x > 0$ and let $\alpha(x) : x, \alpha_0, \alpha_1, \cdots$ be the associated augmented weight sequence. Let $t_k := t_k(\alpha)$ be the unique zero of $\det D_k(t)$, where $t := \frac{1}{x^2}$. Then*

$$t_{k+1}(\alpha) = t_k(\alpha) + \frac{[\det A(0,k)]^2}{\det A(1,k-1) \cdot \det A(1,k)} \tag{2.1}$$

for all $k = 1, 2, \cdots$.

For an $n \times n$ matrix $A = [a_{ij}]_{1 \le i,j \le n}$, we write

$$A\begin{pmatrix} i_1 & i_2 & \cdots & i_p \\ k_1 & k_2 & \cdots & k_p \end{pmatrix} = \begin{vmatrix} a_{i_1k_1} & a_{i_1k_2} & \cdots & a_{i_1k_p} \\ a_{i_2k_1} & a_{i_2k_2} & \cdots & a_{i_2k_p} \\ \vdots & \vdots & \ddots & \vdots \\ a_{i_pk_1} & a_{i_pk_2} & \cdots & a_{i_pk_p} \end{vmatrix}$$

for a minor of A of order p. We recall a fundamental result from [Gan, p. 22] as follows.

Lemma 2.2. *Let $A = [a_{ij}]_{1 \le i,j \le n}$ be an $n \times n$ matrix. Then*

$$\begin{aligned} \det A = \sum_{1 \le i_1 < i_2 < \cdots < i_p \le n} & A\begin{pmatrix} k_1 & k_2 & \cdots & k_p \\ i_1 & i_2 & \cdots & i_p \end{pmatrix} \cdot (-1)^{\sum_{v=1}^p i_v + \sum_{v=1}^p k_v} \\ & \cdot A\begin{pmatrix} k_1' & k_2' & \cdots & k_{n-p}' \\ i_1' & i_2' & \cdots & i_{n-p}' \end{pmatrix}, \end{aligned} \tag{2.2}$$

where $i_1 < i_2 < \cdots < i_p$ and $i_1' < i_2' < \cdots < i_{n-p}'$ form a complete system of indices $1, 2, \cdots, n$, as do $k_1 < k_2 < \cdots < k_p$ and $k_1' < k_2' < \cdots < k_{n-p}'$.

Proof of Theorem 2.1. Let $M_k^{(i)}$ be the $k \times k$ matrix obtained by removing the $(i+1)$-th column from the matrix

$$\begin{bmatrix} \gamma_0 & \gamma_1 & \gamma_2 & \cdots & \gamma_k \\ \gamma_1 & \gamma_2 & \gamma_3 & \cdots & \gamma_{k+1} \\ \vdots & \vdots & \vdots & & \vdots \\ \gamma_{k-1} & \gamma_k & \gamma_{k+1} & \cdots & \gamma_{2k-1} \end{bmatrix}$$

and let $d_k^{(i)} := \det M_k^{(i)}$ for $i = 0, 1, 2, \cdots, k$. Let us expand $d_k(t) = \det D_k(t)$ by the first row to obtain

$$d_k(t) = td_k^{(0)} - \gamma_0 d_k^{(1)} + \gamma_1 d_k^{(2)} - \cdots + (-1)^k \gamma_{k-1} d_k^{(k)}.$$

Let $t_k := t_k(\alpha)$ be the zero of $d_k(t)$. Since $d_k^{(0)} > 0$ for $k = 1, 2, \cdots$, we have

$$t_k = \gamma_0 \frac{d_k^{(1)}}{d_k^{(0)}} - \gamma_1 \frac{d_k^{(2)}}{d_k^{(0)}} + \cdots + (-1)^{k+1} \gamma_{k-1} \frac{d_k^{(k)}}{d_k^{(0)}} \quad (k \in \mathbb{N}).$$

Hence

$$
\begin{aligned}
d_k^{(0)} d_{k+1}^{(0)}(t_{k+1}-t_k) = {} & \gamma_0(d_k^{(0)} d_{k+1}^{(1)} - d_k^{(1)} d_{k+1}^{(0)}) - \gamma_1(d_k^{(0)} d_{k+1}^{(2)} - d_k^{(2)} d_{k+1}^{(0)}) \\
& + \cdots + (-1)^{k-1}\gamma_{k-1}(d_k^{(0)} d_{k+1}^{(k)} - d_k^{(k)} d_{k+1}^{(0)}) \\
& + (-1)^k \gamma_k d_k^{(0)} d_{k+1}^{(k+1)}.
\end{aligned} \tag{2.3}
$$

We first denote

$$
\widetilde{A}^{(i)}_{(2k+1)\times(2k+1)} := \left[\begin{array}{cccccc|c}
\gamma_0 & \gamma_1 & \cdots & \gamma_{i+1} & \cdots & \gamma_k & \begin{matrix} & \gamma_{k+1} \\ O_{(k+1)\times(k-1)} & \gamma_{k+2} \\ & \vdots \\ & \gamma_{2k+1} \end{matrix} \\
\gamma_1 & \gamma_2 & \cdots & \gamma_{i+2} & \cdots & \gamma_{k+1} & \\
\vdots & \vdots & & \vdots & & \vdots & \\
\gamma_k & \gamma_{k+1} & \cdots & \gamma_{k+i+1} & \cdots & \gamma_{2k} & \\
\hline
\gamma_0 & & & \gamma_{i+1} & & & \\
\gamma_1 & O_{k\times i} & & \gamma_{i+2} & O_{k\times(k-i-1)} & & B^{(i+1)}_{k+1} \\
\vdots & & & \vdots & & & \\
\gamma_{k-1} & & & \gamma_{i+k} & & &
\end{array}\right],
$$

where $O_{i\times j}$ is the $i \times j$ zero matrix and $B^{(i+1)}_{k+1}$ is the $k \times k$ submatrix obtained by removing the $(i+1)$-column from the matrix

$$
\begin{bmatrix}
\gamma_1 & \cdots & \gamma_{k+1} \\
\vdots & \ddots & \vdots \\
\gamma_k & \cdots & \gamma_{2k}
\end{bmatrix}
$$

$(i = 0, 1, \cdots, k)$. Let $v(i,j) = [\gamma_i, \gamma_{i+1}, \cdots, \gamma_{i+j}]^T$. Since the $k+2$ columns of the submatrix

$$
\begin{bmatrix}
\gamma_0 & \cdots & \gamma_k & \gamma_{k+1} \\
\vdots & \ddots & \vdots & \vdots \\
\gamma_k & \cdots & \gamma_{2k} & \gamma_{2k+1}
\end{bmatrix}
$$

in the upper half submatrix of the matrix $\widetilde{A}^{(i)}_{(2k+1)\times(2k+1)}$ are linearly dependent, there exist real numbers ϕ_i, $i = 0, 1, \cdots, k$, such that $v(k+1,k) = \phi_0 v(0,k) + \cdots + \phi_k v(k,k)$, which proves easily that the columns of $\widetilde{A}^{(i)}_{(2k+1)\times(2k+1)}$ are linearly dependent. Hence

$$
\det \widetilde{A}^{(i)}_{(2k+1)\times(2k+1)} = 0 \quad \text{for} \quad i = 0, 1, \cdots, k. \tag{2.4}
$$

For brevity we write $\widetilde{A} := \widetilde{A}^{(i)}_{(2k+1)\times(2k+1)}$. Then, applying Lemma 2.2 (using $\widetilde{A}$ and the last k rows of $\widetilde{A}$), we have that

$$\begin{aligned}
\det \widetilde{A} = &\sum_{1\le i_1<i_2<\cdots<i_k\le 2k+1} \widetilde{A}\begin{pmatrix} k+2 & \cdots & 2k+1 \\ i_1 & \cdots & i_k \end{pmatrix} \\
&\qquad \cdot (-1)^{\sum_{v=1}^{k} i_v + \sum_{v=k+2}^{2k+1} v} \cdot \widetilde{A}\begin{pmatrix} 1 & \cdots & k+1 \\ i_1^{'} & \cdots & i_{k+1}^{'} \end{pmatrix} \\
&\qquad \text{where } i_1 < i_2 < \cdots < i_k \text{ and } i_1^{'} < i_2^{'} < \cdots < i_{k+1}^{'} \\
&\qquad \text{form a complete system of indices } 1, 2, \cdots, 2k+1, \\
= &(-1)^{1+\sum_{v=k+2}^{2k} v + \sum_{v=k+2}^{2k+1} v} d_k^{(i+1)} d_{k+1}^{(0)} \\
&+ (-1)^{\sum_{v=k+2}^{2k+1} v + \sum_{v=k+2}^{2k+1} v} \det B_{k+1}^{(i+1)} \det A(0,k) \\
&+ (-1)^{(i+2)+\sum_{v=k+2}^{2k} v + \sum_{v=k+2}^{2k+1} v} \\
&\cdot \begin{vmatrix} \gamma_{i+1} & \gamma_1 & \cdots & \gamma_i & \gamma_{i+2} & \cdots & \gamma_k \\ \gamma_{i+2} & \gamma_2 & \cdots & \gamma_{i+1} & \gamma_{i+3} & \cdots & \gamma_{k+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \gamma_{k+i} & \gamma_k & \cdots & \gamma_{k+i-1} & \gamma_{k+i+1} & \cdots & \gamma_{2k-1} \end{vmatrix} \cdot d_{k+1}^{(i+1)} \\
= &d_k^{(i+1)} d_{k+1}^{(0)} + \det B_{k+1}^{(i+1)} \det A(0,k) \\
&- (-1)^{(i+2)} \begin{vmatrix} \gamma_{i+1} & \gamma_1 & \cdots & \gamma_i & \gamma_{i+2} & \cdots & \gamma_k \\ \gamma_{i+2} & \gamma_2 & \cdots & \gamma_{i+1} & \gamma_{i+3} & \cdots & \gamma_{k+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \gamma_{k+2} & \gamma_k & \cdots & \gamma_{k+i-1} & \gamma_{k+i+1} & \cdots & \gamma_{2k-1} \end{vmatrix} \cdot d_{k+1}^{(i+1)} \\
= &d_k^{(i+1)} d_{k+1}^{(0)} + \det B_{k+1}^{(i+1)} \det A(0,k) - d_k^{(0)} d_{k+1}^{(i+1)}.
\end{aligned}$$

Hence by (2.4) we have

$$d_k^{(0)} d_{k+1}^{(i+1)} - d_k^{(i+1)} d_{k+1}^{(0)} = \det A(0,k) \cdot \det B_{k+1}^{(i+1)} \quad (i = 0, \cdots, k-1). \tag{2.5}$$

Since $d_{k+1}^{(k+1)} = \det A(0,k)$, by (2.3) and (2.5) we have

$$\begin{aligned}
d_k^{(0)} d_{k+1}^{(0)} (t_{k+1} - t_k) &= [\det A(0,k)] \cdot [\sum_{i=0}^{k-1} (-1)^i \gamma_i \det B_{k+1}^{(i+1)} + (-1)^k \gamma_k d_k^{(0)}] \\
&= [\det A(0,k)]^2,
\end{aligned}$$

which proves the theorem. □

Since $\det A(1, k-1) > 0$ for $k = 1, 2, \cdots$, $d_k(t) \geq 0 \iff t \geq t_k(\alpha)$ for all $k = 1, 2, \cdots$. Since $\det A(0,k) > 0$ for $k = 1, 2, \cdots$ and $t_1(\alpha) = \frac{{\gamma_0}^2}{\gamma_1}$, by (2.1), $0 < t_1(\alpha) < t_2(\alpha) < \cdots$. Hence we have the following corollary.

Corollary 2.3. *Let $\alpha : \alpha_0 < \alpha_1 < \alpha_2 < \cdots < \alpha_n < \cdots$ be a sequence of positive real numbers. Let $x > 0$ and let $\alpha(x) : x, \alpha_0, \alpha_1, \cdots$ be the associated augmented weight sequence. Assume that W_α is a non-recursively subnormal weighted shift.*

Let $t_k := t_k(\alpha)$ be the unique zero of $\det D_k(t)$*, where* $t := \frac{1}{x^2}$*. Then for any* $k \in \mathbb{N}$*,*

$$HE(\alpha, k) \setminus HE(\alpha, k+1) = (\frac{1}{\sqrt{t_{k+1}(\alpha)}}, \frac{1}{\sqrt{t_k(\alpha)}}]. \tag{2.6}$$

In particular, $HE(\alpha, \infty) = \bigcap_{k=1}^{\infty}(0, \frac{1}{\sqrt{t_k(\alpha)}}]$.

2.2. Recursively generated type. Let W_α be a recursively generated subnormal weighted shift and let $\text{rank}\gamma = r$. Then $\det A(1, i) > 0$ for $i = 1, \cdots, r-1$ and $\det A(1, i) = 0$ for $i \geq r$. According to the proof of Theorem 2.1,

$$t_{k+1}(\alpha) = t_k(\alpha) + \frac{[\det A(0,k)]^2}{\det A(1, k-1) \cdot \det A(1, k)} \tag{2.7}$$

for all $k = 1, 2, \cdots, r-1$. Since rank $D_{k+1}(t) =$ rank $D_k(t)$ $(k \geq r)$ by [CuF1], we have that $D_{k+1}(t) \geq 0 \iff D_k(t) \geq 0$ $(k \geq r)$. So $t_{k+1}(\alpha) = t_k(\alpha)$ for all $k \geq r$. Hence we have the following proposition.

Proposition 2.4. *Assume W_α is a recursively generated subnormal weighted shift and let rank $\gamma = r$. Let $x > 0$ and let $\alpha(x) : x, \alpha_0, \alpha_1, \cdots$ be an augmented weight sequence. Let $t_k := t_k(\alpha)$ be the unique zero of* $\det D_k(t)$*, where $t := \frac{1}{x^2}$. Then we have*

(i) *for $p \leq r-1$,*

$$HE(\alpha, p) \setminus HE(\alpha, p+1) = (\frac{1}{\sqrt{t_{p+1}(\alpha)}}, \frac{1}{\sqrt{t_p(\alpha)}}],$$

(ii) *for any $p \geq r$, $HE(\alpha, p) = HE(\alpha, \infty)$.*

3. Examples

3.1. Non-recursively generated type. Given any non-recursively generated subnormal weighted shift W_α, by Theorem 2.1 and Corollary 2.3 the one step extension of W_α provides several examples to distinguish the classes of k-hyponormal operators. For example, we may recapture Curto's example [Cu1, Proposition 7] as follows.

Example 3.1. Let $\alpha_n := \sqrt{\frac{n+2}{n+3}}$ $(n \geq 0)$. It follows from [Cu1] (or Example 3.2) that $W_{\alpha(x)}$ is subnormal if and only if $0 < x \leq \sqrt{\frac{1}{2}}$. Since the support of Berger measure corresponded by W_α is not finite, W_α is non-recursively generated. Applying Theorem 2.1, we have $t_1 = \frac{3}{2}$, $t_2 = \frac{16}{9}$, $t_3 = \frac{15}{8}$, $t_4 = \frac{48}{25}$, $t_5 = \frac{35}{18}, \cdots$. Hence $HE(\alpha, k) \setminus HE(\alpha, k+1) = (\lambda_{k+1}, \lambda_k]$, where $\lambda_1 = \sqrt{\frac{2}{3}}$, $\lambda_2 = \frac{3}{4}$, $\lambda_3 = \sqrt{\frac{8}{15}}$, $\lambda_4 = \sqrt{\frac{25}{48}}$, $\lambda_5 = \sqrt{\frac{18}{35}}, \cdots$, and $HE(\alpha, \infty) = (0, \sqrt{\frac{1}{2}}]$.

Let W_α be a weighted shift whose restriction to $\vee\{e_1, e_2, \cdots\}$ is subnormal, with associated Berger measure μ. Then it follows from [Cu1, Proposition 8] that W_α is subnormal iff

$$\frac{1}{t} \in L^1(\mu) \quad \text{and} \quad \alpha_0^2 \cdot \left\|\frac{1}{t}\right\|_{L^1(\mu)} \leq 1. \tag{3.1}$$

In particular, W_α is never subnormal when $\mu(\{0\}) > 0$. The following example is useful when considering the behavior of Bergman shift extensions.

Example 3.2. Let

$$\alpha(x_1,\cdots,x_n): x_n,\cdots,x_1,\sqrt{\frac{m}{m+1}},\cdots,\sqrt{\frac{m+k-1}{m+k}},\cdots.$$

(i) If $1 \le n \le m-1$, then

$$\begin{aligned} HE(\alpha,\infty) &= \{(x_1,\cdots,x_n) | W_{\alpha(x_1,\cdots,x_n)} \text{ is subnormal}\} \\ &= \Big\{\Big(\sqrt{\frac{m-1}{m}},\sqrt{\frac{m-2}{m-1}},\cdots,\sqrt{\frac{m-n+1}{m-n+2}},x_n\Big)\Big| 0 < x_n \le \sqrt{\frac{m-n}{m-n+1}}\Big\}. \end{aligned}$$

(ii) If $n \ge m$, then $HE(\alpha,\infty)=\emptyset$.

Proof. (i) First we will find the range of x_1 needed for the subnormality of $W_{\alpha(x_1)}$. Let μ_1 be the probability measure corresponding to the subnormal weighted shift with the weights $\sqrt{\frac{m}{m+1}},\sqrt{\frac{m+1}{m+2}},\cdots,\sqrt{\frac{m+k-1}{m+k}},\cdots$. Then

$$\int_{[0,1]} t^k \, d\mu_1(t) = \frac{m}{m+1}\cdots\frac{m+k-1}{m+k} = \frac{m}{m+k} = \int_0^1 m t^{k+m-1}\, dt.$$

So $d\mu_1 = mt^{m-1}dt$. Since $\Big\|1/t\Big\|_{L^1(\mu_1)} = \frac{m}{m-1}$, by (3.1) we have $x_1 \le \sqrt{\frac{m-1}{m}}$. Let μ_2 be the probability measure corresponding to the weighted shift with the weights $x_1,\sqrt{\frac{m}{m+1}},\cdots,\sqrt{\frac{m+k-1}{m+k}},\cdots$. Since

$$x_1^2\cdot\frac{m}{m+1}\cdots\frac{m+k-2}{m+k-1} = \int_0^1 t^k\, d\mu_2,$$

using a method similar to that described above, we have $d\mu_2 = x_1^2 m t^{m-2}dt$. Furthermore, since $\mu_2[0,1]=1$, we have $x_1 = \sqrt{\frac{m-1}{m}}$. Hence $d\mu_2 = (m-1)t^{m-2}dt$. In general, let μ_i be the probability measure corresponding to the weighted shift with the weights $x_{i-1},\cdots,x_1,\sqrt{\frac{m}{m+1}},\cdots,\sqrt{\frac{m+k-1}{m+k}},\cdots$. Then it follows easily from mathematical induction that $d\mu_i = (m-i+1)t^{m-i}dt$, $x_i = \sqrt{\frac{m-i}{m-i+1}}$ $(1 \le i \le n-1)$ and $x_n \le \sqrt{\frac{m-n}{m-n+1}}$.

(ii) Let μ_n be the probability measure corresponding to the weighted shift with the weights $x_{n-1},\cdots,x_1,\sqrt{\frac{m}{m+1}},\cdots,\sqrt{\frac{m+k-1}{m+k}},\cdots$. Then

$$d\mu_n = (m-n+1)t^{m-n}dt.$$

Since $n \ge m$,

$$\int_0^1 \frac{1}{t} d\mu_n = (m-n+1)\cdot\int_0^1 t^{m-n-1}\, dt = \infty,$$

which implies that $\frac{1}{t} \notin L^1(\mu_n)$. Hence $HE(\alpha,\infty)=\emptyset$. □

Example 3.3. Let W_α be the weighted shift whose weight sequence is given by $\alpha_n := \sqrt{\frac{n+1}{n+2}}$ $(n \ge 0)$. By (ii) of Example 3.2, $W_{\alpha(x)}$ is not subnormal for any $x > 0$. By Theorem 2.1, we have $t_1 = 2,\ t_2 = 3,\ t_3 = \frac{11}{3},\ t_4 = \frac{25}{6},\ t_5 = \frac{137}{30},\cdots$. Hence $HE(\alpha,k)\setminus HE(\alpha,k+1) = (\lambda_{k+1},\lambda_k]$, where $\lambda_1 = \sqrt{\frac{1}{2}},\ \lambda_2 = \sqrt{\frac{1}{3}},\ \lambda_3 = \sqrt{\frac{3}{11}},\ \lambda_4 = \sqrt{\frac{6}{25}},\ \lambda_5 = \sqrt{\frac{30}{137}},\cdots$, and $HE(\alpha,\infty)=\emptyset$.

In addition, we consider an example of non-Bergman shift type.

Example 3.4. Let W_α be the weighted shift whose weight sequence is given by

$$\alpha_n := \sqrt{\frac{n+1}{n+2} \cdot \frac{1}{2} \cdot \frac{2^{n+2}-1}{2^{n+1}-1}} \quad (n \geq 0).$$

Let $d\mu := 2\chi_{[\frac{1}{2},1]}dt$. Since $\gamma_n = 2\int_{\frac{1}{2}}^1 t^n dt = \frac{1}{n+1} \cdot \frac{1}{2^n} \cdot (2^{n+1}-1)$, μ is the probability measure corresponding to the weighted shift with a weight sequence $\alpha := \{\alpha_n\}_{n=0}^\infty$. Hence W_α is subnormal and, by (3.1), $W_{\alpha(x)}$ is subnormal if and only if $x^2 \int_0^1 \frac{1}{t} d\mu \leq 1$, which is equivalent to $0 < x \leq \frac{1}{\sqrt{2\ln 2}}$. By Theorem 2.1, we have $t_1 = \frac{4}{3}$, $t_2 = \frac{18}{13}$, $t_3 = \frac{262}{189}$, $t_4 = \frac{445}{321}$, $t_5 = \frac{34997}{25245}, \cdots$, and $HE(\alpha, k) \setminus HE(\alpha, k+1) = (\lambda_{k+1}, \lambda_k]$, where $\lambda_1 = \sqrt{\frac{3}{4}} \approx 0.866025$, $\lambda_2 = \sqrt{\frac{13}{18}} \approx 0.849837$, $\lambda_3 = \sqrt{\frac{189}{262}} \approx 0.849337$, $\lambda_4 = \sqrt{\frac{321}{445}} \approx 0.849322$, $\lambda_5 = \sqrt{\frac{25245}{34997}} \approx 0.849322, \cdots$, and $HE(\alpha, \infty) = (0, \frac{1}{\sqrt{2\ln 2}}]$, with $\frac{1}{\sqrt{2\ln 2}} \approx 0.849322$.

3.2. Recursively generated type. Using Proposition 2.4, we can recapture a well-known result (cf. [CuF1], [CuF2] or [CuL]).

Example 3.5. We consider $\alpha : (a, b, c)^\wedge$, where $0 < a < b < c$. Since rank$\gamma = 2$, by (2.7) $t_1 = \frac{1}{a^2}$ and $t_2 = \frac{a^4 - 2a^2b^2 + b^2c^2}{a^2b^2(c^2-b^2)}$. Hence

(i) $HE(\alpha, 1) = \{x : 0 < x \leq a\}$,

(ii) $HE(\alpha, 2) = \cdots = HE(\alpha, \infty) = \{x : 0 < x \leq ab\sqrt{\frac{c^2-b^2}{a^4-2a^2b^2+b^2c^2}}\}$.

We close the paper with the following example.

Example 3.6. Let $\alpha : (a, b, c, d, e)^\wedge$ with $0 < a < b < c < d < e$ satisfying

$$\frac{b^2}{c^2} \cdot \frac{c^4 - 2a^2c^2 + a^2b^2}{b^2 - a^2} \leq d^2 \quad \text{and} \quad \frac{c^2}{d^2} \cdot \frac{d^4 - 2b^2d^2 + b^2c^2}{c^2 - b^2} \leq e^2. \tag{3.2}$$

Then by [Li, Corollary 2.12], W_α is subnormal. Assume that rank$\gamma = 3$. Then by (1.1),

$$\alpha_n^2 = \psi_2 + \frac{\psi_1}{\alpha_{n-1}^2} + \frac{\psi_0}{\alpha_{n-1}^2 \cdot \alpha_{n-2}^2} \qquad (n \geq 3)$$

with

$$\psi_0 = \frac{a^2b^2c^2(b^2c^4 - 2b^2c^2d^2 + c^2d^4 + b^2d^2e^2 - c^2d^2e^2)}{a^2b^4 - 2a^2b^2c^2 + b^2c^4 + a^2c^2d^2 - b^2c^2d^2},$$
$$\psi_1 = -\frac{b^2c^2(a^2b^2c^2 - a^2b^2d^2 - a^2c^2d^2 + c^2d^4 + a^2d^2e^2 - c^2d^2e^2)}{a^2b^4 - 2a^2b^2c^2 + b^2c^4 + a^2c^2d^2 - b^2c^2d^2},$$
$$\psi_2 = \frac{c^2(a^2b^4 - a^2b^2c^2 - a^2b^2d^2 + b^2c^2d^2 + a^2d^2e^2 - b^2d^2e^2)}{a^2b^4 - 2a^2b^2c^2 + b^2c^4 + a^2c^2d^2 - b^2c^2d^2}.$$

Hence by (2.7), $t_1 = \frac{1}{a^2}$, $t_2 = \frac{a^4-2a^2b^2+b^2c^2}{a^2b^2(c^2-b^2)}$, and $t_3 = \frac{A}{B}$, where

$$\begin{aligned} A = {} & a^4b^6 - 3a^4b^4c^2 + a^4b^2c^4 + 2a^2b^4c^4 + 2a^4b^2c^2d^2 - 2a^2b^4c^2d^2 - 2a^2b^2c^4d^2 \\ & + b^2c^4d^4 - a^4c^2d^2e^2 + 2a^2b^2c^2d^2e^2 - b^2c^4d^2e^2, \\ B = {} & a^2b^2c^2(b^2c^4 - 2b^2c^2d^2 + c^2d^4 + b^2d^2e^2 - c^2d^2e^2). \end{aligned}$$

Hence

(i) $HE(\alpha, 1) = \{x : 0 < x \leq a\}$,

(ii) $HE(\alpha, 2) = \{x : 0 < x \leq ab\sqrt{\frac{c^2-b^2}{a^4-2a^2b^2+b^2c^2}}\}$,

(iii) $HE(\alpha, 3) = \cdots = HE(\alpha, \infty) = \{x : 0 < x \leq \sqrt{\frac{B}{A}}\}$.

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