

SPLITTING SETS IN INTEGRAL DOMAINS

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(Communicated by Wolmer V. Vasconcelos)

ABSTRACT. Let D be an integral domain. A saturated multiplicatively closed subset S of D is a *splitting set* if each nonzero $d \in D$ may be written as $d = sa$ where $s \in S$ and $s'D \cap aD = s'aD$ for all $s' \in S$. We show that if S is a splitting set in D , then $SU(D_N)$ is a splitting set in D_N , N a multiplicatively closed subset of D , and that $S \subseteq D$ is a splitting set in $D[X] \iff S$ is an lcm splitting set of D , i.e., S is a splitting set of D with the further property that $sD \cap dD$ is principal for all $s \in S$ and $d \in D$. Several new characterizations and applications of splitting sets are given.

I. INTRODUCTION

Throughout this paper D denotes an integral domain with quotient field K . As usual, $D^* = D - \{0\}$ and $U(D)$ is the group of units of D .

A saturated multiplicatively closed subset S of D^* is said to be a *splitting set* if for each $d \in D^*$ we can write $d = sa$ for some $s \in S$ and $a \in D$ with $s'D \cap aD = s'aD$ for all $s' \in S$. A splitting set is said to be an *lcm splitting set* if for each $s \in S$ and $d \in D$, $sD \cap dD$ is principal.

Perhaps the most important example of an lcm splitting set is as follows. A set $\{p_\alpha\}$ of nonzero principal primes is a *splitting set of principal primes* if (a) for each α , $\bigcap_{n=1}^{\infty} p_\alpha^n D = 0$ (or equivalently, $\text{ht } p_\alpha D = 1$) and (b) for any sequence $\{p_{\alpha_n}\}$ of nonassociate members of $\{p_\alpha\}$, $\bigcap_{n=1}^{\infty} p_{\alpha_n} D = 0$. Then $S = \{up_{\alpha_1} \cdots p_{\alpha_n} \mid u \in U(D), p_{\alpha_i} \in \{p_\alpha\}, n \geq 0\}$ is an lcm splitting set. In fact, if D is an atomic integral domain (i.e., every nonzero nonunit of D is a product of irreducible elements), then a multiplicatively closed subset S of D is an lcm splitting set if and only if S is generated by a splitting set of principal primes ([1, Corollary 2.7]). Several characterizations of multiplicatively closed subsets generated by a set of splitting primes may be found in [1, Proposition 2.6]. Splitting sets of primes were probably first used by Nagata [11] to obtain his useful result that if D is Noetherian and S is a multiplicatively closed subset of D generated by principal primes with D_S a UFD, then D is a UFD.

Splitting sets were first considered by Mott and Schexnayder [10]. They gave a condition ([10, Proposition 4.1]), essentially (4) of Theorem 1.2, for a multiplicatively closed subset S of D to have $\langle S \rangle$, the subgroup $\{s_1 s_2^{-1} D \mid s_1, s_2 \in S\}$ of the group $P(D)$ of nonzero principal fractional ideals of D , to be a cardinal summand of $P(D)$. They showed that if S is generated by a splitting set of principal primes,

Received by the editors May 5, 1999 and, in revised form, December 18, 1999.

1991 *Mathematics Subject Classification*. Primary 13A05, 13A15, 13G05.

Key words and phrases. Splitting sets.

then $\langle S \rangle$ is a cardinal summand of $P(D)$. They then observed that this approach could be used to prove known results such as if D is a GCD domain (respectively, UFD), then $D[X]$ is a GCD domain (respectively, UFD); and certain ‘‘Nagata-type’’ theorems: if S is generated by a splitting set of principal primes and if D_S is a GCD domain (respectively, UFD), then D is a GCD domain (respectively, UFD). Other Nagata-type theorems using splitting sets of primes were given by Gabelli and Roitman [7] and Roitman [12]. Additional Nagata-type theorems were given in [2] where an alternative characterization for $\langle S \rangle$ being a cardinal summand was used, namely S being a splitting set. Splitting sets and lcm splitting sets were then considered in detail in [1]. Splitting sets have also been used by Mott [9] to compute the group of divisibility of a Rees ring and by D.F. Anderson et al. [5] to study elasticity of factorizations. Also, lcm splitting sets (not generated by principal primes) were first used by Gilmer and Parker [8] in their study of divisibility properties in semigroup rings.

As previously mentioned, the notion of a splitting set has been useful in studying factorizations and explaining why statements like Nagata’s theorem on UFD’s are possible in various situations. Our aim here is to bring together the various characterizations and applications of splitting sets and their variations and add a few more to the list.

In Section II we show that if S is an (lcm) splitting set of D , then $SU(D_N)$ is an (lcm) splitting set of D_N where N is a multiplicatively closed subset of D and that $S \subseteq D^*$ is a splitting set of $D[X]$ if and only if S is an lcm splitting set of D . Then several new characterizations and applications of splitting sets are given.

We say that $x, y \in D^*$ are v -coprime if $(x, y)_v = D$, or equivalently, $xD \cap yD = xyD$. Here as usual for a nonzero fractional ideal I of D , $I_v = (I^{-1})^{-1}$. Let X be a nonempty subset of D^* . We say that $t \in D^*$ is v -coprime to X if t is v -coprime to each $x \in X$. Then the set $T_X = \{t \in D^* \mid t \text{ is } v\text{-coprime to } X\}$ is easily seen to be a saturated multiplicatively closed subset of D . We refer to T_X as the m -complement of X and may drop the subscript if no confusion is possible. The following result relates v -coprimeness to overrings. As usual, for $s \in D^*$, $D_s = D_S$ where $S = \{s^n \mid n \geq 0\}$.

Proposition 1.1. *Let S and T be multiplicatively closed subsets of D . Then S and T are v -coprime if and only if $D = D_S \cap D_T$ and $s, t \in D^*$ are v -coprime if and only if $D = D_s \cap D_t$.*

Proof. (\Rightarrow) Let $0 \neq x \in D_S \cap D_T$. So there exist $s \in S$ and $t \in T$ with $sx, tx \in D$. Hence $D \supseteq (sx, tx)_v = x(s, t)_v = xD$, and hence $D = D_S \cap D_T$.

(\Leftarrow) Suppose $D = D_S \cap D_T$. Let $s \in S$ and $t \in T$. Then $sD \cap tD = s(D_S \cap D_T) \cap t(D_S \cap D_T) = sD_S \cap sD_T \cap tD_S \cap tD_T = sD_S \cap stD_T \cap stD_S \cap tD_T = stD_S \cap stD_T = st(D_S \cap D_T) = stD$. \square

A multiplicatively closed subset S of D with $U(D) \subseteq S$ is a splitting set if and only if for $T = T_S$, we have $ST = \{st \mid s \in S, t \in T\} = D^*$. Note that if S is a splitting set with m -complement T , then T is also a splitting set with S for its m -complement. The following two theorems from [1] give several characterizations of splitting sets and lcm splitting sets.

Theorem 1.2 ([1, Theorem 2.2]). *The following conditions are equivalent for a saturated multiplicatively closed subset of an integral domain D .*

- (1) S is a splitting set.

- (2) $\langle S \rangle$ is a cardinal summand of $P(D)$, that is, there is a subgroup H of $P(D)$ with $\langle S \rangle \oplus_c H = P(D)$.
- (3) If A is an integral principal ideal of D_S , then $A \cap D$ is a principal ideal of D .
- (4) There exists a multiplicatively closed subset T of D such that:
 - (a) each element $d \in D^*$ may be written as $d = st$ where $s \in S$ and $t \in T$, and
 - (b) one of the following equivalent four conditions holds:
 - (i) If $st = s't'$, where $s, s' \in S$ and $t, t' \in T$, then $s' = su$ and $t' = tu^{-1}$ where $u \in U(D)$.
 - (ii) If $d = st$ where $s \in S$ and $t \in T$, then $dD_S \cap D = tD$.
 - (iii) For each $s \in S$ and $t \in T$, $sD \cap tD = stD$.
 - (iv) For each $t \in T$, $tD_S \cap D = tD$.

Theorem 1.3 ([1, Proposition 2.4]). *The following conditions are equivalent for a splitting multiplicatively closed subset S of D .*

- (1) S is lcm splitting.
- (2) For $s_1, s_2 \in S$, $s_1D \cap s_2D$ is principal.
- (3) For $s_1, s_2 \in S$, $s_1D \cap s_2D = sD$ for some $s \in S$.
- (4) D_T is a GCD domain where T is the m -complement for S .

We follow the notation and terminology of [1]. For a nonzero fractional ideal I of D , $I_t = \bigcup \{(a_1, \dots, a_n)_v \mid a_1, \dots, a_n \in I - \{0\}\}$ and I is a t -ideal if $I = I_t$. The set of maximal t -ideals of D is denoted by $t\text{-max}(D)$. For $f = a_0 + a_1X + \dots + a_nX^n \in D[X]$, $A_f = (a_0, \dots, a_n)$, the content of f .

II. SPLITTING SETS

We begin this section by proving two stability theorems for splitting sets. Theorem 2.1 gives that if S is a splitting set of the integral domain D and N is a multiplicatively closed subset of D , then $SU(D_N)$ is a splitting set in D_N . Theorem 2.2 shows that a multiplicatively closed subset S of D is a splitting set for $D[X]$ if and only if S is an lcm splitting set in D .

Theorem 2.1. *Let D be an integral domain and S a splitting set in D with m -complement T . Let N be a multiplicatively closed subset of D . Then $\bar{S} = SU(D_N)$ is a splitting set in D_N with m -complement $\bar{T} = TU(D_N)$. Moreover, if S is an lcm splitting set in D , then \bar{S} is an lcm splitting set in D_N .*

Proof. Now \bar{S} and \bar{T} are multiplicatively closed subsets of D_N containing $U(D_N)$. Clearly $\bar{S}\bar{T} = D_N^*$. For $su_1 \in \bar{S}$ and $tu_2 \in \bar{T}$ where $s \in S$, $t \in T$ and $u_1, u_2 \in U(D_N)$, $su_1D_N \cap tu_2D_N = sD_N \cap tD_N = (sD \cap tD)_N = stD_N = (su_1)(tu_2)D_N$. So \bar{T} is contained in the m -complement of \bar{S} . By the paragraph after Proposition 1.1, \bar{S} is a splitting set with m -complement \bar{T} .

Suppose that S is an lcm splitting set in D . Then D_T is a GCD domain by Theorem 1.3. Hence $(D_N)_{\bar{T}}$ being a localization of D_T is a GCD domain. Again by Theorem 1.3, \bar{S} is an lcm splitting set in D_N . □

Theorem 2.2. *Let S be a multiplicatively closed subset of the integral domain D . If S is a splitting set in $D[X]$, then S is an lcm splitting set in D and in $D[X]$. Conversely, if S is an lcm splitting set of D with m -complement T , then S is an lcm splitting set of $D[X]$ with m -complement $T' = \{f \in D[X] \mid (A_f D_T)_t = D_T\}$.*

Proof. First, suppose S is a splitting set in $D[X]$. Let T' be the m -complement of S in $D[X]$. Clearly S is a splitting set in D . Let $s_0, s_1 \in S$. For some $s \in S$ and $f \in T'$ we have $s_0 + s_1X = sf$, say $f = t_0 + t_1X$. Let $a \in s_0D \cap s_1D$, thus $a = a_0s_0 = a_1s_1$ with $a_0, a_1 \in D$. Since $f \in T'$ and since $f = t_0 + t_1X$ divides $s_0s_1(a_1 + a_0X)$ in $D[X]$, we see that $t_0 + t_1X \mid a_1 + a_0X$ in $D[X]$. Hence $t_0 \mid a_1$ in D , which implies that $a \in uD$, where $u = t_0s_1 = t_1s_0 = st_0t_1$. We obtain that $s_0D \cap s_1D = uD$, hence $s_0D[X] \cap s_1D[X] = uD[X]$. By Theorem 1.3 we conclude that S is an lcm splitting set in D and in $D[X]$.

Suppose that S is an lcm splitting set of D with m -complement T . Now S is a saturated multiplicatively closed subset of $D[X]$. Since $D = D_S \cap D_T$, $D[X] = D_S[X] \cap D_T[X]$. Let $0 \neq f = a_0 + a_1X + \cdots + a_nX^n \in D[X]$. For each nonzero a_i , $a_i = s_it_i$ where $s_i \in S$ and $t_i \in T$. Now since S is an lcm splitting set, $s = \text{GCD}(\{s_i \mid s_i \neq 0\})$ exists and $sD = (\{s_i \mid s_i \neq 0\})_t$. Let $f = sf'$ where $f' = (\frac{s_0}{s})t_0 + (\frac{s_1}{s})t_1X + \cdots + (\frac{s_n}{s})t_nX^n$. Note that $(A_{f'})_t = ((\frac{s_0}{s})t_0, \dots, (\frac{s_n}{s})t_n)_t = ((\frac{s_0}{s}, \dots, \frac{s_n}{s})(t_0, \dots, t_n))_t = ((\frac{s_0}{s}, \dots, \frac{s_n}{s})_t(t_0, \dots, t_n))_t = (t_0, \dots, t_n)_t$ where the second equality follows from [1, Lemma 3.1]. Hence $(A_{f'}D_T)_t = (A_{f'})_tD_T = D_T$ where the first equality follows from [1, Corollary 3.5]. It remains to show that $s'D[X] \cap f'D[X] = s'f'D[X]$ for all $s' \in S$. It suffices to show that if $s \in S$ and $f \in D[X]$ with $(A_fD_T)_t = D_T$, then $sD[X] \cap fD[X] = sfD[X]$. Now since S is an lcm splitting set, D_T is a GCD domain. Since f is primitive in $D_T[X]$, $sD_T[X] \cap fD_T[X] = sfD_T[X]$. Hence $sD[X] \cap fD[X] = (sD_S[X] \cap sD_T[X]) \cap (fD_S[X] \cap fD_T[X]) = fD_S[X] \cap (sD_T[X] \cap fD_T[X]) = fD_S[X] \cap sfD_T[X] = sfD_S[X] \cap sfD_T[X] = sfD[X]$. So S is a splitting set in $D[X]$. Hence S is an lcm splitting set of $D[X]$. Let T' be the m -complement of S in $D[X]$. We claim that $T' = \{0 \neq f \in D[X] \mid (A_fD_T)_t = D_T\}$. We have already shown the \supseteq containment. Let $g \in T'$. We have shown that $g = sg'$ where $s \in S$ and $(A_{g'}D_T)_t = D_T$. Since $g = 1g = sg'$ where $1, s \in S$ and $g, g' \in T'$, 1 and s are associates. Hence s is a unit, so $(A_gD_T)_t = D_T$. \square

We next give another characterization of splitting sets.

Proposition 2.3. *Let S be a saturated multiplicatively closed subset of D and let T be the m -complement of S . Then S is a splitting set if and only if (1) the set ST is saturated and (2) every nonzero prime ideal P that is disjoint with S intersects T nontrivially.*

Proof. (\Rightarrow) Let S be a splitting set and suppose that T is the m -complement of S . Then as $ST = D^*$, ST is saturated. Next suppose that a nonzero prime ideal P is disjoint with S and let d be a nonzero element of P . Then as S is a splitting set, $d = st$ where $s \in S$ and $t \in T$. Since $P \cap S = \emptyset$, $s \notin P$, so $t \in P$.

(\Leftarrow) Suppose that S is a saturated multiplicatively closed subset of D and that (1) and (2) hold. We claim that $ST = D^*$. For if not, there is a nonzero $x \in D - ST$. Because ST is saturated, a standard argument leads to the conclusion that there is a prime ideal P containing x such that $P \cap ST = \emptyset$. This implies that $P \cap S = \emptyset$. However, by (2) P must intersect T nontrivially which means that $P \cap ST \neq \emptyset$, a contradiction. \square

An element x of D^* is called *primal* if in D , $x \mid ab$ implies that $x = rs$ where $r \mid a$ and $s \mid b$. An element p of D is called *completely primal* if every factor of p is primal. An integral domain with all nonzero elements primal is called *pre-Schreier* [13]. An integrally closed pre-Schreier domain was initially introduced as a *Schreier*

ring by Cohn [6]. In general if S and T are saturated multiplicatively closed sets, it is not necessary that ST should also be one. In fact, there is no relationship between S and T being saturated and ST being saturated. Let D be a Dedekind domain with class group $\text{Cl}(D) = \mathbb{Z}_2$ and nonprincipal maximal ideals M and N . Let $M^2 = aD$, $N^2 = bD$ and $MN = cD$. Then $S = \{ua^n \mid u \in U(D), n \geq 0\}$ and $T = \{ub^n \mid u \in U(D), n \geq 0\}$ are saturated multiplicatively closed subsets of D , but ST is not saturated. For $c^2D = (MN)^2 = M^2N^2 = abD$ gives $c^2 \in ST$ but $c \notin ST$. Conversely, if ST is saturated, then S or T may or may not be saturated, even in the domain \mathbb{Z} . Let $S = \{1\} \cup \{\pm 2^n\}_{n \geq 1}$ and $T = \{\pm 1\} \cup \{\pm 2^n\}_{n \geq 2}$, so $ST = \{\pm 2^n\}_{n \geq 0}$ is saturated while neither S nor T is saturated. If $S = \{2^n\}_{n \geq 0}$ and $T = \{\pm 3^m\}_{m \geq 0}$, T and $ST = \{\pm 2^n \cdot 3^m\}_{n,m \geq 0}$ are saturated, but S is not saturated. Also, if $S = \{\pm 2^n\}_{n \geq 0}$ and $T = \{\pm 3^n\}_{n \geq 0}$, then S , T , and ST are saturated. However, the following proposition gives a case where we can conclude that the product is saturated.

Proposition 2.4. *Let S be a saturated multiplicatively closed subset of D consisting of completely primal elements and let T be any saturated multiplicatively closed subset of D . Then ST is a saturated multiplicatively closed subset of D .*

Proof. Clearly ST is multiplicatively closed. Now let y be a factor of an element st of ST . Then there exists $r \in D$ such that $ry = st$. Since s is completely primal and $s \mid ry$, $s = uv$ where $u \mid r$ and $v \mid y$. Let $y = y_1v$ and $r = r_1u$. Substituting in the original equation and cancelling out the common factors from both sides we conclude that $y_1 \mid t$. □

Corollary 2.5. *Let S be a saturated multiplicatively closed subset of D consisting of completely primal elements. Then S is a splitting set if and only if for each prime P of D that is disjoint with S , P contains an element x such that x is v -coprime to S .*

Proof. Let T be the m -complement of S . As S and T are both saturated, so is ST by Proposition 2.4 and now Proposition 2.3 applies. □

Now obviously every prime element is completely primal and so are the so-called (lcm) extractors, i.e., elements r of D^* such that for each $x \in D$, $rD \cap xD$ is principal. There are a number of results in the literature that can be given somewhat different statements. Of these the following are worth mentioning. The reader may consult [1, Theorem 4.3] for Corollary 2.6 below. Note that the hypothesis of both corollaries gives that an element x is v -coprime to $S \iff x$ is coprime to S , i.e., $\text{GCD}(x, s) = 1$ for all $s \in S$.

Corollary 2.6. *Let S be a saturated multiplicatively closed subset of D consisting of extractors. If D_S is a GCD domain (respectively, PVMD) and every nonzero prime ideal of D that is disjoint from S contains an element coprime to S , then D is a GCD domain (respectively, PVMD).*

Corollary 2.7 (An alternative form of Nagata-like theorems). *Let S be a multiplicatively closed subset of D generated by primes. Suppose also that every nonzero prime ideal of D that is disjoint from S contains an element coprime to every element of S . If D_S is a UFD (respectively, PVMD, GCD domain, atomic domain, etc.), then so is D .*

For the proof of this corollary the reader may note that the hypothesis ensures that S is a splitting set. The rest has been amply established in the literature. The reader may consult [2, Theorem 3.1] for a quick reference/proof.

Corollary 2.7 provides a statement of Nagata's Theorem on UFD's that is worth mentioning because in this statement there is no reference to the ascending chain condition on principal ideals (ACCP) which is usually required for such results. Indeed, Corollary 2.7 also avoids any mention of the condition that S be a splitting set as used in [2].

In an effort to complete the picture we study the effect S being a splitting set has on the prime t -ideals of D . For this we take another look at [14]. Immediately after the statement of [14, Theorem 1] appears the following statement: "As a consequence of the proof of this theorem we conclude that if $D^{(S)} [= D + XD_S[X]]$ is a GCD domain and if P is a PF-prime of D with $P \cap D \neq \emptyset$ then P intersects S in detail". (Recall that P intersects S in detail if for every nonzero prime ideal $Q \subseteq P$, $Q \cap S \neq \emptyset$.) In the more general situation we do not have the luxury of PF-primes, but we do have the advantage of hindsight. Hence we have the following proposition.

Proposition 2.8. *Let S be a multiplicatively closed subset of D with the property that every nonzero prime P disjoint with S contains an element v -coprime to S . Then every prime t -ideal that intersects S intersects it in detail. Consequently, if S is a splitting set, then every prime t -ideal that intersects S , intersects S in detail.*

Proof. Suppose that M is a prime t -ideal such that $M \cap S \neq \emptyset$. Let $s \in M \cap S$ and suppose that there is a nonzero prime ideal Q contained in M such that $Q \cap S = \emptyset$. By the hypothesis there is $d \in Q$ such that d is v -coprime with S . But then $(s, d)_v = D$ and $s, d \in M$, contradicting the assumption that M is a t -ideal. \square

That the converse of Proposition 2.8 is false was established in [14, Example 2.6]. However, the hypothesis of Proposition 2.8 may play an important role in deciding whether a multiplicatively closed set is a splitting set. We now proceed to show that even that is not enough to establish that the set in question is a splitting set. Before we do that, however, it seems pertinent to give the hypothesis of Proposition 2.8 a name. A multiplicatively closed set S of an integral domain D will be said to have *property ** if for every nonzero prime t -ideal P with $P \cap S = \emptyset$, P contains an element v -coprime with S . It is routine to check that property $*$ and the hypothesis of Proposition 2.8 are equivalent. It is also routine to check that if S has property $*$, then S partitions $t\text{-max}(D)$ into two sets: $F = \{P \in t\text{-max}(D) \mid P \cap S = \emptyset\}$ and $G = \{Q \in t\text{-max}(D) \mid Q \cap S \neq \emptyset\} = \{Q \in t\text{-max}(D) \mid Q \text{ intersects } S \text{ in detail}\}$ such that for all $P \in F$ and for all $Q \in G$, P and Q do not contain a common nonzero prime ideal. (Note that $F = \emptyset$ precisely when $S = D^*$ while $G = \emptyset$ precisely when $S = U(D)$.) Now let us establish that even $*$ is not enough to force S to be a splitting set.

Example 2.9. Let D be a Dedekind domain with torsion class group. Then for every nonzero prime ideal P of D there is a smallest natural number n such that $P^n = (d)$. Next, as every power of (d) is P -primary, we conclude that $S = \{ud^m \mid u \in U(D), m \geq 0\}$ is a saturated multiplicatively closed set. Clearly S satisfies property $*$ because for every nonzero prime ideal Q with $Q \cap S = \emptyset$, and for some $m \in \mathbb{N}$, $Q^m = (e)$ where $e \notin P$ and hence is v -coprime to d . Now let $n > 1$. Then there is an atom $x \in P - \{0\}$ that is not P -primary. Obviously $x \notin S$ and being

an atom, x is incapable of being written as $x = st$ where $s \in S$ and t is v -coprime to S .

Now that we know that something less than a splitting set induces a partition of $t\text{-max}(D)$, we want to know what kind of partition of $t\text{-max}(D)$ is induced by a splitting set. The following theorem provides an answer.

Theorem 2.10. *If $S \neq D^*$, $U(D)$ is a splitting set of D , then S partitions $t\text{-max}(D)$ into $F = \{P \in t\text{-max}(D) \mid P \cap S = \emptyset\}$ and $G = \{Q \in t\text{-max}(D) \mid Q \text{ intersects } S \text{ in detail}\}$ such that for $P \in F$ and $Q \in G$ if a nonzero element x belongs to $P \cap Q$, then $x = x_1y_1$ where $x_1 \in P - X$ for every $X \in G$ and $y_1 \in Q - Y$ for every $Y \in F$. Conversely, let $S \neq D^*$, $U(D)$ be a saturated multiplicatively closed set and let $F = \{P \in t\text{-max}(D) \mid P \cap S = \emptyset\}$ and $G = \{Q \in t\text{-max}(D) \mid Q \text{ intersects } S \text{ in detail}\}$. Suppose $t\text{-max}(D) = F \cup G$ and if $0 \neq x \in P \cap Q$ where $P \in F$ and $Q \in G$, then $x = x_1y_1$ where $x_1 \in P - X$ for all $X \in G$ and $y_1 \in Q - Y$ for all $Y \in F$. Then S is a splitting set and $\{F, G\}$ is a partition of $t\text{-max}(D)$.*

Proof. Indeed, if S is a splitting set, then S satisfies $*$ and hence $t\text{-max}(D)$ is partitioned into F and G as described in the statement of the theorem. Now as S is a splitting set, given $P \in F$, $Q \in G$ and given $0 \neq x \in P \cap Q$, $x = x_1y_1$ where $y_1 \in S$ and x_1 is v -coprime to every member of S . Because two v -coprime elements cannot share a prime t -ideal, x_1 does not belong to any member of G . Hence $x_1 \in P - X$ for all $X \in G$. This also implies that $y_1 \in Q$. That $y_1 \in Q - Y$ for all $Y \in F$ follows from the fact that $y_1 \in S$.

Conversely, let $P \in F$. As $P \cap S = \emptyset$, $S \subseteq D - P$. So $S \subseteq \bigcap_{P \in F} (D - P) = D - \bigcup_{P \in F} P$ and as S is a saturated multiplicatively closed set, $S = D - \bigcup_{P \in F} P$. (For if not, there is a nonzero $x \in (D - \bigcup_{P \in F} P) - S$ and as S is saturated, a minimal prime ideal q of (x) that is disjoint with S . Now q being a t -ideal must be contained in one of the maximal t -ideals. Clearly as q is not contained in any members of F , q must be contained in some member of G . But every member of G intersects S in detail, a contradiction.)

Next let $T = D - \bigcup_{Q \in G} Q$ and consider ST . We claim that $ST = D^*$. For if not, let $x \in D^* - ST$. Then as $x \notin ST$, $x \notin S$ and $x \notin T$ and hence there is at least one $P \in F$ and at least one $Q \in G$ such that $x \in P \cap Q$. But then by the given property, $x = x_1y_1$ where $x_1 \in P - X$ for all $X \in G$ and $y_1 \in Q - Y$ for all $Y \in F$. Since $x_1 \in P - X$ implies that $x_1 \in D - X$ we conclude that $x_1 \in T = D - \bigcup_{Q \in G} Q$ and similarly $y_1 \in S = D - \bigcup_{P \in F} P$. But this implies that $x = x_1y_1 \in ST$ and thus $D^* = ST$. So every nonzero nonunit of D can be written as uv where $u \in S$ and $v \in T$. Now to complete the proof we must show that v is v -coprime to every element of S . For this let $s \in S$ and consider (s, v) . Noting that s is not in any member of F and v is not in any member of G and that $F \cup G = t\text{-max}(D)$ we conclude that s and v share no maximal t -ideal of D . Hence $(s, v)_v = (s, v)_t = D$. Thus we have established that S is a splitting set. That $\{F, G\}$ is a partition follows from the very definition of F and G . □

Corollary 2.11. *Let $S \neq D^*$, $U(D)$ be a splitting set of D . Then*

$$S = D - \bigcup \{P \in t\text{-max}(D) \mid P \cap S = \emptyset\}$$

with m -complement

$$T = D - \bigcup \{Q \in t\text{-max}(D) \mid Q \cap S \neq \emptyset\}.$$

Proof. This follows from the proof of Theorem 2.10. \square

Theorem 2.12. *Suppose that every nonzero nonunit of D belongs to only a finite number of maximal t -ideals and suppose that $t\text{-max}(D)$ is partitioned into two sets F and G such that for every $(P, Q) \in F \times G$, and for every nonzero $x \in P \cap Q$, $x = pq$ where $p \in P - Q$ and $q \in Q - P$. Then $S = D - (\bigcup_{P \in F} P)$ is a splitting set with m -complement $T = D - (\bigcup_{Q \in G} Q)$. Conversely, if $S \neq D^*$, $U(D)$ is a splitting set with m -complement T , then S induces a partition of $t\text{-max}(D)$ into two disjoint sets $F = \{P \in t\text{-max}(D) \mid P \cap S = \emptyset\}$ and $G = \{Q \in t\text{-max}(D) \mid Q \text{ intersects } S \text{ in detail}\}$ such that for every $(P, Q) \in F \times G$, and for every nonzero $x \in P \cap Q$, $x = pq$ where $p \in P - Q$ and $q \in Q - P$.*

Proof. Let F and G be as given and let x be a nonzero nonunit of D . Then x belongs to only a finite set of maximal t -ideals, say $x \in P_1, P_2, \dots, P_r, Q_1, Q_2, \dots, Q_s$, where the P_j 's $\in F$ and Q_i 's $\in G$. We show that $x \in ST$ by induction on $r+s$. If $r+s = 1$, the result is clear. If $r = 0$ or $s = 0$, clearly $x \in ST$. So suppose $r \geq 1$ and $s \geq 1$. Then by the condition, $x = x_1 y_1$ where $x_1 \in P_1 - Q_1$ and $y_1 \in Q_1 - P_1$. Now x_1 and y_1 are each contained in at most $r+s-1$ maximal t -ideals. So by induction $x_1, y_1 \in ST$. Hence $x = x_1 y_1 \in ST$. The converse is obvious. \square

Recall that an integral domain D is *weakly Krull* if $D = \bigcap_{P \in X^{(1)}} D_P$ where $X^{(1)} = X^{(1)}(D)$ is the set of height-one primes of D and the intersection is locally finite. Thus a Cohen-Macaulay integral domain is an atomic weakly Krull domain.

Corollary 2.13. *Let D be an atomic weakly Krull domain and let $\{F, G\}$ be a partition of $X^{(1)}(D)$. Then $S = D - (\bigcup_{P \in F} P)$ is a splitting set if and only if no member of F shares an atom with any member of G .*

In [4, Theorem] we showed that for an integral domain D the following conditions are equivalent: (1) each saturated multiplicatively closed subset of D is a splitting set, (2) for each prime ideal P of D , $S = D - P$ is a splitting set, and (3) D is weakly Krull and $\text{Cl}_t(D) = 0$. Here $\text{Cl}_t(D)$ denotes the t -class group of D —the group of t -invertible t -ideals of D modulo the subgroup of principal fractional ideals. Since according to Corollary 2.11 a splitting set $S \neq D^*$ of D must be the complement of a union of maximal t -ideals, it is of interest to characterize the integral domains for which each such multiplicatively closed set is a splitting set. Several conditions equivalent to (3) of Theorem 2.14 are given in [3, Corollary 4.4].

Theorem 2.14. *For an integral domain D the following conditions are equivalent.*

- (1) *For each nonempty collection F of maximal t -ideals of D , $S = D - \bigcup_{P \in F} P$ is a splitting set.*
- (2) *For each maximal t -ideal P of D , $S = D - P$ is a splitting set.*
- (3) *$D = \bigcap_{P \in t\text{-max}(D)} D_P$ has finite character, for distinct $P, Q \in t\text{-max}(D)$, $P \cap Q$ contains no nonzero prime ideal, and $\text{Cl}_t(D) = 0$.*

Proof. (1) \Rightarrow (2) Clear. (2) \Rightarrow (3) Let $P \in t\text{-max}(D)$. Since $S = D - P$ is a splitting set, by Theorem 1.2 $x D_P \cap D$ is principal for each $x \in D$. The implication now follows from [3, Corollary 4.4]. (3) \Rightarrow (1) Suppose that D satisfies the hypotheses of (3). Let F be as in (1) and let $G = t\text{-max}(D) - F$. We may assume $G \neq \emptyset$ for $G = \emptyset$ gives $S = U(D)$. We apply Theorem 2.12. Let $0 \neq x \in P \cap Q$ where $(P, Q) \in F \times G$. By [3, Corollary 4.4] we can write $x = x_1 \cdots x_n$ where each x_i is contained in a unique maximal t -ideal P_i (for by [3, Lemma 4.2] if x_i is P_i -pure

where $P_i \in t\text{-max}(D)$, then P_i is the unique maximal t -ideal containing x_i). Let $p = \prod \{x_i \notin Q\}$ and $q = \prod \{x_i \mid x_i \notin P\}$. Then $x = pq$ where $p \in P - Q$ and $q \in Q - P$. \square

ACKNOWLEDGEMENT

We would like to thank the referee for several observations that simplified the proofs of Theorems 2.1 and 2.2.

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