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INEQUALITIES FOR PRODUCTS OF SPECTRAL RADII

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Abstract. It is shown that submultiplicative inequalities for spectral radii often imply supermultiplicative inequalities, and vice versa.

1. Introduction

The spectral radius of a linear transformation on a finite-dimensional complex space, or of a bounded linear operator on a complex Banach space, or, more generally, of an element of a complex Banach algebra, is the supremum of the moduli of numbers in the spectrum of the element. The famous spectral radius formula of Gelfand states that the spectral radius, $r(A)$, is given by

$$r(A) \;=\; \lim_{n\to\infty} \|A^n\|^{\frac{1}{n}}.$$

In this note we consider semigroups (i.e., subsets closed under multiplication) of operators or elements of a Banach algebra on which the spectral radius is k-submultiplicative, in the sense that there is a positive real number k such that $r(AB) \le kr(A)r(B)$ for all A and B in the semigroup. Our main results give sufficient conditions under which k-submultiplicativity implies that

$$r(AB) \geq \frac{1}{k} r(A)r(B) \quad \text{for all } A \text{ and } B.$$

2. A general result in Banach algebras

Recall that $r(AB) = r(BA)$ for any elements of a Banach algebra (in fact, the spectra of AB and BA differ at most by $\{0\}$). Our basic lemmas hold for any non-negative functions agreeing on AB and BA.

Lemma 1. *If $k > 0$ and f is a non-negative function on a semigroup satisfying $f(AB) = f(BA)$ and $f(AB) \le kf(A)f(B)$ for all A, B, then $f(A^nB^n) \le k^{n-1}(f(AB))^n$ for every positive integer n.*

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Proof. The proof is by induction on n. The case $n = 1$ is trivial, so assume the inequality for $n - 1$. Then

$$\begin{aligned} f(A^nB^n) &= f(AA^{n-1}B^{n-1}B) \\ &= f(A^{n-1}B^{n-1}BA) \\ &\leq kf(A^{n-1}B^{n-1})f(BA) \\ &\leq kk^{n-2}(f(AB))^{n-1}f(AB) \\ &= k^{n-1}(f(AB))^n. \end{aligned}$$

□

Lemma 2. *If $\varepsilon > 0$ and f is a non-negative function on a semigroup satisfying $f(AB) = f(BA)$ and $f(AB) \geq \varepsilon f(A)f(B)$ for all A, B, then $f(A^nB^n) \geq \varepsilon^{n-1}(f(AB))^n$ for every positive integer n.*

Proof. The proof of this lemma is exactly the same as the preceding one except that the inequalities are reversed. □

Lemma 3. *If f satisfies the hypothesis of Lemma 1 (respectively, Lemma 2), and if A and B are elements of the semigroup such that $\{f(A^nB^n)\}$ is bounded and bounded away from 0, then $f(AB) \geq \frac{1}{k}$ $\left(\text{respectively}, f(AB) \leq \frac{1}{\varepsilon}\right)$.*

Proof. Suppose f satisfies the hypothesis of Lemma 1. Choose a subsequence so that $f(A^{n_i}B^{n_i})$ converges to some t; then $t > 0$. Since

$$f(A^{n_i}B^{n_i}) \ \leq \ k^{n_i-1}(f(AB))^{n_i},$$

taking n_ith roots gives

$$(f(A^{n_i}B^{n_i}))^{\frac{1}{n_i}} \ \leq \ k^{1-\frac{1}{n_i}}f(AB).$$

Taking the limit as $\{n_i\} \to \infty$ yields

$$1 \ \leq \ kf(AB).$$

The proof of the consequence of Lemma 2 is exactly the same except that the inequalities are reversed. □

Theorem 1. *If $\mathcal{S}$ is a semigroup contained in a Banach algebra and there exist $\varepsilon > 0$ and $k > 0$ such that $\varepsilon r(A)r(B) \leq r(AB) \leq kr(A)r(B)$ for all $\{A, B\} \subset \mathcal{S}$, then*

$$\frac{1}{k}r(A)r(B) \ \leq \ r(AB) \ \leq \ \frac{1}{\varepsilon}r(A)r(B)$$

for all $\{A, B\} \subset \mathcal{S}$.

Proof. The inequalities of the hypothesis clearly extend to the semigroup $\mathbb{R}^+\mathcal{S} = \{tS : t \geq 0, S \in \mathcal{S}\}$.

Fix A and B in $\mathbb{R}^+\mathcal{S}$; since multiplying each of A and B by positive numbers does not change any of the inequalities, we can assume that $r(A) = r(B) = 1$. We must then show that $\frac{1}{k} \leq r(AB) \leq \frac{1}{\varepsilon}$.

By Lemma 3, it suffices to show that $r(A^nB^n)$ is bounded and is bounded away from 0.

By hypothesis,

$$\varepsilon r(A^n)r(B^n) \leq r(A^nB^n) \leq kr(A^n)r(B^n)$$

for all n. Since $r(A) = r(B) = 1$, $r(A^n) = r(B^n) = 1$, and

$$\varepsilon \leq r(A^n B^n) \leq k$$

for all n. □

Corollary 1. *Let $\mathcal{S}$ be a semigroup contained in a Banach algebra. If r is submultiplicative on $\mathcal{S}$ and if there is an $\varepsilon > 0$ such that $r(AB) \geq \varepsilon r(A)r(B)$ for all A and B in $\mathcal{S}$, then r is multiplicative on $\mathcal{S}$.*

Proof. Submultiplicativity means that $k = 1$ in the hypothesis of Theorem 1, and $\frac{1}{1} = 1$. □

3. Linear transformations on finite-dimensional spaces

In the finite-dimensional case, the existence of an ε such that $r(AB) \geq \varepsilon r(A)r(B)$ can be inferred under various hypotheses.

Theorem 2. *Let $\mathcal{S}$ be a semigroup of linear transformations on a finite-dimensional space which is closed under multiplication by positive real numbers and is also closed in the topological sense. If $\mathcal{S}$ has no non-zero divisors and $r(AB) \leq kr(A)r(B)$ for all A and B in $\mathcal{S}$, then $r(AB) \geq \frac{1}{k}r(A)r(B)$ for all A and B in $\mathcal{S}$.*

Proof. This will follow from Theorem 1 if we show the existence of an $\varepsilon > 0$ such that $r(AB) \geq \varepsilon r(A)r(B)$, for every A and B in S. If there were no such ε, then for every positive integer n there would be A_n and B_n in $\mathcal{S}$ satisfying

$$r(A_n B_n) \; < \; \frac{1}{n} r(A_n) r(B_n).$$

Then

$$r\left(\frac{A_n}{\|A_n\|}\frac{B_n}{\|B_n\|}\right) \; < \; \frac{1}{n} r\left(\frac{A_n}{\|A_n\|}\right) r\left(\frac{B_n}{\|B_n\|}\right).$$

Since the unit sphere of the space of linear transformations is compact, there are transformations E and F and an increasing sequence of positive integers $\{n_i\}$ such that $\left\{\frac{A_{n_i}}{\|A_{n_i}\|}\right\} \to E$ and $\left\{\frac{B_{n_i}}{\|B_{n_i}\|}\right\} \to F$. It follows that $r(EF) = 0$ (spectral radius is continuous in finite-dimensions). Since EF is nilpotent it is a zero divisor and hence $EF = 0$. If $EF = 0$, then, since $\|E\| = \|F\| = 1$, E and F are non-zero zero divisors, which contradicts the hypothesis. □

Definition. We say that *multiplication is bounded below* on a subset $\mathcal{S}$ of a Banach algebra if there is an $\varepsilon > 0$ such that $\|AB\| \geq \varepsilon \|A\| \|B\|$ for all A and B in $\mathcal{S}$.

Corollary 2. *Let $\mathcal{S}$ be a semigroup of linear transformations on a finite-dimensional space, on which multiplication is bounded below. If there exists a k such that*

$$r(AB) \; \leq \; kr(A)r(B) \quad \textit{for all } A \textit{ and } B \textit{ in } \mathcal{S}, \textit{ then}$$

$$r(AB) \; \geq \; \tfrac{1}{k} r(A)r(B) \quad \textit{for all } A \textit{ and } B \textit{ in } \mathcal{S}.$$

Proof. Let $\mathcal{T}$ be the closure of

$$\{tS : t \geq 0, \; S \in \mathcal{S}\}.$$

Then the inequality of the hypothesis extends to the semigroup $\mathcal{T}$ so the corollary will follow from Theorem 2 if we show that $\mathcal{T}$ has no non-zero zero divisors. Note that

$$\|AB\| \geq \varepsilon \|A\| \|B\|$$

for all A and B in $\mathcal{S}$ implies the same for all A and B in $\mathcal{T}$, so $\mathcal{T}$ does not have non-zero zero divisors. □

Corollary 3. *If multiplication is bounded below on a semigroup $\mathcal{S}$ of linear transformations on a finite-dimensional space, and if spectral radius is submultiplicative on $\mathcal{S}$, then spectral radius is multiplicative on $\mathcal{S}$.*

Proof. $\frac{1}{1} = 1$. □

There is a result analogous to Theorem 2 for the reverse inequality.

Theorem 3. *Let $\mathcal{S}$ be a semigroup of linear transformations on a finite-dimensional space which is closed under multiplication by positive real numbers and is also closed in the topological sense. If $\mathcal{S}$ has no non-zero zero divisors and there is an $\varepsilon > 0$ such that $r(AB) \geq \varepsilon r(A)r(B)$ for all A and B in $\mathcal{S}$, then $r(AB) \leq \frac{1}{\varepsilon} r(A)r(B)$ for all A and B in $\mathcal{S}$.*

Proof. It suffices to show that there is some k such that $r(AB) \leq kr(A)r(B)$ for all A and B (by Theorem 1). If not, then for every k there are A_k and B_k in $\mathcal{S}$ satisfying

$$r(A_k B_k) > kr(A_k)r(B_k)$$

with $r(A_k)$ and $r(B_k) \neq 0$. For each k, let $C_k = \frac{A_k}{r(A_k)}$ and $D_k = \frac{B_k}{r(B_k)}$.

If $\{C_k\}$ was not bounded in norm, $\left\{\frac{C_k}{\|C_k\|}\right\}$ would have a subsequence which converged to a linear transformation of norm 1 and spectral radius 0, contradicting the lack of zero divisors in $\mathcal{S}$. Thus $\{C_k\}$ and $\{D_k\}$ are bounded sequences, and so therefore is $\{C_k D_k\}$. But $r(C_k D_k) > k$ for each k, which is a contradiction. □

Theorem 3 has corollaries similar to those of Theorem 2.

4. Semigroups of compact operators

The results of section 3 do not appear to extend to semigroups of compact operators. However, a theorem of [1] can be extended to give a result in the case of irreducible semigroups (i.e., semigroups with no non-trivial invariant (closed) subspaces).

Theorem 4. *If $\mathcal{S}$ is an irreducible semigroup of compact operators on Hilbert space and if there exists $k > 0$ such that $r(AB) \leq kr(A)r(B)$ for all A and B in $\mathcal{S}$, then $r(AB) \geq \frac{1}{k} r(A)r(B)$ for all A and B in $\mathcal{S}$.*

Proof. The proof is a very small modification of the proof of the case $k = 1$ given in Theorem 2.1 of [1]. As in [1], we assume that $\mathcal{S}$ is closed under multiplication by positive numbers and is also closed in the topological sense. Also as in [1], we first consider the case where $A^2 = A$ and $B^2 = B$ and $r(A) = r(B) = 1$. Then

$$r(AB) = r(A^2B^2) = r(BA\,AB) \leq k(r(AB))^2.$$

So, since $r(AB) \neq 0$ (as in [1]), it follows that

$$\frac{1}{k} \leq r(AB).$$

For the general case where $r(A) = r(B) = 1$, choose, as in [1], increasing sequences of positive integers and scalars a and b of modulus 1 such that $\{aA^{n_j}\} \to P$ and $\{bB^{n_j}\} \to Q$ with P and Q non-zero idempotents.

Following [1] but inserting "k" gives

$$\begin{aligned} r(A^{n_j}B^{n_j}) &= r(AB\,B^{n_j-1}A^{n_j-1}) \\ &\le kr(AB)r(B^{n_j-1}A^{n_j-1}) \\ &\le k^2r(AB)r(B^{n_j-1})r(A^{n_j-1}) \\ &= k^2r(AB). \end{aligned}$$

Taking limits yields

$$r(PQ) \le k^2r(AB).$$

Since $\frac{1}{k} \le r(PQ)$ by the first part of this proof, it follows that

$$\frac{1}{k^3} \le r(AB).$$

This holds for all A and B in $\mathcal{S}$ of spectral radius 1. Thus

$$\frac{1}{k^3}r(A)r(B) \le r(AB) \le kr(A)r(B)$$

for all A and B in $\mathcal{S}$. Theorem 1 then gives $\frac{1}{k}r(A)r(B) \le r(AB)$. □

5. An example

Let $\mathcal{S}$ be the semigroup consisting of all positive integral multiples of

$$A = \begin{pmatrix} 2 & 0 \\ 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 2 \\ 0 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix}, \quad D = \begin{pmatrix} 0 & 1 \\ 0 & 2 \end{pmatrix}.$$

Direct multiplication shows that $\mathcal{S}$ is a semigroup, and the inequalities

$$\frac{1}{4}r(E)r(F) \le r(EF) \le 4r(E)r(F)$$

hold and are sharp for $\mathcal{S}$.

References

1. W. Longstaff and H. Radjavi, On permutability and submultiplicativity of spectral radius, *Can. J. Math.* **47** (1995), 1007-1022. MR **97d:**47005

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