

## REAL GROUPS TRANSITIVE ON COMPLEX FLAG MANIFOLDS

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ABSTRACT. Let  $Z = G/Q$  be a complex flag manifold. The compact real form  $G_u$  of  $G$  is transitive on  $Z$ . If  $G_0$  is a noncompact real form, such transitivity is rare but occasionally happens. Here we work out a complete list of Lie subgroups of  $G$  transitive on  $Z$  and pick out the cases that are noncompact real forms of  $G$ .

### 0. THE PROBLEM

Let  $Z = G/Q$  be a complex flag manifold where  $G$  is a complex connected semisimple Lie group and  $Q$  is a parabolic subgroup. Let  $G_0$  be a real form of  $G$ . If  $G_0$  is the compact real form, then it is transitive on  $Z$ . On a number of occasions the question has come up as to whether any noncompact real form of  $G$  can be transitive on  $Z$ . Here I'll record the answer. The rough answer is "yes, but just a few." The precise answer, Corollaries 1.7 and 2.3 below, follows from a more general classification, Theorems 1.6 and 2.2. This more general classification uses a technique of D. Montgomery [M], together with some results of [W1] that depend in an essential way on a classification [O1] of A. L. Onishchik.

After this paper was written I learned of Onishchik's book [O2]. There is some overlap for compact groups, but there are no inclusions.

### 1. THE SOLUTION FOR IRREDUCIBLE FLAGS

We formulate the problem in terms of transitive subgroups. Let  $G_u$  be the compact real form of  $G$ , so  $Z = G_u/(G_u \cap Q)$  and  $G_u \cap Q$  is the compact real form of the reductive part of  $Q$ . Let  $A \subset G$  be a closed subgroup that is transitive on  $Z$ . The identity component  $A^0$  of  $A$  is transitive on  $Z$ , because  $Z$  is connected, so a maximal compact subgroup  $B^0 \subset A^0$  already is transitive on  $Z$ , according to Montgomery [M]. We may replace  $A$  by a conjugate and assume  $B = A \cap G_u$ . So

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now we have several expressions:

$$(1.1) \quad \begin{aligned} Z = G/Q &= G_u/(G_u \cap Q) = A/(A \cap Q) = B/(B \cap Q) \\ &= A^0/(A^0 \cap Q) = B^0/(B^0 \cap Q). \end{aligned}$$

According to [W1, Prop. 3.1] there are just a few possibilities for a homogeneous almost-hermitian manifold  $Z$  to have distinct expressions such as  $G_u/L_u$  and  $B^0/(B^0 \cap L_u)$ , where  $G_u$  is the identity component of the group of all almost-hermitian isometries,  $G_u$  is simple,  $L_u$  is the centralizer of a torus subgroup of  $G_u$ , and  $B^0 \subsetneq G_u$  with  $B^0$  connected. They are :

$$(1.2) \quad Z = P^{2n-1}(\mathbb{C}) = SU(2n)/U(2n-1) = Sp(n)/(Sp(n-1) \cdot U(1)), \text{ complex projective space,}$$

$$(1.3) \quad Z = SO(2r+2)/U(r+1) = SO(2r+1)/U(r), \text{ unitary structures on } \mathbb{R}^{2r+2},$$

$$(1.4) \quad Z = SO(7)/(SO(5) \cdot SO(2)) = G_2/U(2), \text{ 5-dimensional complex quadric, and}$$

$$(1.5) \quad Z = SO(8)/(SO(6) \cdot SO(2)) = \{Spin(7)/Z_2\}/U(3), \text{ 6-dimensional complex quadric.}$$

This applies in our situation because  $L_u = G_u \cap Q$  is the centralizer of a torus subgroup of  $G_u$ , and  $Z$  has a  $G_u$ -invariant hermitian metric.

Now return to the expression  $Z = G/Q$ .  $G$  (and thus  $G_u$ ) is simple. Let  $A \subsetneq G$  be a closed subgroup that is transitive on  $Z$  and let  $B$  be its maximal compact subgroup. We may assume  $B = A \cap G_u$ . Then  $B \subsetneq G_u$ ,  $B^0$  is transitive on  $Z$ , and the expression  $Z = G_u/L_u = B^0/(B^0 \cap L_u)$  is given above. In each case the group  $B^0$  is simple, so  $A^0$  has Levi decomposition  $A^0 = A_{ss}^0 A_{rad}^0$  into semisimple part and solvable radical, where  $B^0$  is a maximal compact subgroup of  $A_{ss}^0$ . We run through the 4 possibilities listed above.

For (1.2),  $G = SL(2n; \mathbb{C})$  and  $B^0 = Sp(n)$ . The semisimple Lie groups with maximal compact subgroup  $Sp(n)$  are  $Sp(n), Sp(n; \mathbb{C})$ , the quaternionic linear group  $SL(n; \mathbb{H})$ , and, for  $n = 4$ , the real group  $F_{4,C_4}$ . But  $F_4$  does not have a representation of degree 8, in other words  $F_4 \not\subset G$ , so now  $A_{ss}^0$  is one of  $Sp(n), Sp(n; \mathbb{C})$  and  $SL(n; \mathbb{H})$ . Each of them is irreducible on  $\mathbb{C}^{2n}$ , so the unipotent radical of the algebraic hull of  $A^0$  acts trivially on  $\mathbb{C}^{2n}$  and the center of the reductive part of  $A^0$  acts by scalars. As  $G$  acts effectively and by transformations of determinant 1 on  $\mathbb{C}^{2n}$  now  $A_{ss}^0 = A^0$ , so  $A^0$  is one of  $Sp(n), Sp(n; \mathbb{C})$  and  $SL(n; \mathbb{H})$ . If  $g \in G$  normalizes  $A^0$ , then some element  $g' \in gA^0$  centralizes  $A^0$ , because  $A^0$  has no rational outer automorphism. As  $A^0$  is irreducible on  $\mathbb{C}^{2n}$  now  $g'$  is scalar (and thus acts trivially on  $Z$ ). Thus  $A = A^0 F$  where  $F$  can be any subgroup of the center  $\{e^{2\pi i k/2n} I \mid 0 \leq k < 2n\}$  of  $G$ .

For (1.3),  $G = SO(2r+2; \mathbb{C})$  and  $B^0 = SO(2r+1)$ . The semisimple Lie groups with maximal compact subgroup  $SO(2r+1)$  are  $SO(2r+1), SO(2r+1; \mathbb{C}), SO(1, 2r+1)$ , and  $SL(2r+1; \mathbb{R})$ . But  $A_{ss}^0 = SL(2r+1; \mathbb{R})$  would give  $SL(2r+1; \mathbb{C}) \subset SO(2r+2; \mathbb{C})$ , so the respective dimensions would satisfy  $4r^2 + 4r \leq 2r^2 + 3r + 1$ , forcing  $r = 0$  and  $Z = (\text{point})$ . Thus<sup>1</sup>  $A_{ss}^0 \neq SL(2r+1; \mathbb{R})$ . Now  $A_{ss}^0$  is one of  $SO(2r+1), SO(2r+1; \mathbb{C}),$  and  $SO(1, 2r+1)$ . The last one acts irreducibly on  $\mathbb{C}^{2r+2}$ , and there  $A_{ss}^0 = A^0$  as above. For the first two, recall that  $SO(2r+1)$  is absolutely irreducible on the tangent space  $\mathfrak{so}(2r+2)/\mathfrak{so}(2r+1)$  of the sphere  $S^{2r+1}$ , so  $A_{rad}^0$  has Lie algebra reduced to 0, and again  $A_{ss}^0 = A^0$ . Now  $A^0$  is one of  $SO(2r+1), SO(2r+1; \mathbb{C}),$  and  $SO(1, 2r+1)$ . If  $g \in G$  normalizes  $A^0$ , then some

<sup>1</sup> The author thanks the referee for a comment that improved and clarified his treatment of this  $SL(2r+1; \mathbb{R})$  case.

element  $g' \in gA^0$  centralizes  $A^0$ , because  $A^0$  has no rational outer automorphism. Thus either  $A = A^0$  or  $A/A^0$  has order 2 where  $A$  is one of  $O(2r + 1)$ ,  $O(2r + 1; \mathbb{C})$ , and  $SO(1, 2r + 1) \cdot \{\pm I\}$ .

For (1.4),  $G = SO(7; \mathbb{C})$  and  $B^0 = G_2$ . The semisimple Lie groups with maximal compact subgroup  $G_2$  are  $G_2$  and its complexification  $G_{2,\mathbb{C}}$ . They are irreducible on  $\mathbb{C}^7$  and have no rational outer automorphisms, so, as before,  $A^0$  is either  $G_2$  or  $G_{2,\mathbb{C}}$ , and if  $g \in G$  normalizes  $A^0$ , then some element  $g' \in gA^0$  centralizes  $A^0$ . This forces  $g'$  to be central in  $SO(7; \mathbb{C})$ , so  $g' = 1$  and  $A = A^0$ . Thus  $A$  is either  $G_2$  or  $G_{2,\mathbb{C}}$ .

Finally, (1.5) is obtained from the case  $r = 3$  of (1.3) by applying the triality automorphism, so it does not give us anything more.

In summary,

**Theorem 1.6.** *Consider a complex flag manifold  $Z = G/Q$ . Suppose that  $Z$  is irreducible, i.e., that  $G$  is simple. Then the closed subgroups  $A \subset G$  transitive on  $Z$ ,  $G_u \neq A \neq G$ , are precisely those given as follows:*

1.  $Z = SU(2n)/U(2n - 1) = P^{2n-1}(\mathbb{C})$  complex projective  $(2n - 1)$ -space;  $G = SL(2n; \mathbb{C})$  and  $A = A^0F$  where  $A^0$  is one of  $Sp(n)$ ,  $Sp(n; \mathbb{C})$  and  $SL(n; \mathbb{H})$ , and  $F$  is any subgroup of the center  $\{e^{2\pi ik/2n} I \mid 0 \leq k < 2n\}$  of  $G$ . Here  $F$  acts trivially on  $Z$ , so  $A$  and  $A^0$  have the same action on  $Z$ .

2.  $Z = SO(2r + 2)/U(r + 1)$ , unitary structures on  $\mathbb{R}^{2r+2}$ ;  $G = SO(2r + 2; \mathbb{C})$  and  $A = A^0F$  where  $A^0$  is one of  $SO(2r + 1)$ ,  $SO(2r + 1; \mathbb{C})$ , and  $SO(1, 2r + 1)$ , and where  $F$  is any subgroup of the center  $\{\pm I\}$  of  $G$ . Here  $F$  acts trivially on  $Z$ , so  $A$  and  $A^0$  have the same action on  $Z$ .

3.  $Z = SO(7)/(SO(5) \cdot SO(2))$ , 5-dimensional complex quadric;  $G = SO(7; \mathbb{C})$  and  $A$  is either the compact connected group  $G_2$  or its complexification  $G_{2,\mathbb{C}}$ .

Picking out the cases where  $A$  is a real form of  $G$  we have

**Corollary 1.7.** *Consider a complex flag manifold  $Z = G/Q$ . Suppose that  $Z$  is irreducible, i.e., that  $G$  is simple. Then the (connected) noncompact real forms  $G_0 \subset G$  transitive on  $Z$  are precisely those given as follows:*

1.  $Z = SU(2n)/U(2n - 1) = P^{2n-1}(\mathbb{C})$  complex projective  $(2n - 1)$ -space;  $G = SL(2n; \mathbb{C})$  and  $G_0$  is the quaternion linear group  $SL(n; \mathbb{H})$ , which has maximal compact subgroup  $Sp(n)$ .

2.  $Z = SO(2r + 2)/U(r + 1)$ , unitary structures on  $\mathbb{R}^{2r+2}$ ;  $G = SO(2r + 2; \mathbb{C})$  and  $G_0$  is the Lorentz group  $SO(1, 2r + 1)$ , which has maximal compact subgroup  $SO(2r + 1)$ .

## 2. THE SOLUTION FOR FLAG MANIFOLDS IN GENERAL

We complete the solution of the problem by reducing it to the case where  $Z$  is irreducible.

**Proposition 2.1.** *Decompose  $G = \prod G_i$ , the local direct product of complex connected simple Lie groups. Thus  $Z = \prod Z_i$ , the product of irreducible flag manifolds  $Z_i = G_i/Q_i$  where  $Q_i = Q \cap G_i$ . Then  $A^0 = \prod A_i^0$  with  $A_i^0 = A^0 \cap G_i$  and  $B^0 = \prod B_i^0$  with  $B_i^0 = B^0 \cap G_i$ . The groups  $A_i^0$  and  $B_i^0$  are connected, simple, and transitive on  $Z_i$ .*

*Proof.* The solvable radical of  $A^0$  is contained in a Borel subgroup of  $G$ , and thus has a fixed point on  $Z$ . It is normal in the transitive group  $A^0$  so it fixes every point. Thus  $A^0$  is semisimple. Similarly  $B^0$  is semisimple.

Let  $\pi_i : G \rightarrow G_i$  denote the projection. The compact connected group  $\pi_i(B^0)$  is transitive on  $Z_i$ . So it must be the compact real form  $G_{u,i} = G_i \cap G_u$  of  $G_i$  or one of the compact connected transitive groups described in (1.2), (1.3) or (1.4). (Recall that (1.5) is in fact a special case of (1.3).) In all cases,  $\pi_i(B^0)$  is nontrivial and simple. Now  $\pi_i$  annihilates all but one of the simple factors of  $B^0$ . Obviously no simple factor of  $B^0$  is annihilated by every  $\pi_i$ . So now  $B^0 = \prod B_\alpha^0$  where the  $B_\alpha^0$  are simple and where the index set  $I$  for  $G = \prod_I G_i$  is a disjoint union of subsets  $I_\alpha$  with  $B_\alpha^0 \subset \prod_{i \in I_\alpha} G_i$ . The proof of Proposition 2.1 is reduced to the case where  $B^0$  (and thus also  $A^0$ ) is simple, and there it is reduced to the proof that  $G_u$  is simple.

We may now assume  $B^0$  simple. Suppose that  $G_u$  is not simple. Projecting to  $G_1 \times G_2$  we may assume  $G = G_1 \times G_2$ . View the isomorphisms  $\pi_i : B^0 \cong \pi_i(B^0)$  as identifications. Denote  $E_i = \pi_i(B_\mathbb{C}^0)$ , the complexification of the image of  $B^0$  in  $G_i$ . Denote  $E_{u,1} = \pi_i(B^0)$ , the compact real form of  $E_i$ . Denote  $P_i = E_i \cap Q_i$ , the parabolic subgroup of  $E_i$  that is its isotropy subgroup in  $Z_i$ , so  $Z_i = E_i/P_i$ . Now  $B_\mathbb{C}^0 = \{(e, e) \mid e \in E_1\}$ ,  $B_\mathbb{C}^0 \cap Q = \{(p, p) \mid p \in (P_1 \cap P_2)\}$ , and  $Z = B_\mathbb{C}^0/(B_\mathbb{C}^0 \cap Q) \cong E_1/(P_1 \cap P_2)$ . In particular  $P_1 \cap P_2$  is a parabolic subgroup of  $E_1$ . Compute complex dimensions:  $\dim E_1 - \dim(P_1 \cap P_2) = \dim B^0 - \dim(B^0 \cap Q) = \dim Z = \dim Z_1 + \dim Z_2 = (\dim E_1 - \dim P_1) + (\dim E_1 - \dim P_2)$ . On the Lie algebra level this says  $\dim \mathfrak{e}_1 = \dim \mathfrak{p}_1 + \dim \mathfrak{p}_2 - \dim(\mathfrak{p}_1 \cap \mathfrak{p}_2)$ , in other words  $\mathfrak{p}_1 + \mathfrak{p}_2 = \mathfrak{e}_1$ . As  $\mathfrak{p}_1 \cap \mathfrak{p}_2$  is a parabolic subalgebra of  $\mathfrak{e}_1$  we have a Cartan subalgebra  $\mathfrak{h}$  and a Borel subalgebra  $\mathfrak{s}$  with  $\mathfrak{h} \subset \mathfrak{s} \subset \mathfrak{p}_1 \cap \mathfrak{p}_2$ . In the root order such that  $\mathfrak{s}$  is the sum of  $\mathfrak{h}$  and the negative root spaces, no parabolic containing  $\mathfrak{s}$  can contain the root space for the maximal root. This contradicts  $\mathfrak{p}_1 + \mathfrak{p}_2 = \mathfrak{e}_1$ . The contradiction proves  $G_u$  simple and completes the proof.  $\square$

Combining Proposition 2.1 with Theorem 1.6 we have

**Theorem 2.2.** *Let  $Z = G/Q$ , the complex flag manifold, where  $G$  is a complex connected semisimple Lie group acting with finite kernel on  $Z$ . Then the closed subgroups  $A \subset G$  transitive on  $Z$  are precisely those given as follows. Decompose  $G = \prod G_i$  with  $G_i$  simple, so  $Z = \prod Z_i$  with  $Z_i = G_i/(Q \cap G_i)$ . Then  $A = A^0 F$  where  $A^0 = \prod A_i$  with  $A_i = (A \cap G_i)^0$ , and  $A_i$  is equal to  $G_i$ , or to its compact real form  $G_{u,i}$ , or to one of the three types listed in Theorem 1.6, and  $F$  is any subgroup of the center of  $G$ . Here  $F$  acts trivially on  $Z$ , so  $A$  and  $A^0$  have the same action on  $Z$ .*

Picking out the cases where  $A$  is a real form of  $G$  we have, as in Corollary 1.7,

**Corollary 2.3.** *Let  $Z = G/Q$ , the complex flag manifold, where  $G$  is a complex connected semisimple Lie group acting with finite kernel on  $Z$ . Then the (connected) real forms  $G_0 \subset G$  transitive on  $Z$  are precisely those given as follows. Decompose  $G = \prod G_i$  with  $G_i$  simple, so  $Z = \prod Z_i$  with  $Z_i = G_i/(Q \cap G_i)$ . Then  $A = \prod A_i$  where  $A_i = A \cap G_i$  either is the compact real form  $G_{u,i}$  of  $G_i$  or is one of the two types listed in Corollary 1.7.*

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