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REAL GROUPS TRANSITIVE ON COMPLEX FLAG MANIFOLDS

JOSEPH A. WOLF

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ABSTRACT. Let $Z = G/Q$ be a complex flag manifold. The compact real form G_u of G is transitive on Z. If G_0 is a noncompact real form, such transitivity is rare but occasionally happens. Here we work out a complete list of Lie subgroups of G transitive on Z and pick out the cases that are noncompact real forms of G.

0. THE PROBLEM

Let $Z = G/Q$ be a complex flag manifold where G is a complex connected semisimple Lie group and Q is a parabolic subgroup. Let G_0 be a real form of G. If G_0 is the compact real form, then it is transitive on Z. On a number of occasions the question has come up as to whether any noncompact real form of G can be transitive on Z. Here I'll record the answer. The rough answer is "yes, but just a few." The precise answer, Corollaries 1.7 and 2.3 below, follows from a more general classification, Theorems 1.6 and 2.2. This more general classification uses a technique of D. Montgomery [M], together with some results of [W1] that depend in an essential way on a classification [O1] of A. L. Onishchik.

After this paper was written I learned of Onishchik's book [O2]. There is some overlap for compact groups, but there are no inclusions.

1. THE SOLUTION FOR IRREDUCIBLE FLAGS

We formulate the problem in terms of transitive subgroups. Let G_u be the compact real form of G, so $Z = G_u/(G_u \cap Q)$ and $G_u \cap Q$ is the compact real form of the reductive part of Q. Let $A \subset G$ be a closed subgroup that is transitive on Z. The identity component A^0 of A is transitive on Z, because Z is connected, so a maximal compact subgroup $B^0 \subset A^0$ already is transitive on Z, according to Montgomery [M]. We may replace A by a conjugate and assume $B = A \cap G_u$. So

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now we have several expressions:

$$\begin{aligned} Z = G/Q &= G_u/(G_u \cap Q) = A/(A \cap Q) = B/(B \cap Q) \\ &= A^0/(A^0 \cap Q) = B^0/(B^0 \cap Q). \end{aligned} \tag{1.1}$$

According to [W1, Prop. 3.1] there are just a few possibilities for a homogeneous almost–hermitian manifold Z to have distinct expressions such as G_u/L_u and $B^0/(B^0 \cap L_u)$, where G_u is the identity component of the group of all almost–hermitian isometries, G_u is simple, L_u is the centralizer of a torus subgroup of G_u, and $B^0 \subsetneqq G_u$ with B^0 connected. They are :

(1.2) $Z = P^{2n-1}(\mathbb{C}) = SU(2n)/U(2n-1) = Sp(n)/(Sp(n-1) \cdot U(1))$, complex projective space,

(1.3) $Z = SO(2r+2)/U(r+1) = SO(2r+1)/U(r)$, unitary structures on $\mathbb{R}^{2r+2}$,

(1.4) $Z = SO(7)/(SO(5) \cdot SO(2)) = G_2/U(2)$, 5–dimensional complex quadric, and

(1.5) $Z = SO(8)/(SO(6) \cdot SO(2)) = \{Spin(7)/Z_2\}/U(3)$, 6–dimensional complex quadric.

This applies in our situation because $L_u = G_u \cap Q$ is the centralizer of a torus subgroup of G_u, and Z has a G_u–invariant hermitian metric.

Now return to the expression $Z = G/Q$. G (and thus G_u) is simple. Let $A \subsetneqq G$ be a closed subgroup that is transitive on Z and let B be its maximal compact subgroup. We may assume $B = A \cap G_u$. Then $B \subsetneqq G_u$, B^0 is transitive on Z, and the expression $Z = G_u/L_u = B^0/(B^0 \cap L_u)$ is given above. In each case the group B^0 is simple, so A^0 has Levi decomposition $A^0 = A^0_{ss}A^0_{rad}$ into semisimple part and solvable radical, where B^0 is a maximal compact subgroup of A^0_{ss}. We run through the 4 possibilities listed above.

For (1.2), $G = SL(2n;\mathbb{C})$ and $B^0 = Sp(n)$. The semisimple Lie groups with maximal compact subgroup $Sp(n)$ are $Sp(n), Sp(n;\mathbb{C})$, the quaternionic linear group $SL(n;\mathbb{H})$, and, for $n = 4$, the real group F_{4,C_4}. But F_4 does not have a representation of degree 8, in other words $F_4 \not\subset G$, so now A^0_{ss} is one of $Sp(n), Sp(n;\mathbb{C})$ and $SL(n;\mathbb{H})$. Each of them is irreducible on $\mathbb{C}^{2n}$, so the unipotent radical of the algebraic hull of A^0 acts trivially on $\mathbb{C}^{2n}$ and the center of the reductive part of A^0 acts by scalars. As G acts effectively and by transformations of determinant 1 on $\mathbb{C}^{2n}$ now $A^0_{ss} = A^0$, so A^0 is one of $Sp(n), Sp(n;\mathbb{C})$ and $SL(n;\mathbb{H})$. If $g \in G$ normalizes A^0, then some element $g' \in gA^0$ centralizes A^0, because A^0 has no rational outer automorphism. As A^0 is irreducible on $\mathbb{C}^{2n}$ now g' is scalar (and thus acts trivially on Z). Thus $A = A^0F$ where F can be any subgroup of the center $\{e^{2\pi ik/2n}I \mid 0 \leqq k < 2n\}$ of G.

For (1.3), $G = SO(2r+2;\mathbb{C})$ and $B^0 = SO(2r+1)$. The semisimple Lie groups with maximal compact subgroup $SO(2r+1)$ are $SO(2r+1), SO(2r+1;\mathbb{C}), SO(1,2r+1)$, and $SL(2r+1;\mathbb{R})$. But $A^0_{ss} = SL(2r+1;\mathbb{R})$ would give $SL(2r+1;\mathbb{C}) \subset SO(2r+2;\mathbb{C})$, so the respective dimensions would satisfy $4r^2+4r \leqq 2r^2+3r+1$, forcing $r = 0$ and $Z =$ (point). Thus[1] $A^0_{ss} \neq SL(2r+1;\mathbb{R})$. Now A^0_{ss} is one of $SO(2r+1), SO(2r+1;\mathbb{C})$, and $SO(1,2r+1)$. The last one acts irreducibly on $\mathbb{C}^{2r+2}$, and there $A^0_{ss} = A^0$ as above. For the first two, recall that $SO(2r+1)$ is absolutely irreducible on the tangent space $\mathfrak{so}(2r+2)/\mathfrak{so}(2r+1)$ of the sphere S^{2r+1}, so A^0_{rad} has Lie algebra reduced to 0, and again $A^0_{ss} = A^0$. Now A^0 is one of $SO(2r+1), SO(2r+1;\mathbb{C})$, and $SO(1,2r+1)$. If $g \in G$ normalizes A^0, then some

[1] The author thanks the referee for a comment that improved and clarified his treatment of this $SL(2r+1;\mathbb{R})$ case.

element $g' \in gA^0$ centralizes A^0, because A^0 has no rational outer automorphism. Thus either $A = A^0$ or A/A^0 has order 2 where A is one of $O(2r+1), O(2r+1;\mathbb{C})$, and $SO(1,2r+1)\cdot\{\pm I\}$.

For (1.4), $G = SO(7;\mathbb{C})$ and $B^0 = G_2$. The semisimple Lie groups with maximal compact subgroup G_2 are G_2 and its complexification $G_{2,\mathbb{C}}$. They are irreducible on $\mathbb{C}^7$ and have no rational outer automorphisms, so, as before, A^0 is either G_2 or $G_{2,\mathbb{C}}$, and if $g \in G$ normalizes A^0, then some element $g' \in gA^0$ centralizes A^0. This forces g' to be central in $SO(7;\mathbb{C})$, so $g' = 1$ and $A = A^0$. Thus A is either G_2 or $G_{2,\mathbb{C}}$.

Finally, (1.5) is obtained from the case $r = 3$ of (1.3) by applying the triality automorphism, so it does not give us anything more.

In summary,

Theorem 1.6. *Consider a complex flag manifold $Z = G/Q$. Suppose that Z is irreducible, i.e., that G is simple. Then the closed subgroups $A \subset G$ transitive on Z, $G_u \neq A \neq G$, are precisely those given as follows:*

1. *$Z = SU(2n)/U(2n-1) = P^{2n-1}(\mathbb{C})$ complex projective $(2n-1)$–space; $G = SL(2n;\mathbb{C})$ and $A = A^0F$ where A^0 is one of $Sp(n)$, $Sp(n;\mathbb{C})$ and $SL(n;\mathbb{H})$, and F is any subgroup of the center $\{e^{2\pi ik/2n}I \mid 0 \leqq k < 2n\}$ of G. Here F acts trivially on Z, so A and A^0 have the same action on Z.*
2. *$Z = SO(2r+2)/U(r+1)$, unitary structures on $\mathbb{R}^{2r+2}$; $G = SO(2r+2;\mathbb{C})$ and $A = A^0F$ where A^0 is one of $SO(2r+1)$, $SO(2r+1;\mathbb{C})$, and $SO(1,2r+1)$, and where F is any subgroup of the center $\{\pm I\}$ of G. Here F acts trivially on Z, so A and A^0 have the same action on Z.*
3. *$Z = SO(7)/(SO(5)\cdot SO(2))$, 5–dimensional complex quadric; $G = SO(7;\mathbb{C})$ and A is either the compact connected group G_2 or its complexification $G_{2,\mathbb{C}}$.*

Picking out the cases where A is a real form of G we have

Corollary 1.7. *Consider a complex flag manifold $Z = G/Q$. Suppose that Z is irreducible, i.e., that G is simple. Then the (connected) noncompact real forms $G_0 \subset G$ transitive on Z are precisely those given as follows:*

1. *$Z = SU(2n)/U(2n-1) = P^{2n-1}(\mathbb{C})$ complex projective $(2n-1)$–space; $G = SL(2n;\mathbb{C})$ and G_0 is the quaternion linear group $SL(n;\mathbb{H})$, which has maximal compact subgroup $Sp(n)$.*
2. *$Z = SO(2r+2)/U(r+1)$, unitary structures on $\mathbb{R}^{2r+2}$; $G = SO(2r+2;\mathbb{C})$ and G_0 is the Lorentz group $SO(1,2r+1)$, which has maximal compact subgroup $SO(2r+1)$.*

2. The solution for flag manifolds in general

We complete the solution of the problem by reducing it to the case where Z is irreducible.

Proposition 2.1. *Decompose $G = \prod G_i$, the local direct product of complex connected simple Lie groups. Thus $Z = \prod Z_i$, the product of irreducible flag manifolds $Z_i = G_i/Q_i$ where $Q_i = Q \cap G_i$. Then $A^0 = \prod A_i^0$ with $A_i^0 = A^0 \cap G_i$ and $B^0 = \prod B_i^0$ with $B_i^0 = B^0 \cap G_i$. The groups A_i^0 and B_i^0 are connected, simple, and transitive on Z_i.*

Proof. The solvable radical of A^0 is contained in a Borel subgroup of G, and thus has a fixed point on Z. It is normal in the transitive group A^0 so it fixes every point. Thus A^0 is semisimple. Similarly B^0 is semisimple.

Let $\pi_i : G \to G_i$ denote the projection. The compact connected group $\pi_i(B^0)$ is transitive on Z_i. So it must be the compact real form $G_{u,i} = G_i \cap G_u$ of G_i or one of the compact connected transitive groups described in (1.2), (1.3) or (1.4). (Recall that (1.5) is in fact a special case of (1.3).) In all cases, $\pi_i(B^0)$ is nontrivial and simple. Now π_i annihilates all but one of the simple factors of B^0. Obviously no simple factor of B^0 is annihilated by every π_i. So now $B^0 = \prod B^0_\alpha$ where the B^0_α are simple and where the index set I for $G = \prod_I G_i$ is a disjoint union of subsets I_α with $B^0_\alpha \subset \prod_{i \in I_\alpha} G_i$. The proof of Proposition 2.1 is reduced to the case where B^0 (and thus also A^0) is simple, and there it is reduced to the proof that G_u is simple.

We may now assume B^0 simple. Suppose that G_u is not simple. Projecting to $G_1 \times G_2$ we may assume $G = G_1 \times G_2$. View the isomorphisms $\pi_i : B^0 \cong \pi_i(B^0)$ as identifications. Denote $E_i = \pi_i(B^0_{\mathbb{C}})$, the complexification of the image of B^0 in G_i. Denote $E_{u,1} = \pi_i(B^0)$, the compact real form of E_i. Denote $P_i = E_i \cap Q_i$, the parabolic subgroup of E_i that is its isotropy subgroup in Z_i, so $Z_i = E_i/P_i$. Now $B^0_{\mathbb{C}} = \{(e,e) \mid e \in E_1\}$, $B^0_{\mathbb{C}} \cap Q = \{(p,p) \mid p \in (P_1 \cap P_2)\}$, and $Z = B^0_{\mathbb{C}}/(B^0_{\mathbb{C}} \cap Q) \cong E_1/(P_1 \cap P_2)$. In particular $P_1 \cap P_2$ is a parabolic subgroup of E_1. Compute complex dimensions: $\dim E_1 - \dim(P_1 \cap P_2) = \dim B^0 - \dim(B^0 \cap Q) = \dim Z = \dim Z_1 + \dim Z_2 = (\dim E_1 - \dim P_1) + (\dim E_1 - \dim P_2)$. On the Lie algebra level this says $\dim \mathfrak{e}_1 = \dim \mathfrak{p}_1 + \dim \mathfrak{p}_2 - \dim(\mathfrak{p}_1 \cap \mathfrak{p}_2)$, in other words $\mathfrak{p}_1 + \mathfrak{p}_2 = \mathfrak{e}_1$. As $\mathfrak{p}_1 \cap \mathfrak{p}_2$ is a parabolic subalgebra of $\mathfrak{e}_1$ we have a Cartan subalgebra $\mathfrak{h}$ and a Borel subalgebra $\mathfrak{s}$ with $\mathfrak{h} \subset \mathfrak{s} \subset \mathfrak{p}_1 \cap \mathfrak{p}_2$. In the root order such that $\mathfrak{s}$ is the sum of $\mathfrak{h}$ and the negative root spaces, no parabolic containing $\mathfrak{s}$ can contain the root space for the maximal root. This contradicts $\mathfrak{p}_1 + \mathfrak{p}_2 = \mathfrak{e}_1$. The contradiction proves G_u simple and completes the proof. □

Combining Proposition 2.1 with Theorem 1.6 we have

Theorem 2.2. *Let $Z = G/Q$, the complex flag manifold, where G is a complex connected semisimple Lie group acting with finite kernel on Z. Then the closed subgroups $A \subset G$ transitive on Z are precisely those given as follows. Decompose $G = \prod G_i$ with G_i simple, so $Z = \prod Z_i$ with $Z_i = G_i/(Q \cap G_i)$. Then $A = A^0F$ where $A^0 = \prod A_i$ with $A_i = (A \cap G_i)^0$, and A_i is equal to G_i, or to its compact real form $G_{u,i}$, or to one of the three types listed in* Theorem 1.6*, and F is any subgroup of the center of G. Here F acts trivially on Z, so A and A^0 have the same action on Z.*

Picking out the cases where A is a real form of G we have, as in Corollary 1.7,

Corollary 2.3. *Let $Z = G/Q$, the complex flag manifold, where G is a complex connected semisimple Lie group acting with finite kernel on Z. Then the (connected) real forms $G_0 \subset G$ transitive on Z are precisely those given as follows. Decompose $G = \prod G_i$ with G_i simple, so $Z = \prod Z_i$ with $Z_i = G_i/(Q \cap G_i)$. Then $A = \prod A_i$ where $A_i = A \cap G_i$ either is the compact real form $G_{u,i}$ of G_i or is one of the two types listed in* Corollary 1.7.

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Institut für Mathematik, Ruhr–Universität Bochum, D-44780 Bochum, Germany

(Permanent address) Department of Mathematics, University of California, Berkeley, California 94720–3840

E-mail address: `jawolf@math.berkeley.edu`