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A PROOF OF THE HOMOTOPY PUSH-OUT AND PULL-BACK LEMMA

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ABSTRACT. The homotopy push-out and pull-back lemma of Iwase (1998) is a generalized version of Theorem 1.1 of Ganea (1965) and the Theorem of Rutter (1971) whose proofs were given under the simply-connectivity condition. The purpose of this paper is to give a proof in the general case.

1. INTRODUCTION

In this paper, we work in the category of Hausdorff compactly generated spaces. Let (X, A) and (Y, B) be NDR-pairs with $i : A{\to}X$ and $j : B{\to}Y$ the inclusions; i.e., $i : A{\to}X$ and $j : B{\to}Y$ are closed cofibrations (see page 22 in [9], for example). For given $f : Z{\to}X$ and $g : Z{\to}Y$, we define some homotopy pull-backs:

$$\Omega_{f,i} = \{(z,l){\in}Z{\times}P(X)|f(z) = l(0),\ l(1){\in}A\},$$
$$\Omega_{g,j} = \{(z,l'){\in}Z{\times}P(Y)|g(z) = l'(0),\ l'(1){\in}B\},$$

where $P(X) = \{l : [0,\infty){\to} X \mid l(t) = l(1),\ \text{for } t{\geq}1\}$ and $P(Y) = \{l' : [0,\infty){\to}Y \mid l'(t) = l'(1),\ \text{for } t{\geq}1\}$. Similarly, for maps $i{\times}j : A{\times}B{\to}X{\times}Y$, $k : X{\times}B{\cup}A{\times}Y{\to} X{\times}Y$ and $(f, g) = (f{\times}g){\circ}\Delta_Z : Z{\to}X{\times}Y$, we define

$$\Omega_{(f,g),i\times j} = \{(z,l,l'){\in}Z{\times}P(X){\times}P(Y)|l(0) = f(z),\ l'(0) = g(z),$$
$$(l(1), l'(1)){\in}A{\times}B\},$$

$$\Omega_{(f,g),k} = \{(z,l,l'){\in}Z{\times}P(X){\times}P(Y)|l(0) = f(z),\ l'(0) = g(z),$$
$$(l(1), l'(1)){\in}X{\times}B{\cup}A{\times}Y\}.$$

Using them, we have the homotopy push-out $W = \Omega_{f,i}{\cup}\{\Omega_{(f,g),i\times j}{\times}[-1, 1]\}{\cup}\Omega_{g,j}$ of natural projections $p_1 : \Omega_{(f,g),i\times j}{\to}\Omega_{f,i}$ and $p_2 : \Omega_{(f,g),i\times j}{\to}\Omega_{g,j}$ which are given by

$$p_1(z,l,l') = (z,l), \quad p_2(z,l,l') = (z,l').$$

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Then we have the following diagram:

$$
\begin{tikzcd}
\Omega_{(f,g),i\times j} \arrow[r, "p_2"] \arrow[d, "p_1"'] \arrow[dr, phantom, "\text{HPO}"] & \Omega_{g,j} \arrow[d] & & \\
\Omega_{f,i} \arrow[r] & W \arrow[r, dashed] & \Omega_{(f,g),k} \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{HPB}"] & X\times B\cup A\times Y \arrow[d, "k"] \\
& & Z \arrow[r, "{(f,g)}"] & X\times Y,
\end{tikzcd}
\tag{1.1}
$$

where $(z, l, l', -1)$ and $(z, l, l', 1)$ are identified with (z, l) and (z, l') in W, respectively.

Theorem 1.1 (Homotopy Push-out and Pull-back Lemma)**.** *Let (X, A) and (Y, B) be NDR-pairs and let Z be a space with maps $f : Z\to X$ and $g : Z\to Y$. Then the homotopy pull-back $\Omega_{(f,g),k}$ of maps (f, g) and k has naturally the homotopy type of the homotopy push-out W of p_1 and p_2, in the diagram* (1.1).

2. Proof of Theorem 1.1

Proof. First, we define $\hat{W}$ a subspace of $\Omega_{(f,g),k}\times[-1, 1]$ by

$$\hat{W} = \Omega_{(f,g),i\times 1_Y}\times\{-1\}\cup\Omega_{(f,g),i\times j}\times[-1, 1]\cup\Omega_{(f,g),1_X\times j}\times\{1\},$$

and a map $\hat{\Phi} : \hat{W}\to\Omega_{(f,g),k}$ by

$$\hat{\Phi}(z, l, l', t) = (z, l_{1-t}, l'_{1+t}) = \begin{cases} (z, l, l'_{1+t}), & t\leq 0, \\ (z, l_{1-t}, l'), & t\geq 0, \end{cases}$$

where $l_{1-t}(s) = l(\mathrm{Min}\{(1-t)s, s\})$, $l'_{1+t}(s) = l'(\mathrm{Min}\{(1+t)s, s\})$ $(s\in I = [0, 1])$, respectively. Let $\pi : \hat{W}\to W$ be the identification map. By definition, $\hat{\Phi}(z, l, l', -1) = (z, l, c_{g(z)})$ doesn't depend on l'. Similarly, $\hat{\Phi}(z, l, l', 1) = (z, c_{f(z)}, l')$ doesn't depend on l. Thus we see that $\hat{\Phi}$ induces a map $\Phi : W\to\Omega_{(f,g),k}$.

Second, to define a homotopy inverse of Φ, let us recall that (X, A) is a NDR-pair: there exist maps $u : X\to I$ and $h : X\times I\to X$ which satisfy the following properties:

(1) $u^{-1}(0) = A$.
(2) $h|_{X\times\{0\}} = 1_X,\ h|_{A\times\{t\}} = 1_A\ (t\in I)$.
(3) If $x\in U = u^{-1}([0, 1)),$ then $h(x, 1)\in A$.

Similarly, the NDR-pair (Y, B) has maps $u' : Y\to I,\ h' : Y\times I\to Y$ as above. Making use of them, we define maps $\hat{\Psi} : \Omega_{(f,g),k}\to\hat{W}$ and $\Psi : \Omega_{(f,g),k}\to W$ as follows:

$$\Psi = \pi\circ\hat{\Psi},$$

$$\begin{aligned} \hat{\Psi}(z, l, l') &= (z, l\oplus h, l'\oplus h', u(l(1)) - u'(l'(1))) \\ &= \begin{cases} (z, l\oplus h, l'\oplus h', -u'(l'(1))), & l(1)\in A, \\ (z, l\oplus h, l'\oplus h', u(l(1))), & l'(1)\in B, \end{cases} \end{aligned}$$

where $l \oplus h$, $l' \oplus h'$ are the composition of paths, i.e.,

$$l \oplus h(s) = \begin{cases} l(2s), & 0 \leq s \leq \frac{1}{2}, \\ h(l(1), 2s-1), & \frac{1}{2} \leq s \leq 1, \end{cases} \quad l' \oplus h'(s) = \begin{cases} l'(2s), & 0 \leq s \leq \frac{1}{2}, \\ h'(l'(1), 2s-1), & \frac{1}{2} \leq s \leq 1. \end{cases}$$

Third, we show that $\Phi \circ \Psi \simeq 1_{\Omega_{(f,g),k}}$. We have the following equation:

$$\begin{aligned} \Phi \circ \Psi(z, l, l') &= \Phi \circ \pi \circ \hat{\Psi}(z, l, l') \\ &= \hat{\Phi} \circ \hat{\Psi}(z, l, l') \\ &= \begin{cases} \hat{\Phi}(z, l \oplus h, l' \oplus h', -u'(l'(1))), & l(1) \in A, \\ \hat{\Phi}(z, l \oplus h, l' \oplus h', u(l(1))), & l'(1) \in B, \end{cases} \\ &= \begin{cases} (z, l \oplus h, (l' \oplus h')_{1-u'(l'(1))}), & l(1) \in A, \\ (z, (l \oplus h)_{1-u(l(1))}, l' \oplus h'), & l'(1) \in B. \end{cases} \end{aligned}$$

Then we define a map $H : \Omega_{(f,g),k} \times I \to \Omega_{(f,g),k}$ by

$$\begin{aligned} H(z, l, l', s) &= (z, (l \oplus h)_{(1-u(l(1)))(1-s)+\frac{1}{2}s}, (l' \oplus h')_{(1-u'(l'(1)))(1-s)+\frac{1}{2}s}) \\ &= \begin{cases} (z, (l \oplus h)_{(1-s)+\frac{1}{2}s}, (l' \oplus h')_{(1-u'(l'(1)))(1-s)+\frac{1}{2}s}), & l(1) \in A, \\ (z, (l \oplus h)_{(1-u(l(1)))(1-s)+\frac{1}{2}s}, (l' \oplus h')_{(1-s)+\frac{1}{2}s}), & l'(1) \in B. \end{cases} \end{aligned}$$

As is easily checked, H gives a homotopy of $\Phi \circ \Psi$ and $1_{\Omega_{(f,g),k}}$, the identity.

Finally, we show that $\Psi \circ \Phi \simeq 1_W$. We have the following equation:

$$\begin{aligned} \hat{\Psi} \circ \hat{\Phi}(z, l, l', t) &= \hat{\Psi}(z, l_{1-t}, l'_{1+t}) \\ &= (z, l_{1-t} \oplus h, l'_{1+t} \oplus h', u(l(1-t)) - u'(l'(1+t))) \\ &= \begin{cases} (z, l \oplus h, l'_{1+t} \oplus h', -u'(l'(1+t))), & t \leq 0, \\ (z, l_{1-t} \oplus h, l' \oplus h', u(l(1-t))), & t \geq 0. \end{cases} \end{aligned}$$

Then we define a map $\hat{H}' : \hat{W} \times I \to W$ by

$$\hat{H}'(z, l, l', t, s)$$
$$= \begin{cases} \pi(z, l_{1-t} \oplus h, l'_{1+t} \oplus h', (1-3s)u(l, l', t) + 3sm(l, l', t)), & 0 \leq s \leq \frac{1}{3}, \\ \pi(z, l_{v(-t,s)} \oplus h, l'_{v(t,s)} \oplus h', m(l, l', t)), & \frac{1}{3} \leq s \leq \frac{2}{3}, \\ \pi(z, (l \oplus h)_{v(\frac{1}{2},s)}, (l' \oplus h')_{v(\frac{1}{2},s)}, (3-3s)m(l, l', t) + (3s-2)t), & \frac{2}{3} \leq s \leq 1, \end{cases}$$

where $v(t,s) = (2-3s)(1+t) + 3s - 1$, $u(l, l', t) = u(l(1-t)) - u'(l'(1+t))$, $m(l,t) = \text{Max}\{u(l([1-t, 2])),\ t\}$, $m(l, l', t) = m(l,t) - m(l', -t)$.

We remark that $u(l, l', t) = u(l(1-t))$ if $t \geq 0$ and $u(l, l', t) = -u'(l'(1+t))$ if $t \leq 0$ for $(z, l, l', t) \in \hat{W}$, and hence $-1 \leq u(l, l', t) \leq 1$, and $u(l, l', 0) = 0$. Similarly, it follows that $m(l,t) = \text{Max}\{0, t\} = 0$ $(-1 \leq t \leq 0)$, and $m(l', -t) = \text{Max}\{0, -t\} = 0$ $(0 \leq t \leq 1)$. We remark that $m(l, l', t) = m(l,t)$ if $t \geq 0$ and $m(l, l', t) = -m(l', -t)$ if $t \leq 0$ for $(z, l, l', t) \in \hat{W}$. Thus $-1 \leq m(l, l', t) \leq 1$, and $m(l, l', 0) = 0$.

Let us recall that $\pi\times 1_I : \hat{W}\times I\to W\times I$ gives also an identification map (see page 20 in [9], for example). By definition, we have the following:

$$
\begin{aligned}
&\hat{H}'(z,l,l',-1,s)\\
&=\begin{cases}
\pi(z,l\oplus h,c_{g(z)}\oplus h',-u'(g(z)))=\hat{\Psi}\circ\hat{\Phi}(z,l,l',-1), & s=0,\\
\pi(z,l\oplus h,c_{g(z)}\oplus h',(1-3s)(-u'(g(z)))-3s), & 0\le s\le \frac{1}{3},\\
\pi(z,l\oplus h,c_{g(z)}\oplus h',-1)=[z,l\oplus h], & s=\frac{1}{3},\\
\pi(z,l\oplus h,l'_{3s-1}\oplus h',-1)=[z,l\oplus h], & \frac{1}{3}\le s\le\frac{2}{3},\\
\pi(z,l\oplus h,l'\oplus h',-1)=[z,l\oplus h], & s=\frac{2}{3},\\
\pi(z,(l\oplus h)_{2-\frac{3}{2}s},(l'\oplus h')_{2-\frac{3}{2}s},-1)=[z,(l\oplus h)_{2-\frac{3}{2}s}], & \frac{2}{3}\le s\le 1,\\
\pi(z,l,l',-1)=[z,l], & s=1,
\end{cases}\\
&\hat{H}'(z,l,l',t,s)\\
&=\begin{cases}
\pi(z,l\oplus h,l'_{1+t}\oplus h',-u'(l'(1+t)))=\hat{\Psi}\circ\hat{\Phi}(z,l,l',t), & t\le 0,\ s=0,\\
\pi(z,l\oplus h,l'_{1+t}\oplus h',(1-3s)(-u'(l'(1+t)))-3sm(l',-t)), & t\le 0,\ 0\le s\le\frac{1}{3},\\
\pi(z,l\oplus h,l'_{1+t}\oplus h',-m(l',-t)), & t\le 0,\ s=\frac{1}{3},\\
\pi(z,l\oplus h,l'_{v(t,s)}\oplus h',-m(l',-t)), & t\le 0,\ \frac{1}{3}\le s\le\frac{2}{3},\\
\pi(z,l\oplus h,l'\oplus h',-m(l',-t)), & t\le 0,\ s=\frac{2}{3},\\
\pi(z,(l\oplus h)_{2-\frac{3}{2}s},(l'\oplus h')_{2-\frac{3}{2}s},(3s-3)m(l',-t)+(3s-2)t), & t\le 0,\ \frac{2}{3}\le s\le 1,\\
\pi(z,l,l',t), & t\le 0,\ s=1,
\end{cases}\\
&\hat{H}'(z,l,l',0,s)\\
&=\begin{cases}
\pi(z,l\oplus h,l'\oplus h',0)=\hat{\Psi}\circ\hat{\Phi}(z,l,l',0), & 0\le s\le\frac{2}{3},\\
\pi(z,(l\oplus h)_{2-\frac{3}{2}s},(l'\oplus h')_{2-\frac{3}{2}s},0), & \frac{2}{3}\le s\le 1,\\
\pi(z,l,l',0), & s=1,
\end{cases}\\
&\hat{H}'(z,l,l',t,s)\\
&=\begin{cases}
\pi(z,l_{1-t}\oplus h,l'\oplus h',u(l(1-t)))=\hat{\Psi}\circ\hat{\Phi}(z,l,l',t), & t\ge 0,\ s=0,\\
\pi(z,l_{1-t}\oplus h,l'\oplus h',(1-3s)u(l(1-t))+3sm(l,t)), & t\ge 0,\ 0\le s\le\frac{1}{3},\\
\pi(z,l_{1-t}\oplus h,l'\oplus h',m(l,t)), & t\ge 0,\ s=\frac{1}{3},\\
\pi(z,l_{v(-t,s)}\oplus h,l'\oplus h',m(l,t)), & t\ge 0,\ \frac{1}{3}\le s\le\frac{2}{3},\\
\pi(z,l\oplus h,l'\oplus h',m(l,t)), & t\ge 0,\ s=\frac{2}{3},\\
\pi(z,(l\oplus h)_{2-\frac{3}{2}s},(l'\oplus h')_{2-\frac{3}{2}s},(3-3s)m(l,t)+(3s-2)t), & t\ge 0,\ \frac{2}{3}\le s\le 1,\\
\pi(z,l,l',t), & t\ge 0,\ s=1,
\end{cases}\\
&\hat{H}'(z,l,l',1,s)\\
&=\begin{cases}
\pi(z,c_{f(z)}\oplus h,l'\oplus h',u(f(z)))=\hat{\Psi}\circ\hat{\Phi}(z,l,l',1), & s=0,\\
\pi(z,c_{f(z)}\oplus h,l'\oplus h',(1-3s)(u(f(z)))+3s), & 0\le s\le\frac{1}{3},\\
\pi(z,c_{f(z)}\oplus h,l'\oplus h',1)=[z,l'\oplus h'], & s=\frac{1}{3},\\
\pi(z,l_{3s-1}\oplus h,l'\oplus h',1)=[z,l'\oplus h'], & \frac{1}{3}\le s\le\frac{2}{3},\\
\pi(z,l\oplus h,l'\oplus h',1)=[z,l'\oplus h'], & s=\frac{2}{3},\\
\pi(z,(l\oplus h)_{2-\frac{3}{2}s},(l'\oplus h')_{2-\frac{3}{2}s},1)=[z,(l'\oplus h')_{2-\frac{3}{2}s}], & \frac{2}{3}\le s\le 1,\\
\pi(z,l,l',1)=[z,l'], & s=1.
\end{cases}
\end{aligned}
$$

Therefore, $\hat{H}'(z,l,l',-1,s)$ doesn't depend on l' and $\hat{H}'(z,l,l',1,s)$ doesn't depend on l. Thus $\hat{H}'$ induces a map $H' : W\times I\to W$, which gives a homotopy of $\Psi\circ\Phi$ and 1_W, the identity. □

3. A generalization of Theorem 1.1

In this section, we extend Theorem 1.1 in the category of quasi-fibrations *q-Fib*.

Definition 3.1. An object of the category *q-Fib* is a pair consisting of a topological space X and a continuous map $p : X \to B$ which satisfy the following conditions:
(1) $p : X \to B$ is a quasi-fibration over polyhedra.
(2) (local-triviality condition) For each simplex Δ_α of B, there exists a homeomorphism ϕ_α which makes the following diagram commute:

$$\begin{tikzcd} \Delta_\alpha \times F \arrow[r, "\phi_\alpha"] \arrow[dr, "pr_1"'] & \iota_\alpha^* X \arrow[r] \arrow[d] \arrow[dr, phantom, "\mathrm{PB}"] & X \arrow[d, "p"] \\ & \Delta_\alpha \arrow[r, "\iota_\alpha : \text{embedding}"] & B \end{tikzcd}$$

(3) A is closed in $X \Longleftrightarrow$ for each $\alpha \in \Lambda$, $A \cap \iota_\alpha^* X$ is closed in $\iota_\alpha^* X$.
A morphism of the category of *q-Fib* is a pair (f, g) of maps which makes the following diagram commute:

$$\begin{tikzcd} X_1 \arrow[r, "f"] \arrow[d, "p_1"'] & X_2 \arrow[d, "p_2"] \\ B_1 \arrow[r, "g"] & B_2 \end{tikzcd}$$

where g is a simplicial map.

Definition 3.2. Let $p_X : X \to B$ be an object of the category of *q-Fib*, and A a subspace of X with the restriction $p_X|_A$ an object of it. Then a pair (X, A) is called a fibrewise cofibred pair when the following fibrewise homotopy extension property is satisfied. Let $p_E : E \to B$ be an object of *q-Fib* and $(f, 1_B)$ a morphism of *q-Fib*. Let $g_t : A \to E$ be a homotopy of f_A which makes the following diagram commute:

$$\begin{tikzcd} A \arrow[rr, "g_t"] \arrow[dr, "p_A"'] & & E \arrow[dl, "p_E"] \\ & B & \end{tikzcd}$$

Then there exists a homotopy $h_t : X \to E$ of f such that $(h_t, 1_B)$ is a morphism of *q-Fib* and $g_t = h_t|_A$.

Some arguments on continuous functors (see [7]) yield the following result.

Theorem 3.3. *Let (X_1, A_1) and (X_2, A_2) be closed fibrewise cofibred pairs and $p_Z : Z \to B$ an object of q-Fib with morphisms $(f_1, 1_B)$ and $(f_2, 1_B)$. Then the homotopy pull-back $\Omega_{(f_1,f_2),k}$ of $(f_1, f_2) : Z \to X_1 \times_B X_2$ and $k : X_1 \times_B A_2 \cup A_1 \times_B X_2 \to X_1 \times_B X_2$*

has naturally fibrewise homotopy type of the homotopy push-out of $p_1 : \Omega_{(f_1,f_2),i_1\times_B i_2} \to \Omega_{f_1,i_1}$ *and* $p_2 : \Omega_{(f_1,f_2),i_1\times_B i_2} \to \Omega_{f_2,i_2}$

$$\begin{array}{ccccc}
\Omega_{(f_1,f_2),i_1\times_B i_2} & \xrightarrow{p_1} & \Omega_{f_1,i_1} & & \\
{\scriptstyle p_2}\downarrow & & \downarrow & & \\
\Omega_{f_2,i_2} & \longrightarrow & \Omega_{(f_1,f_2),k} & \longrightarrow & X_1\times_B A_2\cup A_1\times_B X_2 \\
 & & \downarrow & & \downarrow{\scriptstyle k} \\
 & & Z & \xrightarrow{(f_1,f_2)} & X_1\times_B X_2
\end{array}$$

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References

1. M. C. Crabb and I. M. James, *Fibrewise Homotopy Theory*, Springer (1998). MR **99k:**55001
2. T. Ganea, *A generalization of the homology and homotopy suspension*, Comm. Math. Helv. **39** (1965), 295–322. MR **31:**4033
3. N. Iwase, *Ganea's conjecture on Lusternik-Schnirelmann category*, Bull. London Math. Soc. **30** (1998), 623–634. MR **99j:**55003
4. I. M. James, *General topology and Homotopy theory*, Springer (1984). MR **86d:**55001
5. I. M. James, *Handbook of Algebraic Topology*, North-Holland (1995). MR **96g:**55002
6. J. W. Rutter, *On a theorem of T. Ganea*, J. London Math. Soc. (2) **3** (1971), 190–192. MR **43:**5532
7. M. Sakai, *Functors on the category of quasi-fibrations*, Kyushu University preprint series in mathematics, 1999-25.
8. N. E. Steenrod, *A convenient category of topological spaces*, Michigan Math. J. **14** (1967), 133–152. MR **35:**970
9. G. W. Whitehead, *Elements of Homotopy Theory*, Springer Verlag, Berlin, **GTM** series **61** (1978). MR **80b:**55001

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