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ON THE COMMUTANT OF OPERATORS OF MULTIPLICATION BY UNIVALENT FUNCTIONS

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ABSTRACT. Let $\mathcal{B}$ be a certain Banach space consisting of continuous functions defined on the open unit disk. Let $\phi \in \mathcal{B}$ be a univalent function defined on $\overline{\mathbf{D}}$, and assume that M_ϕ denotes the operator of multiplication by ϕ. We characterize the structure of the operator T such that $M_\phi T = TM_\phi$. We show that $T = M_\varphi$ for some function φ in $\mathcal{B}$. We also characterize the commutant of M_{ϕ^2} under certain conditions.

1. INTRODUCTION

Let $\mathcal{B}$ be a Banach space consisting of continuous functions defined on the open unit disk $\mathbf{D}$ such that $\mathcal{B}$ satisfies conditions (1)–(6).

(1) $1 \in \mathcal{B}, z\mathcal{B} \subset \mathcal{B}$.

(2) For every $\lambda \in \mathbf{D}$ the evaluation functional at λ, $e_\lambda : \mathcal{B} \to \mathbf{C}$, given by $f \mapsto f(\lambda)$, is bounded.

(3) $\dim \ker(M_z - \lambda)^* = 1$ for every $\lambda \in \mathbf{D}$.

(4) If $f \in \mathcal{B}$ and f has an analytic extention to a neighborhood of $\lambda \in \mathbf{D}$, then $\frac{f-f(\lambda)}{z-\lambda} \in \mathcal{B}$. Also for every $\lambda \in \mathbf{D}$ the subspace of $\mathcal{B}$ consisting of those functions in $\mathcal{B}$ that have analytic extention to a neighborhood of λ is dense in $\mathcal{B}$.

(5) For every $f \in \mathcal{B}$ the function $\tilde{f}$ defined by $\tilde{f}(\lambda) = f(-\lambda)$ is in $\mathcal{B}$ and $\|\tilde{f}\| = \|f\|$.

(6) If $f \in \mathcal{B}$ and $|f(\lambda)| > c > 0$ for every $\lambda \in \mathbf{D}$, then $\frac{1}{f}$ is a multiplier of $\mathcal{B}$.

Throughout this article by a Banach space of continuous functions $\mathcal{B}$ we mean one satisfying the above conditions. A complex valued function ϕ defined on $\mathbf{D}$ is called a multiplier of $\mathcal{B}$ if $\phi\mathcal{B} \subset \mathcal{B}$, i.e., ϕf is in $\mathcal{B}$ for every f in $\mathcal{B}$, and the set of all multipliers of $\mathcal{B}$ is denoted by $\mathcal{M}(\mathcal{B})$. As it is shown in [6] each multiplier ϕ is bounded on $\mathbf{D}$. Given a multiplier ϕ, let M_ϕ, defined by $M_\phi(f) = \phi f$, denote the operator of multiplication by ϕ. By the closed graph theorem M_φ is bounded. The algebra of all bounded operators on $\mathcal{B}$ is denoted by $L(\mathcal{B})$. Let $X \in L(\mathcal{B})$ be a bounded operator on $\mathcal{B}$ and $XM_z = M_zX$. It is easy to see that $X = M_\varphi$ for some function $\varphi \in \mathcal{M}(\mathcal{B})$.

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Throughout this article $\{M_\varphi\}'$ denotes the set of all bounded linear operators X on $\mathcal{B}$ such that $M_\varphi X = XM_\varphi$, i.e., the commutant of M_φ. Assume $T \in \mathcal{B}^*$ and $f \in \mathcal{B}$. We denote the value of T at f by $\langle f, T\rangle$. We define $\hat{M}_\varphi : \mathcal{B} \to \mathcal{B}$ by $\hat{M}_\varphi(f) = \varphi\hat{f}$. By the closed graph theorem $\hat{M}_\varphi$ is bounded.

In what follows we present some examples of such spaces.

Examples. a) Cole and Gamelin [2] proved that if $\mathcal{A}$ is a T-invariant algebra on a compact set K, then for each $\lambda \in K$, $\operatorname{ran}(M_z - \lambda)$ is dense in $\ker e_\lambda$. Hence $\dim \ker(M_z^* - \lambda) = 1$ for every $\lambda \in K$. Also they have shown that every T-invariant algebra satisfies condition (4). Therefore the algebra of all continuous functions defined on $\mathbf{D}$, i.e., $C(\overline{\mathbf{D}})$, is a Banach space of continuous functions.

b) The disk algebra $A(\mathbf{D})$ which is the algebra of all continuous functions on the closure of disk that are analytic on $\mathbf{D}$.

c) The Bergman space of analytic functions defined on the unit disk $L_a^P(\mathbf{D})$ for $1 \leq p \leq \infty$.

d) The spaces D_α of all functions $f(z) = \sum \hat{f}(n)z^n$, holomorphic in $\mathbf{D}$, for which

$$\|f\|_\alpha^2 = \sum (n+1)^\alpha \mid \hat{f}(n) \mid^2 < \infty$$

for every $\alpha \geq 1$ or $\alpha \leq 0$.

e) The analytic Lipschitz spaces $\mathcal{A}_\alpha$ for $0 < \alpha < 1$, i.e., the space of all analytic functions defined on $\mathbf{D}$ that satisfy a Lipschitz condition of order α.

f) The subspace $\mathcal{A}^\alpha$ of $\mathcal{A}_\alpha$ consisting of functions f in $\mathcal{A}_\alpha$ for which

$$\lim_{z\to w} \frac{\mid f(z) - f(w) \mid}{\mid z - w \mid^\alpha} = 0.$$

g) The classical Hardy spaces H^p for $1 \leq p \leq \infty$.

Shields and Wallen [6] studied the commutant of the operator M_z on the Hilbert spaces of analytic functions. By a slight change in their methods one can obtain the commutant of M_z on the Banach spaces of analytic functions. The commutant of a Toeplitz operator on certain Hilbert spaces of functions was studied by many mathematicians. See for example [1], [7], [8]. Cuckovic in [3] investigated the commutant of M_{z^n} on the Bergman space ${L^2}_a(\mathbf{D})$. Seddighi and Vaezpour [5] have shown that under certain conditions on the reproducing kernels of a functional Hilbert spaces every operator S essentially commuting with M_z and commuting with M_{z^n} for some $n > 1$ is a multiplication operator. Also the commutant of M_{z^2} on a Banach space of analytic functions and the commutant of M_{z^n} on a certain Hilbert space of functions were studied in [4]. In section 2 of this article we characterize the commutant of M_ϕ for a univalent function $\phi \in \mathcal{M}(\mathcal{B}) \cap A(\mathbf{D})$ on a Banach space of continuous functions and we investigate the commutant of M_{ϕ^2} under certain conditions.

2. The main results

Lemma 2.1. *If $\phi \in \mathcal{M}(\mathcal{B})$ and $T \in \{M_\phi\}'$, then $T^*(e_\lambda) \in ker(M_\phi - \phi(\lambda))^*$ for every $\lambda \in \mathbf{D}$.*

Proof. Let $f \in \mathcal{B}$. We have $\langle f, {M_\phi}^* T^*(e_\lambda)\rangle = \langle f, T^* {M_\phi}^*(e_\lambda)\rangle = \langle M_\phi T(f), e_\lambda\rangle = \phi(\lambda)Tf(\lambda) = \phi(\lambda)\langle T(f), e_\lambda\rangle = \phi(\lambda)\langle f, T^*(e_\lambda)\rangle = \langle f, \phi(\lambda)T^*(e_\lambda)\rangle$; hence ${M_\phi}^* T^*(e_\lambda) = \phi(\lambda)T^*(e_\lambda)$ which implies that $T^*(e_\lambda) \in \ker(M_\phi - \phi(\lambda))^*$. $\square$

Theorem 2.2. *Let $\phi \in \mathcal{M}(\mathcal{B}) \cap A(\mathbf{D})$ be a univalent map. If $T \in \{M_\phi\}'$, then $T = M_\psi$ for some function $\psi \in \mathcal{M}(\mathcal{B})$.*

Proof. Let $\lambda \in \mathbf{D}$. We show that $\overline{\operatorname{ran}(M_\phi - \phi(\lambda))} = \ker e_\lambda$. It is easy to see that $\overline{\operatorname{ran}(M_\phi - \phi(\lambda))} \subset \ker e_\lambda$.

To show the converse, let $\phi - \phi(\lambda) = (z-\lambda)g(z)$ by the properties of $\mathcal{B}$, $g \in \mathcal{B}$. Since ϕ is univalent, $g(z) \neq 0$ on $\overline{\mathbf{D}}$ and hence $\frac{1}{g}$ is in $\mathcal{M}(\mathcal{B})$. Now assume that $f \in \ker e_\lambda$ so $f(\lambda) = 0$. Since the subspace of $\mathcal{B}$ consisting of functions which are analytic in a neighborhood of λ is dense in $\mathcal{B}$, it follows that there is a sequence $\{f_n\}$ of functions in this subspace such that f_n tends to f. Now assume that $f_n - f_n(\lambda) = (z-\lambda)g_n$ by property (4) of $\mathcal{B}$, $g_n \in \mathcal{B}$. Hence

$$f_n - f_n(\lambda) = \frac{\phi - \phi(\lambda)}{g(z)} g_n(z) = (\phi - \phi(\lambda)) \frac{g_n(z)}{g(z)},$$

which implies that $(f_n - f_n(\lambda)) \in \operatorname{ran}(M_\phi - \phi(\lambda))$. Since $f_n - f_n(\lambda)$ tends to f in $\mathcal{B}$, it follows that $f \in \overline{\operatorname{ran}(M_\phi - \phi(\lambda))}$. Now since $(M_\phi - \phi(\lambda))^*(e_\lambda) = (M_\phi - \phi(\lambda))^* T^*(e_\lambda) = 0$ and $\dim \ker(M_\phi - \phi(\lambda))^* = 1$, we conclude that $T^*(e_\lambda) = \psi(\lambda) e_\lambda$ for some constant $\psi(\lambda)$. Therefore, we have

$$T(f)(\lambda) = \langle T(f), e_\lambda \rangle = \langle f, T^*(e_\lambda) \rangle = \psi(\lambda)\langle f, e_\lambda \rangle = \psi(\lambda) f(\lambda).$$

Hence $T(f) = \psi f$ for every $f \in \mathcal{B}$ and the proof is complete. □

In the remainder of this section we investigate the commutant of M_{ϕ^2} for some univalent function ϕ.

Corollary 2.3. *If $\phi \in \mathcal{M}(\mathcal{B}) \cap A(\mathbf{D})$ isa univalent map such that $\phi(\overline{\mathbf{D}})$ has no distinct points which are symmetric with respect to the origin, then $\{M_{\phi^2}\}' = \{M_\psi : \psi \in \mathcal{M}(\mathcal{B})\}$. In particular if $|\lambda| > 1$, then $\{M_{(z-\lambda)^2}\}' = \{M_\psi : \psi \in \mathcal{M}(\mathcal{B})\}$.*

Remark. In the proofs of Lemma 2.1 and Theorem 2.2 we did not use property (5) of Banach space $\mathcal{B}$. Also $\mathbf{D}$ can be replaced by every bounded open set G.

Lemma 2.4. *Let ϕ be a univalent odd function in $\mathcal{M}(\mathcal{B}) \cap A(\mathbf{D})$, $S \in L(\mathcal{B})$ and $SM_\phi = -M_\phi S$. Then there exists $\psi \in \mathcal{M}(\mathcal{B})$ such that $S = \hat{M}_\psi$.*

Proof. Since $T^* M_\phi^* = -M_\phi^* T^*$ by a similar argument asin the proof of Lemma 2.1, we have $M_\phi^* T^*(e_\lambda) = -\phi(\lambda) T^*(e_\lambda)$; hence $(M_\phi + \phi(\lambda))^*(T^*(e_\lambda)) = 0$ which yields $T^*(e_\lambda) \in \ker(M_\phi + \phi(\lambda))^*$. By the proof of Theorem 2.2, $e_{-\lambda}$ spans $\ker(M_\phi + \phi(\lambda))^*$ so $T^*(e_\lambda) = \psi(\lambda) e_{-\lambda}$ for some constant $\psi(\lambda)$. Now

$$\langle T(f), e_\lambda \rangle = \langle f, T^*(e_\lambda) \rangle = \psi(\lambda) \langle f, e_{-\lambda} \rangle = \psi(\lambda) f(-\lambda)$$

which implies that $T = \hat{M}_\psi$. Also $T(\hat{f}) = \psi f$; hence $\psi \in \mathcal{M}(\mathcal{B})$. □

Theorem 2.5. *Let $\phi \in \mathcal{M}(\mathcal{B}) \cap A(\mathbf{D})$ be an odd univalent map. Let $S \in \{M_{\phi^2}\}'$ and $SM_\phi - M_\phi S$ be a compact operator. Then there is some $\Psi \in \mathcal{M}(\mathcal{B})$ such that $S = M_\Psi$.*

Proof. We have $(SM_\phi - M_\phi S)M_\phi = (SM_{\phi^2} - M_\phi S M_\phi) = M_{\phi^2} S - M_\phi S M_\phi = -M_\phi(SM_\phi - M_\phi S)$. Hence, by Lemma 2.4 there exists some $\psi \in \mathcal{B}$ such that $SM_\phi - M_\phi S = \hat{M}_\psi$. Now we show that M_ψ is compact. Let the operator T be defined by $T(f) = \hat{f}$; it is obvious that T is continuous. Now we have $M_\psi(f) = \hat{M}_\psi T(f)$ for every $f \in \mathcal{B}$ and so M_ψ is compact and by the Fredholm alternative theorem $\psi = 0$. This implies that $M_\phi S = SM_\phi$; hence $S \in \{M_\phi\}'$ and we conclude that $S = M_\Psi$ for some $\Psi \in \mathcal{M}(\mathcal{B})$. □

Theorem 2.6. *Let $\phi \in \mathcal{M}(\mathcal{B}) \cap A(\mathbf{D})$ be an odd univalent map. Suppose T is an operator in $\{M_{\phi^2}\}'$ and let $TM_\phi + M_\phi T$ be a compact operator. Then there exists a $\Psi \in \mathcal{M}(\mathcal{B})$ such that $T = \hat{M}_\Psi$.*

Proof. We have $(TM_\phi + M_\phi T)M_\phi = M_\phi(TM_\phi + M_\phi T)$; hence by Theorem 2.2, there is a function $\psi \in \mathcal{M}(\mathcal{B})$ such that $TM_\phi + M_\phi T = M_\psi$. Since $TM_\phi + M_\phi T$ is compact, we have $\psi = 0$, so $M_\phi T = -M_\phi T$. Now, by Lemma 2.4, there is $\Psi \in \mathcal{M}(\mathcal{B})$ such that $T = \hat{M}_\Psi$. □

Theorem 2.7. *Let $\phi \in \mathcal{M}(\mathcal{B}) \cap A(\mathbf{D})$ be a univalent map of $\mathbf{D}$ onto $\mathbf{D}$ such that $f \circ \phi$ and $f \circ \phi^{-1}$ are in $\mathcal{B}$ for every $f \in \mathcal{B}$. Let $S \in \{M_{\phi^2}\}'$. If polynomials are dense in $\mathcal{B}$, then $S(f) = \Phi f + \psi \frac{f + \widehat{(f \circ \Phi^{-1})} \circ \Phi}{2\phi}$, where $S(1) = \Phi$ and $(SM_\phi - M_\phi S)(1) = \psi$.*

Proof. We define $T : \mathcal{B} \to \mathcal{B}$ by $T(f) = f \circ \phi^{-1}$. Clearly $T \in L(\mathcal{B})$ with inverse $T^{-1}(f) = f \circ \phi$. Since $M_z T = TM_\phi$, by induction we have $M_{z^n} T = TM_{\phi^n}$ for every positive integer n. Since $SM_{\phi^2} = M_{\phi^2} S$, it follows that $ST^{-1}M_{z^2}T = T^{-1}M_{z^2}TS$ and so $TST^{-1} \in \{M_{z^2}\}'$. Now by a similar argument as in the proof of [4, Theorem 2.6] we have

$$TST^{-1}(f) = TST^{-1}(1)(f) + (TST^{-1}M_z - M_z TST^{-1})(1)(\frac{f + \hat{f}}{2z}).$$

If $S(1) = \Phi$ and $(SM_\phi - M_\phi S)(1) = \psi$, then $TST^{-1}(1) = \Phi \circ \phi^{-1}$. Since

$$T(SM_\phi - M_\phi S)T^{-1} = TST^{-1}M_z - M_z TST^{-1},$$

it follows that $(TST^{-1}M_z - M_z TST^{-1})(1) = \psi \circ \phi^{-1}$. Hence

$$S(f) = T^{-1}TST^{-1}T(f) = \Phi f + \psi \frac{f + \widehat{(f \circ \Phi^{-1})} \circ \Phi}{2\phi}.$$

□

Remark. If ϕ in Theorem 2.7 is an odd function, then $S(f) = \Phi f + \psi \frac{f + \hat{f}}{2\phi}$.

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