

## THE FUCIK SPECTRUM AND CRITICAL GROUPS

KANISHKA PERERA AND MARTIN SCHECHTER

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ABSTRACT. We compute critical groups of zero for variational functionals arising from semilinear elliptic boundary value problems with jumping nonlinearities when the asymptotic limits of the nonlinearity fall in certain parts of Type (II) regions between curves of the Fucik spectrum.

### 1. INTRODUCTION

In this paper we consider the problem of determining the critical groups of zero for the functional

$$(1.1) \quad I(u) = I(u, a, b) = \int_{\Omega} |\nabla u|^2 - a(u^-)^2 - b(u^+)^2, \quad u \in H = H_0^1(\Omega),$$

associated with the problem

$$(1.2) \quad \begin{cases} -\Delta u &= b u^+ - a u^- & \text{in } \Omega, \\ u &= 0 & \text{on } \partial\Omega \end{cases}$$

where  $\Omega$  is a smooth bounded domain in  $\mathbb{R}^n$  and  $u^{\pm}(x) = \max\{\pm u(x), 0\}$ .

The set  $\Sigma$  of those points  $(a, b) \in \mathbb{R}^2$  for which (1.2) has nontrivial solutions is called the Fucik spectrum of  $-\Delta$ . It was shown in Schechter [7] that, if  $0 < \lambda_1 < \lambda_2 < \dots$  are the distinct Dirichlet eigenvalues of  $-\Delta$ , there are decreasing curves  $C_{l1}, C_{l2}$  (which may coincide) passing through the point  $(\lambda_l, \lambda_l)$  such that all points on the curves are in  $\Sigma$ , while points in the square  $Q_l := (\lambda_{l-1}, \lambda_{l+1})^2$  that are either in the region  $I_{l1}$  below the lower curve  $C_{l1}$  or in the region  $I_{l2}$  above the upper curve  $C_{l2}$  are not in  $\Sigma$ . When the curves do not coincide, points in the region  $II_l$  between them may or may not belong to  $\Sigma$ . We set  $I_l = I_{l1} \cup I_{l-1, 2}$ .

When  $(a, b) \notin \Sigma$ , the origin is an isolated critical point of  $I$  and hence the critical groups  $C_*(I, 0)$  are defined. The following partial result on the critical groups was obtained by Dancer [2, 1].

**Theorem 1.1.** *Let  $(a, b) \in Q_l \setminus \Sigma$  and let  $d_l$  denote the dimension of the subspace  $N_l$  spanned by the eigenfunctions corresponding to  $\lambda_1, \dots, \lambda_l$ .*

$$(i) \text{ If } (a, b) \in I_l, \text{ then } C_q(I, 0) = \begin{cases} \mathbb{Z}, & q = d_{l-1}, \\ 0, & q \neq d_{l-1}. \end{cases}$$

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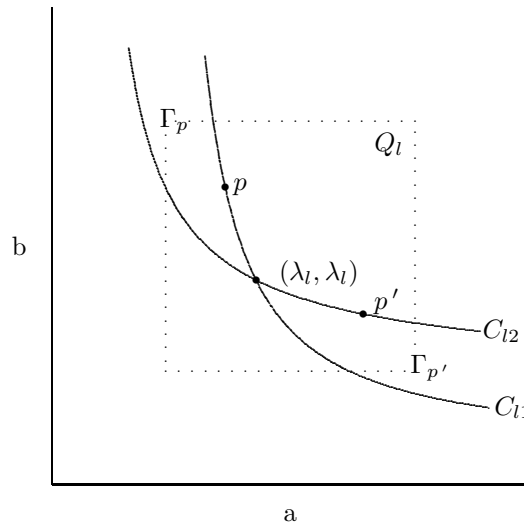
- (ii) If  $(a, b) \in \Pi_l$ , then  $C_q(I, 0) = 0$  for  $q \leq d_{l-1}$  and for  $q \geq d_l$ . In particular,  $C_q(I, 0) = 0$  for all  $q$  when  $\lambda_l$  is a simple eigenvalue.

A different proof of Theorem 1.1 based on some ideas developed in Schechter [7, 8] was given in Perera and Schechter [5]. Note that this theorem does not determine  $C_q(I, 0)$  for  $d_{l-1} < q < d_l$  when  $(a, b) \in \Pi_l$  (and  $\lambda_l$  is a multiple eigenvalue). Here we compute these critical groups for some of the points in  $\Pi_l$ .

**Definition 1.2.** For  $p \in Q_l \cap C_{l2}$ , denote by  $\Gamma_p$  the set of those points  $(a, b) \in \mathbb{R}^2 \setminus \Sigma$  for which there is a curve  $\gamma = (\gamma_1, \gamma_2) \in C([0, 2]) \cap C^1([0, 1])$ ,  $\gamma(0) = p$ ,  $\gamma(2) = (a, b)$  such that

- (i)  $\gamma((0, 2]) \cap \Sigma = \emptyset$ ,
- (ii)  $\gamma'_1, \gamma'_2 \leq 0$  on  $[0, 1]$ ,
- (iii)  $\gamma(0) + \gamma'(0) = \gamma(1)$ .

We can take  $\gamma$  to be the line segment joining  $p = (p_1, p_2)$  and  $(a, b)$  if it intersects  $\Sigma$  only at  $p$  and  $a \leq p_1, b \leq p_2$ . For example, suppose  $C_{l1}$  and  $C_{l2}$  intersect only at  $(\lambda_l, \lambda_l)$  and  $\Pi_l \cap \Sigma = \emptyset$ , as in the figure below. Then the region  $\Pi_l$  consists of two connected components. The component to the left (resp. right) of  $(\lambda_l, \lambda_l)$  is contained in  $\Gamma_p$  for each  $p \in Q_l \cap C_{l2}$  to the left (resp. right) of  $(\lambda_l, \lambda_l)$ . Also note that  $\Gamma_{(\lambda_l, \lambda_l)}$  always contains  $I_l$ .



Observe that, by (i) and the homotopy invariance of the critical groups, the  $C_*(I(\cdot, \gamma(t)), 0)$  are defined for all  $t \in (0, 2]$  and are independent of  $t$ . Let  $K_p$  denote the set of critical points of  $I_p = I(\cdot, p)$  and let  $\widehat{K}_p = \{u \in K_p : \|u\| = 1\}$ .

**Theorem 1.3.** If  $p \in Q_l \cap C_{l2}$  and  $(a, b) \in \Gamma_p$ , then

$$(1.3) \quad C_q(I, 0) \cong \begin{cases} H^{d_l - q - 1}(\widehat{K}_p), & q \neq d_l - 1, \\ H^0(\widehat{K}_p)/\mathbb{Z}, & q = d_l - 1. \end{cases}$$

Theorem 1.3 is proved in Section 2 and should be compared with Theorem 2 of Dancer [2]. Note that Theorem 1.3 completely determines the critical groups for all  $(a, b) \in \Pi_l$  when the region  $\Pi_l$  is free of  $\Sigma$ .

As an application we consider the problem

$$(1.4) \quad \begin{cases} -\Delta u = f(x, u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

where  $f \in C(\bar{\Omega} \times \mathbb{R})$  and

$$(1.5) \quad f(x, t) = \begin{cases} b_0 t^+ - a_0 t^- + o(t) & \text{as } t \rightarrow 0, \\ b t^+ - a t^- + o(t) & \text{as } t \rightarrow \infty \end{cases}$$

with  $(a_0, b_0), (a, b) \notin \Sigma$ . It was shown in Perera and Schechter [4] that this problem has a nontrivial solution if, for some  $q$ ,

$$(1.6) \quad C_q(I_0, 0) \not\cong C_q(I, 0)$$

where  $I_0 = I(\cdot, a_0, b_0)$ . By Theorem 1.1, this is the case when  $(a_0, b_0) \in I_j$  and  $(a, b) \in I_l$  with  $j \neq l$  and when  $(a_0, b_0) \in I_j$  and  $(a, b) \in \Pi_l$  (or  $(a, b) \in I_l$  and  $(a_0, b_0) \in \Pi_j$ ). We use Theorem 1.3 to consider the case where  $(a_0, b_0) \in \Pi_j$ ,  $(a, b) \in \Pi_l$ , and  $\lambda_l$  is a multiple eigenvalue. Let  $i_q(p)$  denote the right hand side of (1.3) and let  $Q = \bigcup_l Q_l$ .

**Theorem 1.4.** *If  $(a_0, b_0), (a, b) \notin \Sigma$ , then (1.4) has a nontrivial solution in each of the following cases:*

- (i)  $(a_0, b_0) \in \Gamma_p$  for some  $p \in (Q_l \cap C_{l2}) \setminus C_{l1}$  with  $i_q(p) \neq 0$  for some  $q$ , and  $(a, b) \in Q \setminus \Pi_l$ ,
- (ii)  $(a, b) \in \Gamma_p$  for some  $p \in (Q_l \cap C_{l2}) \setminus C_{l1}$  with  $i_q(p) \neq 0$  for some  $q$ , and  $(a_0, b_0) \in Q \setminus \Pi_l$ .

Note that, since  $p \notin C_{l1}$  and  $\gamma'_1, \gamma'_2 \leq 0$  on  $[0, 1]$ ,  $\gamma(t) \in \Pi_l$  for  $t > 0$  sufficiently small, and hence  $i_q(p) \neq 0$  implies  $d_{l-1} < q < d_l$  by the remark following Definition 1.2 and part (ii) of Theorem 1.1. For example,  $i_{d_{l-1}}(p) \neq 0$  if  $\widehat{K}_p$  is disconnected (this may be caused by a symmetry of  $\Omega$ , and hence of  $I_p$ ).

Theorem 1.3 also implies that  $H^*(\widehat{K}_{p_1}) \cong H^*(\widehat{K}_{p_2})$  whenever  $p_1, p_2 \in Q_l \cap C_{l2}$  and  $\Gamma_{p_1} \cap \Gamma_{p_2} \neq \emptyset$ . Moreover, it can be combined with Theorem 1.1 to obtain extra information on  $K_p$ . Let  $M_l = N_l^\perp$ . It was shown in Schechter [8] that there are continuous and positive homogeneous functions  $\tau : N_l \rightarrow M_l$ ,  $\theta : M_{l-1} \rightarrow N_{l-1}$  such that  $w_0 = \tau(v)$ ,  $v_0 = \theta(w)$  are the unique solutions of

$$(1.7) \quad I_p(v + w_0) = \inf_{w \in M_l} I_p(v + w), \quad v \in N_l,$$

$$(1.8) \quad I_p(v_0 + w) = \sup_{v \in N_{l-1}} I_p(v + w), \quad w \in M_{l-1},$$

respectively. Let

$$(1.9) \quad S_{l1} = \{v + \tau(v) : v \in N_l\},$$

$$(1.10) \quad S_{l2} = \{\theta(w) + w : w \in M_{l-1}\},$$

$$(1.11) \quad S_l = S_{l1} \cap S_{l2}.$$

We will see in Section 3 that  $S_l$  is a radial manifold modeled on the eigenspace  $E(\lambda_l)$ . Let  $\widehat{S}_l = \widehat{S}_l(p) = \{u \in S_l : \|u\| = 1\}$ . It follows from Lemmas 3.10 and 3.16 of Schechter [7] that  $\widehat{K}_p \subset \widehat{S}_l$ . Moreover, by Lemmas 3.6 and 3.7 of Schechter [6],  $I_p \leq 0$  on  $S_{l1}$  and  $K_p = S_{l1} \cap I_p^{-1}(0)$  if  $p \in C_{l2}$ , while  $I_p \geq 0$  on  $S_{l2}$  and  $K_p = S_{l2} \cap I_p^{-1}(0)$  if  $p \in C_{l1}$ , so  $\widehat{K}_p = \widehat{S}_l$  if  $p \in Q_l \cap C_{l1} \cap C_{l2}$ . We shall prove

**Theorem 1.5.** *Let  $p \in (Q_l \cap C_{l2}) \setminus C_{l1}$ . If  $\Gamma_p \neq \emptyset$ , then  $\widehat{K}_p$  is a proper subset of  $\widehat{S}_l$ . In particular,  $\widehat{K}_p$  is a single point if  $\lambda_l$  is simple.*

As we will see in Section 2, the proof of Theorem 1.3 involves considering  $I|_{S_{l1}}$ . We suspect that there is a counterpart of this theorem for the lower curve  $C_{l1}$ , but the proof seems to be more complicated because the manifold  $S_{l2}$  is infinite dimensional. However, we believe that such a result can be obtained by looking at  $I|_{S_l}$ . In preparation for a future work carrying out this idea, we prove the following theorem in Section 4.

**Theorem 1.6.** *If  $(a, b) \in \Pi_l \setminus \Sigma$ , then*

$$(1.12) \quad C_q(I, 0) \cong \begin{cases} H^{d_l - q - 1}(\widehat{S}_l^+), & q \neq d_l - 1, \\ H^0(\widehat{S}_l^+)/\mathbb{Z}, & q = d_l - 1, \end{cases}$$

where  $\widehat{S}_l^+ = \{u \in \widehat{S}_l(a, b) : I(u) > 0\}$ .

*Notation.* We will use the customary notation

$$(1.13) \quad I^\alpha := \{u \in H : I(u) \leq \alpha\}, \quad \alpha \in \mathbb{R},$$

for the sublevel sets.

## 2. PROOF OF THEOREM 1.3

Let

$$(2.1) \quad J(u, t) = I(u, \gamma(t)), \quad u \in H, t \in [0, 2],$$

where  $\gamma$  is as in Definition 1.2. Then  $J(\cdot, 0) = I_p$ ,  $J(\cdot, 2) = I(\cdot, a, b) = I$ , and the  $C_*(J(\cdot, t), 0)$  are independent of  $t \in (0, 2]$ , so

$$(2.2) \quad C_q(I, 0) \cong C_q(\tilde{I}, 0)$$

where  $\tilde{I} = J(\cdot, 1)$ . By (ii),  $\gamma(t) \in I_{l1} \cup \Pi_l$  for  $t > 0$  sufficiently small, so both sides of (1.3) are zero for  $q \geq d_l$  by Theorem 1.1, and hence we may assume that  $q \leq d_l - 1$ .

Fix  $\alpha < 0$ . Then

$$(2.3) \quad C_q(\tilde{I}, 0) \cong H_q(H, \tilde{I}^\alpha)$$

since 0 is the only critical point of  $\tilde{I}$ . By (ii),  $J(u, t)$  is nondecreasing in  $t$  on  $[0, 1]$ , so  $\tilde{I}^\alpha \subset I_p^\alpha \subset I_p^0 \setminus K_p$ . We claim that  $\tilde{I}^\alpha$  is a strong deformation retract of  $I_p^0 \setminus K_p$ . To see this let  $\eta = \eta(u, t)$  be the flow generated by

$$(2.4) \quad \begin{cases} \frac{d\eta}{dt} = \frac{\alpha - I_p(u) - \partial_t J(\eta, t)}{\|\partial_u J(\eta, t)\|^2} \partial_u J(\eta, t) & \text{for } t \in [0, 1], \\ \eta(u, 0) = u & \text{on } I_p^0 \setminus (K_p \cup \tilde{I}^\alpha). \end{cases}$$

Then

$$(2.5) \quad \frac{d}{dt} J(\eta, t) = (\partial_u J(\eta, t), \frac{d\eta}{dt}) + \partial_t J(\eta, t) = \alpha - I_p(u),$$

so

$$(2.6) \quad J(\eta(u, t), t) = (1 - t) I_p(u) + \alpha t.$$

In particular,  $I_p(\eta) = J(\eta, 0) \leq J(\eta, t) < 0$  for  $t \in (0, 1]$  since  $I_p(u) \leq 0$ , so the flow stays in  $I_p^0 \setminus K_p$ . Also,  $\tilde{I}(\eta(u, 1)) = J(\eta(u, 1), 1) = \alpha$ . Moreover, at  $t = 0$ ,

$$(2.7) \quad \begin{aligned} \alpha - I_p(u) - \partial_t J(\eta, t) &= \alpha - I_p(u) - \partial_t J(u, 0) = \alpha - I(u, p + \gamma'(0)) \\ &= \alpha - \tilde{I}(u) \end{aligned}$$

by (iii), so  $\eta(u, t) = u$  for all  $t \in [0, 1]$  if  $\tilde{I}(u) = \alpha$ . It follows that  $\eta$  can be continuously extended to  $(I_p^0 \setminus K_p) \times [0, 1]$  by setting  $\eta(u, t) = u$  on  $\tilde{I}^\alpha \times [0, 1]$ , defining a strong deformation retraction of  $I_p^0 \setminus K_p$  onto  $\tilde{I}^\alpha$ . Hence

$$(2.8) \quad H_q(H, \tilde{I}^\alpha) \cong H_q(H, I_p^0 \setminus K_p).$$

Since  $p \in C_{l_2}$ ,

$$(2.9) \quad I_p(u) \leq 0, \quad u \in S_{l_1},$$

and

$$(2.10) \quad K_p = \{u \in S_{l_1} : I_p(u) = 0\}.$$

Moreover,  $I_p(v + w)$  is convex in  $w \in M_l$  for fixed  $v \in N_l$  and  $I_p(u + w) > 0$  for  $u \in K_p, w \in M_l \setminus \{0\}$ . Hence the mapping  $(I_p^0 \setminus K_p) \times [0, 1] \rightarrow I_p^0 \setminus K_p$ ,

$$(2.11) \quad (v + w, t) \mapsto v + (1 - t)w + t\tau(v),$$

is a strong deformation retraction of the pair  $(H, I_p^0 \setminus K_p)$  onto the pair  $(S_{l_1}, S_{l_1} \setminus K_p)$ . Since  $S_{l_1}$  is a radial manifold and  $I$  is positive homogeneous,  $(S_{l_1}, S_{l_1} \setminus K_p)$  is homotopic to  $(B_{l_2}, \widehat{S}_{l_1} \setminus \widehat{K}_p)$  where  $B_{l_2} = \{u \in S_{l_1} : \|u\| \leq 1\}$  and  $\widehat{S}_{l_1} = \partial B_{l_2}$ . Hence

$$(2.12) \quad H_q(H, I_p^0 \setminus K_p) \cong H_q(B_{l_2}, \widehat{S}_{l_1} \setminus \widehat{K}_p).$$

Combining (2.2), (2.3), (2.8), and (2.12) we have

$$(2.13) \quad C_q(I, 0) \cong H_q(B_{l_2}, \widehat{S}_{l_1} \setminus \widehat{K}_p).$$

Now consider the exact sequence of the triple  $(B_{l_2}, \widehat{S}_{l_1}, \widehat{S}_{l_1} \setminus \widehat{K}_p)$ :

$$(2.14) \quad \begin{aligned} \cdots \longrightarrow H_{q+1}(B_{l_2}, \widehat{S}_{l_1} \setminus \widehat{K}_p) &\longrightarrow H_{q+1}(B_{l_2}, \widehat{S}_{l_1}) \longrightarrow H_q(\widehat{S}_{l_1}, \widehat{S}_{l_1} \setminus \widehat{K}_p) \\ &\longrightarrow H_q(B_{l_2}, \widehat{S}_{l_1} \setminus \widehat{K}_p) \longrightarrow H_q(B_{l_2}, \widehat{S}_{l_1}) \longrightarrow \cdots \end{aligned}$$

Since  $B_{l_2}$  is homeomorphic to the unit ball in  $N_l$ ,

$$(2.15) \quad H_q(B_{l_2}, \widehat{S}_{l_1}) = \begin{cases} \mathbb{Z}, & q = d_l, \\ 0, & q \neq d_l, \end{cases}$$

so it follows that

$$(2.16) \quad H_q(B_{l_2}, \widehat{S}_{l_1} \setminus \widehat{K}_p) \cong H_q(\widehat{S}_{l_1}, \widehat{S}_{l_1} \setminus \widehat{K}_p), \quad q < d_l - 1.$$

Since  $H_{d_l}(B_{l_2}, \widehat{S}_{l_1} \setminus \widehat{K}_p) \cong C_{d_l}(I, 0) = 0$ , for  $q = d_l - 1$  we get the short exact sequence

$$(2.17) \quad 0 \longrightarrow \mathbb{Z} \longrightarrow H_{d_l-1}(\widehat{S}_{l_1}, \widehat{S}_{l_1} \setminus \widehat{K}_p) \longrightarrow H_{d_l-1}(B_{l_2}, \widehat{S}_{l_1} \setminus \widehat{K}_p) \longrightarrow 0,$$

so

$$(2.18) \quad H_{d_l-1}(B_{l_2}, \widehat{S}_{l_1} \setminus \widehat{K}_p) \cong H_{d_l-1}(\widehat{S}_{l_1}, \widehat{S}_{l_1} \setminus \widehat{K}_p) / \mathbb{Z}.$$

Since

$$(2.19) \quad H_q(\widehat{S}_{l1}, \widehat{S}_{l1} \setminus \widehat{K}_p) \cong H^{d_l - q - 1}(\widehat{K}_p)$$

by the Alexander duality theorem, (1.3) follows from (2.13), (2.16), and (2.18).

### 3. PROOF OF THEOREM 1.5

**Lemma 3.1.** *There are continuous and positive homogeneous functions  $\xi : E(\lambda_l) \rightarrow N_{l-1}$ ,  $\eta : E(\lambda_l) \rightarrow M_l$  such that*

$$(3.1) \quad S_l = \{\xi(y) + y + \eta(y) : y \in E(\lambda_l)\}.$$

*Proof.* We have to show that for each  $y \in E(\lambda_l)$ , there is exactly one  $u = v + y + w \in N_{l-1} \oplus E(\lambda_l) \oplus M_l$  that is in  $S_l$ . Fix  $y \in E(\lambda_l)$  and let

$$(3.2) \quad L(v, w) = I_p(u).$$

Then  $u \in S_l$  if and only if

$$(3.3) \quad L(v, w) = \inf_{h \in M_l} L(v, h),$$

while  $u \in S_{l2}$  if and only if

$$(3.4) \quad L(v, w) = \sup_{g \in N_{l-1}} L(g, w).$$

Thus  $u \in S_l$  if and only if

$$(3.5) \quad L(v, h) \geq \alpha, \quad h \in M_l, \quad L(g, w) \leq \alpha, \quad g \in N_{l-1},$$

where

$$(3.6) \quad \alpha = \sup_{g \in N_{l-1}} \inf_{h \in M_l} L(g, h) = \inf_{h \in M_l} \sup_{g \in N_{l-1}} L(g, h).$$

Let  $u_i = v_i + y + w_i \in S_l$ ,  $i = 1, 2$ , so

$$(3.7) \quad L(v_i, h) \geq \alpha, \quad h \in M_l, \quad L(g, w_i) \leq \alpha, \quad g \in N_{l-1}.$$

In particular

$$(3.8) \quad L(v_i, w_1) \geq \alpha, \quad i = 1, 2.$$

Since  $L(v, w)$  is strictly concave in  $v$ , if  $v_1 \neq v_2$  this implies

$$(3.9) \quad L((1 - t)v_1 + tv_2, w_1) > \alpha, \quad t \in (0, 1),$$

contradicting (3.7). Similarly  $w_1 = w_2$ . □

We are now ready to prove Theorem 1.5. Since  $\Gamma_p \neq \emptyset$  and  $p \notin C_{l1}$ , using (ii) of Definition 1.2 we can take a point  $(a, b) \in \Gamma_p \cap \Pi_l$ . Then

$$(3.10) \quad C_{d_{l-1}}(I, 0) = 0$$

by Theorem 1.1, so

$$(3.11) \quad i_{d_{l-1}}(p) = 0$$

by Theorem 1.3. Since  $\widehat{K}_p \subset \widehat{S}_l$  and  $\widehat{S}_l$  is homeomorphic to the  $(d_l - d_{l-1} - 1)$ -dimensional sphere by Lemma 3.1, this implies that  $\widehat{K}_p$  is a proper subset of  $\widehat{S}_l$ . If  $\lambda_l$  is simple,  $\Pi_l \cap \Sigma = \emptyset$  by Gallouët and Kavian [3], so  $\Gamma_p \neq \emptyset$  for any  $p \in Q_l \cap C_{l2}$ .

4. PROOF OF THEOREM 1.6

Since 0 is the only critical point of  $I$ ,

$$(4.1) \quad C_q(I, 0) \cong H_q(H, I^0 \setminus \{0\}).$$

Arguments similar to those in the proof of Theorem 1.3 show that (2.11) defines a strong deformation retraction of  $(H, I^0 \setminus \{0\})$  onto  $(S_{l1}, (I^0 \cap S_{l1}) \setminus \{0\})$  and

$$(4.2) \quad H_q(S_{l1}, (I^0 \cap S_{l1}) \setminus \{0\}) \cong \begin{cases} H^{d_l - q - 1}(\widehat{S}_{l1}^+), & q \neq d_l - 1, \\ H^0(\widehat{S}_{l1}^+)/\mathbb{Z}, & q = d_l - 1, \end{cases}$$

where  $\widehat{S}_{l1}^+ = \{u \in \widehat{S}_{l1} : I(u) > 0\}$ . For  $u \in N_l$ , let  $\zeta(u) = u + \tau(u)$ . Then  $\widehat{S}_{l1} = \{\zeta(v + y) : v + y \in N_{l-1} \oplus E(\lambda_l), \|\zeta(v + y)\| = 1\}$ . We claim that

$$(4.3) \quad (\zeta(v + y), t) \mapsto \frac{\zeta((1 - t)v + t\xi(y) + y)}{\|\zeta((1 - t)v + t\xi(y) + y)\|}, \quad \zeta(v + y) \in \widehat{S}_{l1}^+, t \in [0, 1],$$

where  $\xi$  is as in Lemma 3.1, defines a strong deformation retraction of  $\widehat{S}_{l1}^+$  onto  $\widehat{S}_l^+$ . We have to show that

$$(4.4) \quad I(\zeta((1 - t)v + t\xi(y) + y)) > 0.$$

First note that

$$(4.5) \quad \begin{aligned} I(\zeta(\xi(y) + y)) &\geq I(v + y + \tau(\xi(y) + y)) \\ &\geq I(\zeta(v + y)). \end{aligned}$$

So, since  $I(v + y + w)$  is concave in  $v$  for any  $w \in M_l$ ,

$$(4.6) \quad \begin{aligned} I((1 - t)v + t\xi(y) + y + w) &\geq (1 - t)I(v + y + w) + tI(\xi(y) + y + w) \\ &\geq (1 - t)I(\zeta(v + y)) + tI(\zeta(\xi(y) + y)) \\ &\geq I(\zeta(v + y)) > 0, \end{aligned}$$

and (4.4) follows. This completes the proof of Theorem 1.6.

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DEPARTMENT OF MATHEMATICAL SCIENCES, FLORIDA INSTITUTE OF TECHNOLOGY, MELBOURNE, FLORIDA 32901-6975

*E-mail address:* [kperera@winnie.fit.edu](mailto:kperera@winnie.fit.edu)

*URL:* <http://winnie.fit.edu/~kperera/>

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA-IRVINE, IRVINE, CALIFORNIA 92697-3875

*E-mail address:* [mschecht@math.uci.edu](mailto:mschecht@math.uci.edu)

*URL:* <http://www.math.uci.edu/mschecht.html>