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A RANDOM VARIATIONAL PRINCIPLE WITH APPLICATION TO WEAK HADAMARD DIFFERENTIABILITY OF CONVEX INTEGRAL FUNCTIONALS

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Abstract. We present a random version of the Borwein-Preiss smooth variational principle, stating that under suitable conditions, the set of minimizers of a perturbed function depending on a random variable, admits a measurable selection. Two applications are given. The first one shows that if E is a superreflexive Banach space, then any convex continuous integral functional on $L^1(T,\mu;E)$ from a certain class (in particular the usual L^1 norm), is weak Hadamard differentiable on a subset whose complement is σ-very porous. The second application is a random version of the Caristi fixed point theorem for multifunctions.

Ekeland's variational principle and its smooth analogues are useful tools in the study of non-linear problems in various areas of mathematics (see for instance [E1], [E2], [B-P], [D-G-Z1], [D-G-Z2]).

In this paper we present a random version of the smooth variational principle of Borwein-Preiss [B-P]. Namely, we prove that a suitable perturbed function of a given one admits, as in [B-P], a minimum point, which in our setting is a measurable function of a random variable.

We give two applications of our random smooth variational principle. The first one is about weak Hadamard differentiability of some convex integral functionals in the Lebesgue-Bochner space $L^1(T,\mu;E)$.

Borwein and Fitzpatrick [B-F] have shown that in $L^1(T,\mu)$, where μ is sigma finite, there exists an equivalent weak Hadamard differentiable norm; hence using Preiss-Phelps-Namioka's theorem [P-P-N], they establish that $L^1(T,\mu)$ is a weak Hadamard Asplund space. Examination of their proofs will convince the reader that the results of Borwein-Fitzpatrick remain valid in the space $L^1(T,\mu;E)$, provided E is a reflexive Banach space and μ is sigma finite, since the Dunford-Pettis criterium for weak compactness in L^1 is crucial in their proofs.

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Here we prove that if E is a separable Banach space with a uniformly Fréchet differentiable norm, then any convex continuous integral functional on $L^1(T,\mu;E)$ from a certain class (in particular the usual L^1 norm) is weak Hadamard differentiable on a subset whose complement is σ-very porous. The proof of this result is direct and, unlike in [B-F], does not relay on the deep theorem in [P-P-N].

As a second application, a random version of the Caristi fixed point theorem for multifunctions is obtained.

Let $(E,\|.\|)$ be a Banach space with dual E^* and let $S=\{x\in E:\|x\|=1\}$.

The norm $\|.\|$ of E is said to be *uniformly Fréchet differentiable* if for every $x\in S$ there exists an element $\nabla\|x\|\in E^*$ such that the following condition holds: for every $\varepsilon>0$ there exists $\delta>0$ such that

$$\frac{\|x+th\|-\|x\|}{t}-\langle\nabla\|x\|,h\rangle<\varepsilon \quad \text{for every} \quad x\in S, h\in S, t\in(0,\delta).$$

The norm $\|.\|$ of E is said to be *weak Hadamard differentiable* at $x\in E$ if there exists an element $\nabla\|x\|\in E^*$ such that for every weakly compact subset W of E the following holds: for every $\varepsilon>0$ there exists $\delta>0$ such that

$$\frac{\|x+th\|-\|x\|}{t}-\langle\nabla\|x\|,h\rangle<\varepsilon \quad \text{for every} \quad h\in W \text{ and } t\in(0,\delta).$$

About measurability we retain notation and terminology of Himmelberg [H].

The following theorem is a random version of the Borwein-Preiss smooth variational principle.

Theorem 1. *Suppose that $(E,\|.\|)$ is a separable Banach space and $(T,\mathcal{A},\mu)$ is a measurable space with a complete σ-finite measure μ. Let $F:T\to 2^E$ be a measurable multifunction with non-empty closed values and $f:T\times E\to\mathbf{R}$ be a function with the following properties:*

(1) $\inf_{x\in F(t)} f(t,x)>-\infty$, *for every* $t\in T$;
(2) $f(.,x)$ *is measurable for every* $x\in E$;
(3) $f(t,.)$ *is continuous for every* $t\in T$.

Let $x_0:T\to E$ be a measurable single-valued selection of F such that

$$f(t,x_0(t))<\inf_{x\in F(t)} f(t,x)+\varepsilon_0(t) \quad \textit{for every} \quad t\in T,$$

where $\varepsilon_0:T\to(0,+\infty)$ is a given measurable function. Let $p\geq 1$ be given and let $\varepsilon:T\to(0,+\infty)$ and $\lambda:T\to(0,+\infty)$ be measurable functions with $\varepsilon(t)>\varepsilon_0(t)$, $t\in T$.

Then there exist measurable selections of F, say $x_n, v:T\to E$ and measurable functions $\mu_n:T\to(0,1)$ such that, for every $t\in T$, we have: $\sum_{n=0}^{\infty}\mu_n(t)=1$, and

(4) $x_n(t)\to v(t)$ *as* $n\to\infty$;
(5) $f(t,v(t))+\Delta(t,v(t))\leq f(t,x)+\Delta(t,x)$, *for every* $x\in F(t)$, *where*
(6) $\Delta(t,x)=\frac{\varepsilon(t)}{\lambda(t)^p}\sum_{n=0}^{\infty}\mu_n(t)\|x-x_n(t)\|^p$;
(7) $\|x_n(t)-v(t)\|<\lambda(t)$ *for every* $n=0,1,2,\dots$.

Proof. Set $q(t)=\frac{1}{2}\left[\frac{\varepsilon(t)-\varepsilon_0(t)}{\varepsilon(t)+\varepsilon_0(t)}\right]^{1/2}$. For $n=0,1,\dots$ and $t\in T, x\in E$, define inductively

$$G_n(t)=\{x\in F(t): f_n(t,x)\leq\inf_{z\in F(t)} f_n(t,z)+\varepsilon_n(t)\}, \tag{8}$$

where

$$f_{n+1}(t,x) = f_n(t,x) + \frac{\varepsilon(t)}{\lambda(t)^p}\mu_n(t)\|x - x_n(t)\|^p, f_0(t,x) = f(t,x), \tag{9}$$

$$\varepsilon_n(t) = \varepsilon_0(t)q(t)^{2n}, \quad \mu_n(t) = (1-q(t))q(t)^n,$$

and $x_n : T \to E$ is a measurable function satisfying

$$x_n(t) \in G_n(t) \tag{10}$$

for every $t \in T$.

We shall prove by induction that the definitions of x_n and f_n are correct. For $n = 0$ this is true by assumption. Suppose that x_{n-1} and f_{n-1} are defined. Now, by (9), f_n is well defined. By [H, Theorem 6.5] the multifunction $t \mapsto f_n(t, F(t))$ is weakly measurable (in fact it is measurable, by [H, Theorem 3.5 (iii)]) and by [H, Theorem 6.6] the function $t \mapsto \inf_{z\in F(t)} f_n(t,z)$ is measurable. By [H, Theorem 6.4] and [H, Theorem 3.5 (iii)] the multifunction $t \mapsto G_n(t)$ is measurable and Kuratowski and Ryll-Nardzewski's theorem [K-RN] (see also [H, Theorem 5.1]) produces a measurable function x_n satisfying (10), completing the induction.

We shall prove that $\{x_n(t)\}$ is a fundamental sequence for every $t \in T$. By (9) and (10) we have

$$\begin{aligned}
&\tfrac{\varepsilon(t)}{\lambda(t)^p}\mu_n(t)\|x_{n+1}(t) - x_n(t)\|^p \\
&= f_{n+1}(t, x_{n+1}(t)) - f_n(t, x_{n+1}(t)) \\
&= f_{n+1}(t, x_{n+1}(t)) - f_{n+1}(t, x_n(t)) + f_n(t, x_n(t)) - f_n(t, x_{n+1}(t)) \\
&\le \varepsilon_{n+1}(t) + \varepsilon_n(t) \\
&= \varepsilon_0(t)q(t)^{2n}(q(t)^2 + 1).
\end{aligned}$$

Hence for $m > n$ we obtain

$$\|x_m(t) - x_n(t)\| \le \lambda(t)\Big(\frac{\varepsilon_0(t)}{\varepsilon(t)}.\frac{1+q(t)^2}{1-q(t)^2}\Big)^{\frac{1}{p}} q(t)^{\frac{n}{p}} < \lambda(t)q(t)^{\frac{n}{p}}. \tag{11}$$

Therefore $\{x_n(t)\}_{n=0}^{\infty}$ is a fundamental sequence and so converges to $v(t)$ and clearly v is measurable. From (11), letting $m \to +\infty$, we obtain $\|v(t) - x_n(t)\| < \lambda(t)$, which is (7).

To establish (5), let $\gamma > 0$ be given. As $(f + \Delta)(t, .)$ is continuous, there exists $\delta(t) > 0$ such that

$$f(t, v(t)) + \Delta(t, v(t)) < f(t,x) + \Delta(t,x) + \gamma/3, \tag{12}$$

whenever $\|x - v(t)\| < \delta(t)$. For fixed $t \in T$, choose n sufficiently large so that the following inequalities hold: $\varepsilon_n(t) < \gamma/3$, $\|v(t) - x_k(t)\| < \delta(t)$ for every $k \ge n$, and $\frac{\varepsilon(t)}{\lambda(t)^p}\sum_{k=n}^{\infty}\mu_k(t)\|x_n(t) - x_k(t)\|^p < \gamma/3$.

For every $x \in F(t)$, using (12), (8), (9) and (10), we can write

$$\begin{aligned} & f(t,v(t)) + \Delta(t,v(t)) \\ & < f(t,x_n(t)) + \Delta(t,x_n(t)) + \gamma/3 \\ & = f_n(t,x_n(t)) + \frac{\varepsilon(t)}{\lambda(t)^p} \sum_{k=n}^{\infty} \mu_k(t) \|x_n(t) - x_k(t)\|^p + \gamma/3 \\ & < f_n(t,x) + \varepsilon_n(t) + \gamma/3 + \gamma/3 \\ & < f(t,x) + \Delta(t,x) + \gamma \end{aligned}$$

and (5) is proved. □

Remark. It is clear that if the conditions of Theorem 1 are satisfied for a.e. $t \in T$, then its conclusions are satisfied for a.e. $t \in T$, too.

Theorem 2 (Random Ekeland's variational principle). *Under the assumptions of Theorem 1, there exists a measurable selection $v : T \to E$ of F such that:*

(5′) $f(t,v(t)) < f(t,x) + \frac{\varepsilon(t)}{\lambda(t)} \|x - v(t)\|$ *for every* $t \in T$ *and* $x \in F(t), x \neq v(t)$;

(7′) $\|x_0(t) - v(t)\| < \lambda(t)$ *for every* $t \in T$;

(well posedness) $x_n \to v(t)$ *whenever* $f(t,x_n) + \frac{\varepsilon(t)}{\lambda(t)} \|x_n - v(t)\| \to f(t,v(t))$.

Proof. (5′) follows immediately from (5) of Theorem 1 with $p = 1$, and $\varepsilon'(t) = \frac{1}{2}(\varepsilon_0(t) + \varepsilon(t))$ in the place of $\varepsilon(t)$. □

Now we recall the following definition.

A subset P of E is said to be *very porous* if there exists $\alpha > 0$ for which the following holds: for every $x \in E$ and every $r \in (0,\alpha)$ there is $y \in E$ such that

$$B(y,\alpha r) \subset B(x,r) \cap (E \setminus P).$$

P is called *σ-very porous* if it is a countable union of very porous sets.

In the sequel $L^1(T,\mu;E)$ will denote the usual Lebesgue-Bochner space, i.e. the set of all (equivalence classes of) μ-Bochner integrable functions $f : T \to E$ with the norm $\|f\|_{L^1} = \int_T \|f(t)\| d\mu(t)$.

Theorem 3. *Let $(E, \|.\|)$ be a separable superreflexive Banach space and let $(T, \mathcal{A}, \mu)$ be a measurable space, with a complete σ-finite measure μ, $\int_T d\mu(t) = 1$. Suppose that $f : T \times E \to \mathbf{R}$ is a function satisfying conditions* (1), (2), (3) *of Theorem 1, and*

a) $f(t,.)$ *is convex for every* $t \in T$;

b) *there exists a function* $L \in L^\infty(T,\mu;\mathbf{R})$ *such that for all* $t \in T$

$$|f(t,x_1) - f(t,x_2)| \leq L(t)\|x_1 - x_2\| \qquad \text{for every } x_1, x_2 \in E; \tag{13}$$

c) $f(.,0) \in L^1(T,\mu;\mathbf{R})$.

Define the function $g : L^1(T,\mu;E) \to \mathbf{R}$ by $g(x) = \int_T f(t,x(t))d\mu(t)$. Then g is weak Hadamard differentiable on a subset X_0 of $L^1(T,\mu;E)$, whose complement is σ-very porous.

Proof. Since every superreflexive Banach space has an equivalent uniformly Fréchet differentiable norm (see [D-G-Z2, Corollary 4.6, page 152]), we may suppose without loss of generality that the norm $\|.\|$ is uniformly Fréchet differentiable.

In view of b) and c), the definition of g makes sense. The absolute continuity of the Lebesgue integral (see [K-F, Theorem V.5.5]) allows us to define, for every $x, h \in L^1(T,\mu;E)$,

(14)
$$\begin{aligned}\gamma_n(x,h) = \sup\Big\{ \quad & \gamma > 0 : \int_\Gamma \Big(f(t,x(t)+h(t)) + f(t,x(t)-h(t)) \\ & -2f(t,x(t))\Big) d\mu(t) \le \tfrac{1}{n} \\ & \text{for every } \ \Gamma \subset T \ \text{ with } \quad \mu(\Gamma) \le \gamma \|h\|_{L^1} \ \Big\}.\end{aligned}$$

Put

$$H_{n,m}(x) = \{h \in L^1(T,\mu;E) : \gamma_n(x,h) \ge \frac{1}{m}\} \tag{15}$$

and

$$\Gamma_\gamma(h) = \{t \in T : \|h(t)\| \ge \frac{1}{\gamma}\}. \tag{16}$$

Obviously

(17)
$$\mu(\Gamma_\gamma(h)) \le \gamma \int_T \|h(t)\| d\mu(t) = \gamma \|h\|_{L^1} \qquad \text{for every } \ h \in L^1(T,\mu;E).$$

Since the norm $\|.\|$ is uniformly Fréchet differentiable, it is easy to see that there exists $s_{n,m} \in (0,1/m)$ such that

(18)
$$\frac{\|x + s_{n,m}h\|^2 - \|x\|^2}{s_{n,m}} - \langle \nabla \|x\|^2, h\rangle < \frac{1}{n} \qquad \text{whenever } \ \|x\| < 1, \|h\| \le m.$$

Define the set

$$\begin{aligned} X_{n,m} \quad = \quad & \{x \in L^1(T,\mu;E) : \quad \text{there exists } s \in (0,1/m) \ \text{ such that} \\ & \frac{g(x+sh) + g(x-sh) - 2g(x)}{s} < 14/n, \quad \text{for every } h \in H_{n,m}(x)\}.\end{aligned}$$

Claim. *$L^1(T,\mu;E) \setminus X_{n,m}$ is very porous for every integer n, m.*

Assume the contrary. Then for some integer n, m, for $\alpha \in (0, s_{n,m}/2n)$, there exist $x_0 \in L^1(T,\mu,E)$ and $r \in (0,\alpha)$ such that

$$B(v;\alpha r) \cap (L^1(T,\mu;E) \setminus X_{n,m}) \ne \emptyset \qquad \text{for every } \ v \in B(x_0;r). \tag{19}$$

By Himmelberg [H] it follows that the multivalued mapping $F : t \mapsto B[x_0(t);r]$ is measurable.

Without loss of generality we may assume that $\|L\|_\infty < 1$. In view of (13) we have

$$\|f(t,x) - f(t,x_0(t))\| \le r \qquad \text{for a.e. } t \in T \ \text{ and for every } \ x \in F(t) \ t \in T.$$

By Theorem 1, with the above F and $\varepsilon_0 = r = \varepsilon/2, l = r/2, p = 2$, we obtain measurable selections $v, x_n : T \to E$ of F, and measurable functions $\mu_n : T \to (0,1)$ such that for the function $\Delta(t,x)$ given by (6) (with the above constructed x_n), we have

(20)
$$f(t,x) - f(t,v(t)) \le \Delta(t,x) - \Delta(t,v(t)) \quad \text{for a.e. } t \in T \ \text{ and for every } x \in F(t),$$

and

$$\|v(t) - x_0(t)\| < \lambda \qquad \text{for a.e. } t \in T. \tag{21}$$

So by (19) we can find $z \in B(v; \alpha r) \cap (L^1(T, \mu; E) \setminus X_{n,m})$.

Hence, by definition of $X_{n,m}$, for $s = s_{n,m} r/2$, there exists $h \in H_{n,m}(z)$ such that , setting $\Gamma = \Gamma_{\gamma_n(z,h)}(h)$, we have $\mu(\Gamma) \leq \gamma_n(z,h)\|h\|_{L^1}$ (from (17)) and

$$\begin{aligned}
14/n \quad &\leq \quad \frac{g(z+sh) + g(z-sh) - 2g(z)}{s} \\
&\leq \quad \int_T \frac{f(t, v(t)+sh(t)) + f(t, v(t)-sh(t)) - 2f(t, v(t))}{s} d\mu(t) + 4/n \\
&\qquad \text{(by (13) and by the choice of } \alpha \text{ and } s) \\
&< \quad \int_{T\setminus\Gamma} \frac{f(t, v(t)+sh(t)) + f(t, v(t)-sh(t)) - 2f(t, v(t))}{s} d\mu(t) + 5/n \\
&\qquad \text{(by (14) and monotonicity of the differential quotient of a convex function)} \\
&\leq \quad \int_{T\setminus\Gamma} \frac{1}{s}\Big(\Delta(t, v(t)+sh(t)) + \Delta(t, v(t)-sh(t)) - 2\Delta(t, v(t))\Big) d\mu(t) + 5/n \\
&\qquad \text{(by (20), since (16) and (15) imply that } h(t) \leq m \text{ for } t \notin \Gamma \\
&\qquad \text{and (21) implies that } v(t)+sh(t) \in F(t)) \\
&= \quad \frac{\varepsilon}{\lambda} \int_{T\setminus\Gamma} \sum_{n=0}^{\infty} \mu_n \Big[\frac{\|\frac{v(t)-x_n(t)}{\lambda} + \frac{s}{\lambda}h(t)\|^2 - \|\frac{v(t)-x_n(t)}{\lambda}\|^2}{\frac{s}{\lambda}} \\
&\quad + \frac{\|\frac{v(t)-x_n(t)}{\lambda} - \frac{s}{\lambda}h(t)\|^2 - \|\frac{v(t)-x_n(t)}{\lambda}\|^2}{\frac{s}{\lambda}} \Big] d\mu(t) + 5/n \\
&< \quad 13/n \quad \text{(by (18), since } \frac{s}{\lambda} \leq s_{n,m}),
\end{aligned}$$

which is a contradiction.

Therefore the set $L^1(T, \mu; E) \setminus X_{n,m}$ is very porous.

We need the following.

Proposition 4. *If K is a weakly compact subset in $L^1(T, \mu; E)$, then for every integer n and every $v \in E$ there exists an integer m such that $K \subset H_{n,m}(v)$.*

Proof. Assuming the contrary, there exist an integer n and a $v \in E$ such that for every m there exists $h_m \in K \setminus H_{n,m}(v)$, i.e. there exists Γ_m such that $\mu(\Gamma_m) =$

$\frac{1}{m}\|h_m\|$ and

(22)
$$\int_{\Gamma_m}(f(t,v(t)+h_m(t))+f(t,v(t)-h_m(t))-2f(t,v(t))d\mu(t)\geq\frac{1}{n}.$$

Since K is weakly compact, it is bounded, so $\mu(\Gamma_m)\to 0$. Now by Dunford-Pettis' theorem of weak compactness in $L^1(T,\mu;E)$ (see [D-U], page 105), K is uniformly integrable, i.e. $\lim_{\mu(E)\to 0}\int_E\|h(t)\|d\mu(t)=0$ uniformly for $h\in K$. So, by (22) and (13), when m tends to infinity, we obtain a contradiction. ☐

The proof of the following proposition is the same as that of [Ph, Proposition 1.23] (which concerns Fréchet differentiability).

Proposition 5. *A convex continuous function $f:E\to\mathbf{R}$ is weak Hadamard differentiable at $x\in E$ if and only if for every $\varepsilon>0$ and every weakly compact subset $W\subset E$ there exists $t>0$ such that*

$$\frac{f(x+th)+f(x-th)-2f(x)}{t}<\varepsilon\quad\textit{for every}\quad h\in W.$$

Now we can complete the proof of Theorem 3. By Proposition 4 and Proposition 5, it follows that g is weak Hadamard differentiable on the set $X_0=\bigcap_{n,m=1}^{\infty}X_{n,m}$. By the Claim, it follows that the set $L^1(T,\mu;E)\setminus X_0$ is σ-very porous and the theorem is proved.

Comparing the Fréchet and the weak Hadamard differentiability of the usual norm $\|.\|_{L^1}$ of $L^1(T,\mu;E)$, as an interesting corollary of Theorem 3, we obtain that $\|.\|_{L^1}$ is weak Hadamard differentiable on a complement of a σ-very porous subset of $L^1(T,\mu;E)$, while $\|.\|_{L^1}$ is nowhere Fréchet differentiable (see [Ph]).

The following theorem is a random version of Caristi's fixed point theorem [A-E, Theorem 14, Ch.5 Sec.1].

Theorem 6 (Multivalued Caristi's random fixed point theorem)**.** *Suppose that $(E,\|.\|)$ is a separable Banach space and $(T,\mathcal{A},\mu)$ is a measurable space with a complete σ-finite measure μ. Let X be a closed subset of E, $F:T\times X\to X$ be a multivalued mapping. Assume that there exists a function $f:T\times X\to\mathbf{R}$ such that $f(.,x)$ is measurable, $f(t,.)$ is continuous, and for every $x\in X$ and every $t\in T$ there exists $y_{t,x}\in F(t,x)$ such that*

(23)
$$f(t,y_{t,x})+\|x-y_{t,x}\|\leq f(t,x).$$

Then there exists a measurable mapping $v:T\to X$ such that $v(t)\in F(t,v(t))$, for every $t\in T$.

Proof. Apply Theorem 2 with $\varepsilon_0<\varepsilon<1,\lambda=1$ and obtain a measurable mapping $v:T\to X$ such that

(24)
$$f(t,v(t))<f(t,x)+\varepsilon\|v(t)-x\|\quad\text{for every}\quad t\in T\quad\text{and}\quad x\in X,x\neq v(t).$$

By (23), with $x=v(t)$, and by (24), with $x=y_{t,v(t)}$, we obtain

$$\|v(t)-y_{t,v(t)}\|\leq f(t,v(t))-f(t,y_{t,v(t)})\leq\varepsilon\|v(t)-y_{t,v(t)}\|$$

for every $t\in T$. Hence $v(t)=y_{t,v(t)}$ and the theorem is proved. ☐

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