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AN EXAMPLE ON THE MANN ITERATION METHOD FOR LIPSCHITZ PSEUDOCONTRACTIONS

C. E. CHIDUME AND S. A. MUTANGADURA

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Abstract. An example of a Lipschitz pseudocontractive map with a unique fixed point is constructed for which the Mann iteration sequence fails to converge. This resolves a long standing open problem.

1. Introduction

Let K be a nonempty closed convex and bounded subset of a real uniformly convex Banach space and $T : K \to K$ be a *nonexpansive* mapping (i.e. $||Tx - Ty|| \leq ||x - y||$ for all $x, y \in K$). Then T has a fixed point $x^* \in K$ (see, e.g., [1, 5, 8]). Unlike in the case of the Banach contraction mapping principle, trivial examples show that the sequence of successive approximations $x_{n+1} = Tx_n, x_0 \in K, n \geq 0$, for a nonexpansive map T even with a unique fixed point may fail to converge to the fixed point. It suffices, for example, to take for T, a rotation of the unit ball in the plane around the origin of coordinates (see, e.g., [10]). Krasnoselski [9], however, has shown that in this example, one can obtain a convergent sequence of successive approximations if instead of T one takes the auxiliary nonexpansive mapping $\frac{1}{2}(I + T)$, where I denotes the identity transformation of the plane, i.e., if the sequence of successive approximations is defined, for arbitrary $x_0 \in K$, by

$$x_{n+1} = \frac{1}{2}(x_n + Tx_n),\ n \geq 0, \tag{1.1}$$

instead of by the usual so−called *Picard iterates,* $x_{n+1} = Tx_n, x_0 \in K, n \geq 0$. It is easy to see that the mappings T and $\frac{1}{2}(I + T)$ have the same set of fixed points, so that the limit of a convergent sequence defined by (1.1) is necessarily a fixed point of T.

More generally, if E is a normed linear space and K is a convex subset of E, a generalization of (1.1) which has proved successful in the approximation of fixed points of nonexpansive mappings $T : K \to K$ (when they exist) is the following scheme (Schaefer [12]):

$$x_0 \in K, x_{n+1} = (1 - \lambda)x_n + \lambda Tx_n,\ n \geq 0, \lambda \in (0, 1).$$

However, the most general iterative scheme now studied is the following:

$$x_0 \in K,\ x_{n+1} = (1 - c_n)x_n + c_nTx_n,\ n \geq 0, \tag{1.2}$$

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where $\{c_n\} \subset (0,1)$ is a real sequence satisfying appropriate conditions. Under the following additional assumptions: (i) $\lim c_n = 0$; (ii) $\sum_{n=0}^{\infty} c_n = \infty$, the sequence $\{x_n\}$ generated from (1.2) is generally referred to as the *Mann sequence* in light of [10]. The recursion formula (1.2) has also been used to approximate solutions of numerous nonlinear operator equations and nonlinear variational inclusions in Banach spaces (see, e.g., [3, 4, 6, 11]). A class of nonlinear mappings more general than and including the nonexpansive mappings is the class of *pseudocontractions*. A mapping T with domain $D(T)$ and range $R(T)$ in E is called *pseudocontractive* if $\langle Tx - Ty, j(x-y)\rangle \leq \|x-y\|^2$ for each $x, y \in D(T)$ and some $j(x-y) \in J(x-y)$ where $J : E \to 2^{E^*}$ is the normalized duality mapping (see, e.g., [4]). If K is a compact convex subset of a Hilbert space, and $T : K \to K$ is Lipschitz and pseudocontractive, then by the Schauder fixed point theorem, T has a fixed point in K. All efforts to approximate such a fixed point by means of the Mann iteration sequence proved abortive.

In 1974, Ishikawa introduced a new iteration scheme (defined below) and proved the following theorem.

Theorem 1.1 ([7]). *If K is a compact convex subset of a Hilbert space H, $T : K \mapsto K$ is a Lipschitzian pseudocontractive map and x_0 is any point of K, then the sequence $\{x_n\}_{n\geq 0}$ converges strongly to a fixed point of T, where x_n is defined iteratively for each positive integers $n \geq 0$ by*

$$\begin{aligned} x_{n+1} &= (1-\alpha_n)x_n + \alpha_n T y_n, \\ y_n &= (1-\beta_n)x_n + \beta_n T x_n, \end{aligned} \tag{1.3}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ are sequences of real numbers satisfying the conditions (i) $0 \leq \alpha_n \leq \beta_n < 1$*;* (ii) $\lim_{n\to\infty} \beta_n = 0$*;* (iii) $\sum_{n\geq 0} \alpha_n \beta_n = \infty$.

Since its publication in 1974, it has remained an open question (see, e.g., [6]) of whether or not the Mann recursion formula defined by (1.2), which is certainly simpler than the Ishikawa recursion formula (1.3), converges under the setting of Theorem 1.1 to a fixed point of T if the operator T is pseudocontractive and continuous (or even Lipschitzian with constant $L > 1$). In [2, Proposition 8], Borwein and Borwein gave an example of a Lipschitz map (which is not pseudocontractive) with a unique fixed point for which the Mann sequence fails to converge; and in [6], Hicks and Kubicek gave an example of a *discontinuous* pseudocontraction with a unique fixed point for which the Mann iteration does not always converge. The problem for *continuous* pseudocontraction still remained open.

It is our purpose in this paper to resolve this problem by constructing an example of a *Lipschitz* pseudocontraction with a unique fixed point for which every nontrivial Mann sequence fails to converge. This settles the above open question.

2. The example

Let X be the real Hilbert space $\mathbb{R}^2$ under the usual Euclidean inner product. If $x = (a,b) \in X$ we define $x^\perp \in X$ to be $(b, -a)$. Trivially, we have $\langle x, x^\perp\rangle = 0$, $\|x^\perp\| = \|x\|$, $\langle x^\perp, y^\perp\rangle = \langle x, y\rangle$, $\|x^\perp - y^\perp\| = \|x - y\|$ and $\langle x^\perp, y\rangle + \langle x, y^\perp\rangle = 0$ for all x, $y \in X$. We take our closed and bounded convex set K to be the closed unit ball in X and put $K_1 = \{x \in X : \|x\| \leq \frac{1}{2}\}$, $K_2 = \{x \in X : \frac{1}{2} \leq \|x\| \leq 1\}$. We define the map $T : K \longrightarrow K$ as follows:

$$Tx = \begin{cases} x + x^{\perp}\,, & \text{if} \quad x \in K_1, \\ \frac{x}{\|x\|} - x + x^{\perp}\,, & \text{if} \quad x \in K_2. \end{cases}$$

We notice that, for $x \in K_1 \cap K_2$, the two possible expressions for Tx coincide and that T is continuous on both K_1 and K_2. Hence T is continuous on all of K. We now show that T is, in fact, Lipschitz: One easily shows that $\|Tx-Ty\| = \sqrt{2}\|x-y\|$ for $x,\, y \in K_1$. For $x,\, y \in K_2$, we have

$$\begin{aligned} \left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\|^2 &= \frac{2}{\|x\|\|y\|}(\|x\|\|y\| - \langle x, y\rangle) \\ &= \frac{1}{\|x\|\|y\|}\{\|x-y\|^2 - (\|x\| - \|y\|)^2\} \\ &\leq \frac{1}{\|x\|\|y\|}\, 2\|x-y\|^2 \\ &\leq 8\|x-y\|^2\,. \end{aligned}$$

Hence, for $x,\, y \in K_2$, we have

$$\|Tx - Ty\| \leq \left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| + \|x-y\| + \|x^{\perp} - y^{\perp}\| \leq 5\|x-y\|\,,$$

so that T is Lipschitz on K_2. Now let x and y be in the interiors of K_1 and K_2 respectively. Then there exist $\lambda \in (0,1)$ and $z \in K_1 \cap K_2$ for which $z = \lambda x + (1-\lambda)y$. Hence

$$\begin{aligned} \|Tx - Ty\| &\leq \|Tx - Tz\| + \|Tz - Ty\| \\ &\leq \sqrt{2}\|x-z\| + 5\|z-y\| \\ &\leq 5\|x-z\| + 5\|z-y\| \\ &= 5\|x-y\|\,. \end{aligned}$$

Thus $\|Tx - Ty\| \leq 5\|x-y\|$ for all $x,\, y \in K$, as required.

The origin is clearly a fixed point of T. For $x \in K_1$, $\|Tx\|^2 = 2\|x\|^2$, and for $x \in K_2$, $\|Tx\|^2 = 1 + 2\|x\|^2 - 2\|x\|$. From these expressions and from the fact that $Tx = x^{\perp} \neq x$ if $\|x\| = 1$, it is easy to show that the origin is the only fixed point of T. We now show that no Mann iteration sequence for T is convergent for any nonzero starting point:

First, we show that no such Mann sequence converges to the fixed point. Let $x \in K$ be such that $x \neq 0$. Then, in case $x \in K_1$, any Mann iterate of x is actually further away from the fixed point of T than x is. This is because $\|(1-\lambda)x + \lambda Tx\|^2 = (1+\lambda^2)\|x\|^2 > \|x\|^2$ for $\lambda \in (0,1)$. If $x \in K_2$, then, for any $\lambda \in (0,1)$,

$$\begin{aligned} \|(1-\lambda)x + \lambda Tx\|^2 &= \left\| \left(\frac{\lambda}{\|x\|} + 1 - 2\lambda \right) x + \lambda x^{\perp} \right\|^2 \\ &= \left[\left(\frac{\lambda}{\|x\|} + 1 - 2\lambda \right)^2 + \lambda^2 \right] \|x\|^2 \\ &> 0\,. \end{aligned}$$

More generally, it is easy to see that for the recursion formula (1.2), if $x_0 \in K_1$, then $\|x_{n+1}\| > \|x_n\|$ for all integers $n \geq 0$, and if $x_0 \in K_2$, then $\|x_{n+1}\| \geq \frac{\sqrt{2}}{2}\|x_n\|$ for all integers $n \geq 0$. We therefore conclude that, in addition, any Mann iterate of any nonzero vector in K is itself nonzero. Thus any Mann sequence $\{x_n\}$, starting from a nonzero vector, must be infinite. For such a sequence to converge to the origin, x_n would have to lie in the neighbourhood K_1 of the origin for all $n > N_0$, for some real N_0. This is not possible because, as already established for K_1, $\|x_n\| < \|x_{n+1}\|$ for all $n > N_0$.

We now show that no Mann sequence converges to $x \neq 0$. We do this in the form of a general lemma.

Lemma 2.1. *Let M be a nonempty, closed and convex subset of a real Banach space E and let $S : M \to M$ be any continuous function. If a Mann sequence for S is norm convergent, then the corresponding limit is a fixed point for S.*

Proof. Let x_n be a Mann sequence in M for S, as defined in the recursion formula (1.2). Assume, for proof by contradiction, that the sequence converges, in norm, to x in M, where $Sx \neq x$. For each $n \in \mathbb{N}$, put $\epsilon_n = x_n - Sx_n - x + Sx$. Since S is continuous, the sequence ϵ_n converges to 0. Pick $p \in \mathbb{N}$ such that, if $m \geq p$ and $n \geq p$, then $||\epsilon_n|| < \frac{||x-Sx||}{3}$ and $||x_n - x_m|| < \frac{||x-Sx||}{3}$. Pick any positive integer q such that $\sum_{n=p}^{p+q} c_n \geq 1$. We have that

$$
\begin{aligned}
\|x_p - x_{p+q+1}\| &= \left\|\sum_{n=p}^{p+q}(x_n - x_{n+1})\right\| \\
&= \left\|\sum_{n=p}^{p+q}c_n(x - Sx + \epsilon_n)\right\| \\
&\geq \left\|\sum_{n=p}^{p+q}c_n(x - Sx)\right\| - \left\|\sum_{n=p}^{p+q}c_n\epsilon_n\right\| \\
&\geq \sum_{n=p}^{p+q}c_n\left(\|x - Sx\| - \frac{\|x - Sx\|}{3}\right) \geq \frac{2\|x - Sx\|}{3}.
\end{aligned}
$$

The contradiction proves the result.

We now show that T is a pseudocontraction. First, we note that we may put $j(x) = x$, since X is Hilbert. For $x, y \in K$, put $\Gamma(x;y) = ||x-y||^2 - \langle Tx-Ty, x-y\rangle$ and, if x and y are both nonzero, put $\lambda(x;y) = \dfrac{\langle x, y\rangle}{||x||||y||}$. Hence, to show that T is a pseudo$-$contraction, we need to prove that $\Gamma(x;y) \geq 0$ for all $x, y \in K$. We only need examine the following three cases:

1. $x, y \in K_1$: An easy computation shows that $\langle Tx - Ty, x - y\rangle = ||x - y||^2$ so that $\Gamma(x;y) = 0$; thus we are home and dry for this case.
2. $x, y \in K_2$: Again, a straightforward calculation shows that

$$
\begin{aligned}
\langle Tx - Ty, x - y\rangle &= ||x|| - ||x||^2 + ||y|| - ||y||^2 + \langle x, y\rangle\left(2 - \frac{1}{||x||} - \frac{1}{||y||}\right) \\
&= ||x|| - ||x||^2 + ||y|| - ||y||^2 + \lambda(x;y)(2||x||||y|| - ||x|| - ||y||).
\end{aligned}
$$

Hence $\Gamma(x;y) = 2||x||^2 + 2||y||^2 - ||x|| - ||y|| - \lambda(x;y)(4||x||||y|| - ||x|| - ||y||)$. It is not hard to establish that $(4||x||||y|| - ||x|| - ||y||) \geq 0$ for all $x, y \in K_2$. Hence, for fixed $||x||$ and $||y||$, $\Gamma(x;y)$ has a minimum when $\lambda(x;y) = 1$. This minimum is therefore $2\|x\|^2 + 2\|y\|^2 - 4\|x\|\|y\| = 2(\|x\| - \|y\|)^2$. Again, we have that $\Gamma(x; y) \geq 0$ for all x, $y \in K_2$ as required.

3. $x \in K_1$, $y \in K_2$: We have

$$\langle Tx - Ty, x - y\rangle = \|x\|^2 + \|y\| - \|y\|^2 - \lambda(x; y)\|x\|.$$

Hence $\Gamma(x; y) = 2\|y\|^2 - \|y\| + (\|x\| - 2\|x\|\|y\|)\lambda(x; y)$. Since $\|x\| - 2\|x\|\|y\| \leq 0$ for $x \in K_1$ and $y \in K_2$, $\Gamma(x; y)$ has its minimum, for fixed $\|x\|$ and $\|y\|$ when $\lambda(x; y) = 1$. We conclude that

$$\begin{aligned}\Gamma(x; y) \quad &\geq 2\|y\|^2 - \|y\| + \|x\| - 2\|x\|\|y\| \\ &= (\|y\| - \|x\|)(2\|y\| - 1) \\ &\geq 0 \text{ for all } x \in K_1,\ y \in K_2.\end{aligned}$$

This completes the proof. □

Remark 2.2. In [11], Qihou proved that if K is a compact convex subset of a Hilbert space and $T : K \to K$ is a continuous pseudocontractive map with a finite number of fixed points, then the Ishikawa iteration sequence defined by (1.3) converges strongly to a fixed point of T. Consequently, while the Mann sequence does not converge to the fixed point of T in our example, the Ishikawa sequence does.

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The Abdus Salam International Centre for Theoretical Physics, 34100 Trieste, Italy
E-mail address: chidume@ictp.trieste.it

Department of Physics, University of Zimbabwe, Harare, Zimbabwe