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THE FEFFERMAN-STEIN TYPE INEQUALITY FOR THE KAKEYA MAXIMAL OPERATOR

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Abstract. Let K_δ, $0 < \delta << 1$, be the Kakeya maximal operator defined as the supremum of averages over tubes of the eccentricity δ. We shall prove the so-called Fefferman-Stein type inequality for K_δ,

$$\|K_\delta f\|_{L^p(\mathbf{R}^d,w)} \le C_{d,p}(\frac{1}{\delta})^{d/p-1}(\log(\frac{1}{\delta}))^{\alpha(d)}\|f\|_{L^p(\mathbf{R}^d,K_\delta w)},$$

in the range $(1 < p \le (d^2-2)/(2d-3)$, $d \ge 3$, with some constants $C_{d,p}$ and $\alpha(d)$ independent of f and the weight w.

1. Introduction

The purpose of this note is to investigate the so-called Fefferman-Stein type inequality for the Kakeya maximal operator. Throughout this note $0 < \delta << 1$ will be a small parameter. For f a locally integrable function on $\mathbf{R}^d$, $d \ge 2$, define

$$(K_{h,\delta} f)(x) = \sup_T \frac{1}{|T|}\int_T |f(y)|dy,$$

where the supremum is taken over all tubes T containing $x \in \mathbf{R}^d$ with the length h and the radius of the cross section $h\delta$. We define the Kakeya maximal operator K_δ by

$$(K_\delta f)(x) = \sup_{h>0}(K_{h,\delta} f)(x).$$

We call a non-negative Borel measurable function w a weight if it is a locally integrable function on $\mathbf{R}^d$. By $w(A)$ we mean the $w(x)dx$ measure of a set A.

It is verified for $d = 2$ that in the range $1 < p \le d$ the Fefferman-Stein type inequality

$$(1.1)\quad (\int_{\mathbf{R}^d}(K_\delta f)(x)^p w(x)dx)^{1/p} \le C_{d,p,\epsilon}(\frac{1}{\delta})^{d/p-1+\epsilon}(\int_{\mathbf{R}^d}|f(x)|^p(K_\delta w)(x)dx)^{1/p}$$

holds for all $\epsilon > 0$ (Müller and F. Solia, [MS]). But in higher dimensions this fact has been verified only in the range $1 < p \le (d+1)/2$ (A. M. Vargas, [Va]). The main difficulty of this problem lies in making the exponent p as high as possible.

Bourgain proved that an unweighted version of (1.1) (putting $w \equiv 1$) holds in the range $1 < p \le p_d$, where $(d+1)/2 < p_d < (d+2)/2$ is some exponent given by

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a recursive formula starting from $p_3 = 7/3$ [Bo1]. Wolff improved this result [Wo]. He proved that an unweighted version of (1.1) holds in the range $1 < p \leq (d+2)/2$. Recently, in higher dimensions Bourgain improved it further to $1 < p \leq (1/2+c)d$ ($c > 0$ independent of d) [Bo2].

A different approach to this problem (an unweighted version) was given by Igari. He investigated the most difficult case $p = d$. He proved that an unweighted version of (1.1) with $p = d$ holds for a special basis [Ig]. He restricted the bases for taking the supremum to only tubes T of which the axis intersects a fixed line. The author proved the weighted version of this restricted result [Ta2]. In this note we shall improve Vargas's result by using this restricted estimates.

The main theorem of this note is the following.

Theorem 1. *Let $d \geq 3$. There exist constants $C_{d,p}$ and $\alpha(d)$ independent of δ, f, and w such that*

$$\|K_\delta f\|_{L^p(\mathbf{R}^d,w)} \leq C_{d,p}(\frac{1}{\delta})^{d/p-1}(\log(\frac{1}{\delta}))^{\alpha(d)}\|f\|_{L^p(\mathbf{R}^d,K_\delta w)}$$

holds in the range $1 < p \leq (d^2-2)/(2d-3)$.

By using sieve arguments and three-points interpolation lemma our result can be reduced to the discrete analogue as stated in the following theorem. (See [MS], [Va], and also [Ta2].)

Let $Q = (-1/2, 1/2)^d$ and $\tilde{Q} = (-2,2)^d$. We divide $\tilde{Q}$ into δ-tubes, Q_i centered at $i \in \mathcal{I}$, where $\mathcal{I}$ is the set of lattice points with the δ-separation.

Theorem 2. *Let $d \geq 3$. For a measurable set $A \subset Q$ and $0 < \lambda \leq 1$ define*

$$I = \{i \in \mathcal{I} : (K_{1,\delta}\chi_A)(i) > \lambda\}.$$

Then

$$\sum_{i\in I} w(Q_i) \leq C_d(\frac{1}{\delta})^{d-p(d)}(\frac{1}{\lambda})^{p(d)}(\log(\frac{1}{\delta}))^{\beta(d)}(K_\delta w)(A), \tag{1.2}$$

where

$$p(d) = (d^2-2)/(2d-3)$$

and

$$\beta(d) = (d+1)(d-2)/(2d-3).$$

In the following C's will denote constants which may be different in each occasion but depend only on the dimension d.

2. Proof of Theorem 2

2.1. **Preliminaries.** We summarize some known results for later use.

Given any line L in $\mathbf{R}^d$ define

$$(K^L_{1,\delta}f)(x) = \sup_T \frac{1}{|T|}\int_T |f(y)|dy,$$

where the supremum is taken over all δ-tubes T which contain x and of which the axis intersects L. Here, δ-tube is the tube with the length 1 and the radius of the cross section δ.

Lemma 3 (Theorem 2 in [Ta2])**.** *Let* $d \geq 3$*. Let* $\lambda > 0$ *and* L *be any line in* $\mathbf{R}^d$*. Then*

$$w(\{x \in \mathbf{R}^d : (K^L_{1,\delta} f)(x) > \lambda\}) \leq C(\frac{1}{\lambda})^d (\log(\frac{1}{\delta}))^{d+1} \|f\|^d_{L^d(\mathbf{R}^d, K_\delta w)}.$$

Let $\mathcal{B}_{\leq 1/\delta}$ be the class of all rectangles in $\mathbf{R}^d$ which satisfy

$$1 \leq (\text{the length of longest sides})/(\text{the length of shortest sides}) \leq \frac{1}{\delta}.$$

The corresponding maximal operator associated to this base $\mathcal{B}_{\leq 1/\delta}$ will be denoted by $K_{\leq 1/\delta}$.

Lemma 4 (Theorem 3 in [Ta1])**.** *Let* $d \geq 2$*. There exist constants* C_1 *and* C_2 *depending only on* d *such that*

$$C_1(K_\delta f)(x) \leq (K_{\leq 1/\delta} f)(x) \leq C_2(K_\delta f)(x)$$

holds for every $x \in \mathbf{R}^d$*.*

2.2. **Main argument.** Write $W = K_\delta w$. Fix $A \subset Q$ and $0 < \lambda \leq 1$. Recall $I = \{i \in \mathcal{I} : (K_{1,\delta}\chi_A)(i) > \lambda\}$. Then for every $i \in I$ we can select a δ-tube T_i, which contains i, such that

$$|A \cap T_i| > C\delta^{d-1}\lambda. \tag{2.1}$$

Then, it follows from (2.1) and the Schwarz inequality that

$$\begin{aligned}
&(C\delta^{d-1}\lambda \sum_{i\in I} w(Q_i))^2 \\
&\leq (\sum_{i\in I} w(Q_i)|A \cap T_i|)^2 = (\int_A \sum_{i\in I} w(Q_i)\chi_{T_i})^2 \\
&\leq \{\int_A (\sum_{i\in I} w(Q_i)\chi_{T_i})^2 W^{-1}\} W(A) \\
&\leq \{\sum_{i\in I} w(Q_i) \sum_{j\in I} w(Q_j) W^{-1}(T_i \cap T_j)\} W(A) \\
&\leq \max_{i\in I}(\sum_{j\in I} w(Q_j) W^{-1}(T_i \cap T_j)) \cdot (\sum_{i\in I} w(Q_i)) W(A).
\end{aligned}$$

Hence

$$N^{-1} \sum_{i\in I} w(Q_i) \leq C(\frac{1}{\delta})^{2(d-1)} (\frac{1}{\lambda})^2 W(A), \tag{2.2}$$

which corresponds to low multiplicity of Wolff (see the proof of Lemma 3.1 in [Wo]), where

$$N = \max_{i\in I}(\sum_{j\in I} w(Q_j) W^{-1}(T_i \cap T_j)).$$

By the fact that $p(d) > 2$ and $\frac{1}{\lambda} \geq 1$, we may assume that

$$C\delta^{2(d-1)} \leq N. \tag{2.3}$$

The following proposition, corresponding to high multiplicity of Wolff, will be proven later.

Proposition 5. *With previous setup we have*

$$N\{\sum_{i\in I} w(Q_i)\}^{(d-2)/(d-1)} \le C(\frac{1}{\delta})^{-d}\{(\log(\frac{1}{\delta}))^{d+1}(\frac{1}{\lambda})^{d} W(A)\}^{(d-2)/(d-1)}. \tag{2.4}$$

Multiplying both sides of (2.2) and (2.4) together, we obtain the desired inequality (1.2).

2.3. **Proof of Proposition 5.** Take some $i_0 \in I$ so that

$$N = \sum_{j\in I} w(Q_j) W^{-1}(T_{i_0} \cap T_j). \tag{2.5}$$

Let

$$I_0 = \{j \in I : T_{i_0} \cap T_j \neq \emptyset\} \tag{2.6}$$

and

$$s_0 = \inf_{y\in T_{i_0}} W(y). \tag{2.7}$$

By the geometric observation of Córdoba [Co] one sees that

$$|T_{i_0} \cap T_j| \le C\frac{\delta^d}{\delta + \operatorname{dist}(T_{i_0}, j)}. \tag{2.8}$$

From (2.5)–(2.8) we have

$$N \le C(s_0)^{-1} \sum_{j\in I_0} w(Q_j)\frac{\delta^d}{\delta + \operatorname{dist}(T_{i_0}, j)}. \tag{2.9}$$

Define the subset of I_0 as

$$\sigma_k = \{j \in I_0 : (k-1)\delta \le \operatorname{dist}(T_{i_0}, j) < k\delta\}, \quad k = 1, 2, \cdots,$$

and rewrite

$$(s_0)^{-1} \sum_{j\in I_0} w(Q_j)\frac{\delta^d}{\delta + \operatorname{dist}(T_{i_0}, j)} = (s_0)^{-1} \sum_{k}\sum_{j\in\sigma_k} w(Q_j)\frac{\delta^d}{\delta + \operatorname{dist}(T_{i_0}, j)}.$$

Then

$$N \le C(s_0)^{-1}\delta^{d-1} \sum_{k}\sum_{j\in\sigma_k} \frac{w(Q_j)}{k}. \tag{2.10}$$

It follows for some k_0 to be specified later that

$$\begin{aligned}
&\delta^{d-1}\sum_{k=1}^{k_0}\sum_{j\in\sigma_k}\frac{w(Q_j)}{k} \\
&= \delta^{d-1}\sum_{k=1}^{k_0}\sum_{j\in\sigma_k} w(Q_j)(\sum_{l=k}^{k_0}\frac{1}{l(l+1)} + \frac{1}{k_0+1}) \\
&= \{\delta^{d-1}\sum_{k=1}^{k_0}\sum_{j\in\sigma_k} w(Q_j)(\sum_{l=k}^{k_0}\frac{1}{l(l+1)})\} + \{\frac{\delta^{d-1}}{k_0+1}\sum_{k=1}^{k_0}\sum_{j\in\sigma_k} w(Q_j)\} \\
&= \mathrm{I} + \mathrm{II}.
\end{aligned}$$

By reversing the order of summation we have

$$\begin{aligned} \mathrm{I} &= \delta^{d-1}\sum_{l=1}^{k_0}(\sum_{k=1}^{l}\sum_{j\in\sigma_k} w(Q_j))/(l(l+1)) \\ &\leq C\delta^{2d-2}\sum_{l=1}^{k_0} l^{d-3}(\sum_{k=1}^{l}\sum_{j\in\sigma_k} w(Q_j))/((l\delta)^{d-1}). \end{aligned}$$

By using Lemma 4 we see that

$$(s_0)^{-1}(\sum_{k=1}^{l}\sum_{j\in\sigma_k} w(Q_j))/((l\delta)^{d-1}) \leq C(s_0)^{-1}\frac{\int_{R_l} w}{|R_l|} \leq C,$$

where

$$R_l = \{x \in \mathbf{R}^d : \operatorname{dist}(T_{i_0}, x) \leq l\delta\}.$$

Hence

$$(s_0)^{-1}\mathrm{I} \leq C\delta^{2d-2}\sum_{l=1}^{k_0} l^{d-3} \leq C\delta^d(k_0\delta)^{d-2}.$$

Combining these inequalities we obtain

$$N \leq C\delta^d\{(k_0\delta)^{d-2} + (k_0\delta)^{-1}(s_0)^{-1}\sum_{j\in I_0} w(Q_j)\}. \tag{2.11}$$

Now, we can choose some k_0 so that

$$(k_0\delta)^{d-1} \sim (s_0)^{-1}\sum_{j\in I_0} w(Q_j)$$

by (2.3) and (2.10). Then the two terms in the right-hand side of (2.11) balance and hence

$$N \leq C\delta^d\{(s_0)^{-1}\sum_{j\in I_0} w(Q_j)\}^{(d-2)/(d-1)}. \tag{2.12}$$

Applying Lemma 3 with $L =$ the axis of T_{i_0} and $f = \chi_A$, we clearly obtain

$$\sum_{j\in I_0} w(Q_j) \leq C(\frac{1}{\lambda})^d(\log(\frac{1}{\delta}))^{d+1}W(A). \tag{2.13}$$

Thus, from (2.12) and (2.13) we have

$$N(s_0)^{(d-2)/(d-1)} \leq C\delta^d\{(\frac{1}{\lambda})^d(\log(\frac{1}{\delta}))^{d+1}W(A)\}^{(d-2)/(d-1)}. \tag{2.14}$$

Finally, again by Lemma 4 we observe that

$$\sum_{i\in I} w(Q_i) \leq C\frac{w(\tilde{Q})}{|\tilde{Q}|} \leq C's_0. \tag{2.15}$$

Thus, from (2.14) and (2.15) we obtain

$$N\{\sum_{j\in I} w(Q_j)\}^{(d-2)/(d-1)} \leq C\delta^d\{(\frac{1}{\lambda})^d(\log(\frac{1}{\delta}))^{d+1}W(A)\}^{(d-2)/(d-1)}. \qquad \square$$

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