

ITERATIVE APPROXIMATION OF FIXED POINTS OF LIPSCHITZ PSEUDOCONTRACTIVE MAPS

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ABSTRACT. Let E be a q -uniformly smooth Banach space possessing a weakly sequentially continuous duality map (e.g., ℓ_p , $1 < p < \infty$). Let T be a Lipschitzian pseudocontractive selfmapping of a nonempty closed convex and bounded subset K of E and let $\omega \in K$ be arbitrary. Then the iteration sequence $\{z_n\}$ defined by $z_0 \in K$, $z_{n+1} = (1 - \mu_{n+1})\omega + \mu_{n+1}y_n$; $y_n = (1 - \alpha_n)z_n + \alpha_n Tz_n$, converges strongly to a fixed point of T , provided that $\{\mu_n\}$ and $\{\alpha_n\}$ have certain properties. If E is a Hilbert space, then $\{z_n\}$ converges strongly to the unique fixed point of T closest to ω .

1. INTRODUCTION

Suppose E is an arbitrary real Banach space with dual E^* . Given a gauge function $\phi : [0, \infty) \mapsto [0, \infty)$ the mapping $J_\phi : E \mapsto 2^{E^*}$ defined by

$$J_\phi x := \{u^* \in E^* : \langle x, u^* \rangle = \|x\| \cdot \|u^*\|; \|u^*\| = \phi(\|x\|)\}$$

is called the *duality map with gauge function* ϕ . In the particular case $\phi(t) = t$, the duality map $J = J_\phi$ is called the *normalized duality map*. For $q > 1$, let $\phi(t) = t^{q-1}$ be a gauge function. We define the *generalized duality map* $J_q : E \mapsto 2^{E^*}$ by

$$J_{\phi(t)}(x) := J_q(x) = \{x^* \in E^* : \langle x, x^* \rangle = \|x\| \cdot \|x^*\|; \|x^*\| = \|x\|^{q-1}\},$$

and observe that for $q = 2$ we write $J_q = J_2 = J$ (the normalized duality map). It is well known that if E^* is strictly convex, then J is single-valued. In the sequel, we shall denote the single-valued generalized duality map by j_p . Let K be a nonempty subset of E and $T : K \mapsto E$ any map. Then T is called *pseudocontractive* if and only if for all $x, y \in K$ there exists $j(x - y) \in J(x - y)$ such that $\langle Tx - Ty, j(x - y) \rangle \leq \|x - y\|^2$, and is called *strongly pseudocontractive* if and only if there exists $k \in [0, 1)$ such that for all $x, y \in K$ there exists $j(x - y) \in J(x - y)$ with $\langle Tx - Ty, j(x - y) \rangle \leq k\|x - y\|^2$. The map T is called *nonexpansive* if and only if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in K$.

Interest in pseudocontractive maps stems mainly from their firm connection with the important class of nonlinear *accretive* operators where a mapping U with domain $D(U)$ and range $R(U)$ in E is called accretive if the inequality

$$\|x - y\| \leq \|x - y + s(Ux - Uy)\|$$

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holds for every $x, y \in D(U)$ and for all $s > 0$. It is now well known (see, e.g., [3]) that if T is accretive, then the solutions of the equation $Tx = 0$ correspond to the equilibrium points of some evolution systems. Consequently, considerable research efforts, especially within the past 15 years or so, have been devoted to iterative methods for approximating solutions of $Tu = 0$ when T is accretive (or correspondingly, fixed points of pseudocontractions). In the special case in which the operator is Lipschitz and *strongly* pseudocontractive (or correspondingly Lipschitz and *strongly* accretive), the *Mann iteration process* (see, e.g., [8]) has been successfully employed. Results which had been known only in *Hilbert spaces* and only for *Lipschitz maps* (thanks to several geometric inequalities for general Banach spaces developed within the past 15 years or so) have been extended to more general Banach spaces and to more general classes of maps (see, e.g., [2] for relevant references). As has been pointed out in [2], this success has not carried over to arbitrary *Lipschitz pseudocontractions* T even when the domain of the operator T is a *compact convex subset of a Hilbert space*. In fact, it is still an open question whether or not the Mann iteration process converges under this setting (see, e.g., [2], [6]). In 1974, Ishikawa [7] introduced an iteration process which, in some sense, is more general than that of Mann and which converges, under this setting, to a fixed point of T . He proved the following theorem.

Theorem 1.1 (Ishikawa, [7]). *If K is a compact convex subset of a Hilbert space H , $T : K \mapsto K$ is a Lipschitzian pseudocontractive map and x_0 is any point of K , then the sequence $\{x_n\}_{n \geq 0}$ converges strongly to a fixed point of T , where x_n is defined iteratively for each positive integer $n \geq 0$ by*

$$(1) \quad x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T y_n; \quad y_n = (1 - \beta_n)x_n + \beta_n T x_n,$$

where $\{\alpha_n\}, \{\beta_n\}$ are sequences of positive numbers satisfying the conditions (i) $0 \leq \alpha_n \leq \beta_n < 1$; (ii) $\lim_{n \rightarrow \infty} \beta_n = 0$; (iii) $\sum_{n \geq 0} \alpha_n \beta_n = \infty$.

Since its publication in 1974, Theorem 1.1, as far as we know, has never been extended to more general Banach spaces. All extensions proved so far are to slightly more general classes of operators and are still confined to Hilbert spaces (see, e.g., [2], [10], [11]). Other iteration processes have also been introduced. In particular, Schu [14] introduced an iteration process with a recursion formula similar to (1) and proved that if K is a nonempty closed convex and bounded subset of a *Hilbert space* and $T : K \mapsto K$ is pseudocontractive and Lipschitzian, then the sequence generated from the recursion formula converges strongly to the unique fixed point of T of minimum norm. By a different approach, Bruck [1] found another iteration procedure which converges, still in Hilbert spaces, to the same fixed point of T . Both Schu [14] and Bruck [1] made essential use of ideas which are originally due to Halpern [5] who was concerned with fixed points of nonexpansive mappings.

It is our purpose in this paper to prove that the iteration process studied by Schu [14] converges strongly, to a fixed point of a pseudocontractive Lipschitz map in certain real Banach spaces more general than Hilbert spaces. No compactness assumption will be imposed on the domain of our operator. Theorems 3.1, 3.2 and Corollary 3.3 are the first convergence theorems for iterative methods for fixed points of Lipschitz pseudocontractions in Banach spaces more general than Hilbert spaces.

2. PRELIMINARIES

Let E be a real normed linear space of dimension $\dim E \geq 2$. The *modulus of smoothness* of E is defined by

$$\rho_E(\tau) := \sup \left\{ \frac{\|x+y\| + \|x-y\|}{2} - 1 : \|x\| = 1, \|y\| = \tau \right\}; \quad \tau > 0.$$

If $\rho_E(\tau) > 0 \forall \tau > 0$, then E is said to be *smooth*. If there exist a constant $c > 0$ and a real number $1 < q < \infty$, such that $\rho_E(\tau) \leq c\tau^q$, then E is said to be *q-uniformly smooth*. Typical examples of such spaces are the Lebesgue L_p , the sequence ℓ_p and the Sobolev W_p^m spaces for $1 < p < \infty$. In particular, for $1 < p \leq 2$, these spaces are p -uniformly smooth and for $2 \leq p < \infty$, they are 2-uniformly smooth (see, e.g., [15]).

If E is a real q -uniformly smooth Banach space, then (see, e.g., [15]) the following geometric inequality holds:

$$(2) \quad \|x+y\|^q \leq \|x\|^q + q\langle y, j_q(x) \rangle + C_q \|y\|^q$$

for all $x, y \in E$ and some real constant $C_q \geq 1$.

In the sequel, \rightarrow and \rightharpoonup will denote strong and weak convergence, respectively. The set of fixed points $F(T)$ of an operator T is defined as follows: $F(T) := \{x \in E : Tx = x\}$. In what follows, we shall make use of the following lemmas, due to Schu [14], whose proofs we sketch for completeness.

Lemma 2.1. *Let E be a normed space possessing a duality map $j : E \mapsto E^*$ which is weakly sequentially continuous. Let (i) K be a nonempty closed subset of E ; (ii) $T : K \mapsto K$ be continuous and pseudocontractive; (iii) $\lambda_n \in (0, 1)$ be such that $\lim \lambda_n = 1$; (iv) $\{x_n\} \subseteq K$ be such that $x_n = \lambda_n T x_n$, $\forall n \in N$; $x_n \rightharpoonup x^*$ for some $x^* \in E$. Then (a) $x_n \rightarrow x^*$ and $T x^* = x^*$; (b) $\langle x_n, j(y - x_n) \rangle \geq 0 \quad \forall y \in F(T)$; (c) $\langle x^*, j(y - x^*) \rangle \geq 0 \quad \forall y \in F(T)$.*

Proof. Since E possesses a duality map that is weakly sequentially continuous, it is smooth (see, e.g., [4]). Then, part (a) follows from [9], Lemma 2.5 and the proof of [9], Lemma 2.7. For part (b), fix $n \in N$, $y \in F(T)$ and define $\alpha_n := \lambda_n^{-1} - 1 > 0$. Then, using the pseudocontractivity of T , we obtain the following estimate:

$$\begin{aligned} \langle \lambda_n^{-1} x_n - y, j(y - x_n) \rangle &= \langle T x_n - T y, j(y - x_n) \rangle \\ &\geq -\|x_n - y\|^2 = \langle x_n - y, j(y - x_n) \rangle \end{aligned}$$

so that $\langle \alpha_n x_n, j(y - x_n) \rangle \geq 0$, which establishes (b) since $\alpha_n > 0$. Since $x_n \rightharpoonup x^*$, it follows from part (b) and the assumption on j that for each fixed $y \in F(T)$, $\langle x^*, j(y - x^*) \rangle = \lim \langle x_n, j(y - x_n) \rangle \geq 0$. \square

Lemma 2.2. *Let E be a reflexive Banach space possessing a weakly sequentially continuous duality map $J : E \mapsto E^*$. Let (i) K be a nonempty closed convex and bounded subset of E , and ω be an arbitrary vector in K ; (ii) $T : K \mapsto K$ be a continuous and pseudocontractive map; (iii) $\{\lambda_n\}$ be a sequence in $(0, 1)$ such that $\lim \lambda_n = 1$. Then, (a) For each $n \in N$ there exists exactly one $x_n \in K$ such that $x_n = (1 - \lambda_n)\omega + \lambda_n T x_n$; (b) $\{x_n\}$ converges strongly to some fixed point of T .*

Proof. Without loss of generality, we may assume $\omega = 0$. For $n \in N$, $T_n := \lambda_n T$ is clearly strongly pseudocontractive and since $0 \in K$ and K is convex, it follows that T_n maps K into itself. Corollary 1 of [3] then yields that T_n possesses exactly one fixed point $x_n \in K$, completing the proof of part (a). For part (b), since E is

reflexive and $\{x_n\}$ is bounded, Eberlein-Smul'yan theorem yields that there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ which converges weakly to some $x^* \in E$. By Lemma 2.1 applied to $\{x_{n_j}\}$, we obtain that x^* is a fixed point of T , $\{x_{n_j}\}$ converges strongly to x^* and $\langle x^*, j(y - x^*) \rangle \geq 0$ for all $y \in F(T)$.

Claim 1. x^* is unique.

To see this, assume there exists $y^* \neq x^*$ such that $y^* \in F(T)$ and $\langle y^*, j(y - y^*) \rangle \geq 0 \forall y \in F(T)$. Substituting x^* for y in this inequality, y^* for y in $\langle x^*, j(y - x^*) \rangle \geq 0$ and adding the resulting inequalities lead to an obvious contradiction. Hence $\{x_n\}$ possesses a unique strong cluster point and thus $x_n \rightarrow x^*$ as $n \rightarrow \infty$. \square

Remark 2.3. It was shown implicitly in [1] that, in the case of $E = H$, a Hilbert space, there is exactly one fixed point x^* of T such that

$$(3) \quad \|\omega - x^*\| = \inf_{u \in F(T)} \|u - \omega\|$$

and $\{x_n\}$ converges strongly to x^* . To see this, we first note that by Lemma 2.2, $F(T) \neq \emptyset$ and by Corollary 2.3 of [12], $F(T)$ is closed and convex. So, if $P : H \mapsto F(T)$ denotes the proximity map of H onto $F(T)$ (i.e., $\forall x \in H$, $Px = y \in F(T)$ where $\|x - y\| = \inf_{u \in F(T)} \|x - u\|$), then $x^* := P(\omega)$ is clearly the unique fixed point of T satisfying (3). To show that $\{x_n\}$ converges to x^* , assume, without loss of generality, that $\omega = 0$ and set $s^* = \lim x_n$. Then, for each $y^* \in F(T)$, it follows from Lemma 2.1(c) that $\langle s^*, y^* - s^* \rangle \geq 0$ and so $\|s^*\| \leq \|y^*\|$. Hence, $s^* = x^*$.

Before we state our next lemma, we introduce the following definition which is partly due to Halpern [5].

Definition 2.4. Let $\alpha_n \in (0, \infty)$, $\mu_n \in (0, 1)$ for all nonnegative integers n . Then $(\{\alpha_n\}, \{\mu_n\})$ is said to have *property (A)* if and only if the following conditions hold: (i) $\{\alpha_n\}$ is decreasing and $\{\mu_n\}$ is strictly increasing; (ii) there is a sequence $\{\beta_n\} \subset \mathbb{N}$, strictly increasing, such that

$$(a) \quad \lim_{n \rightarrow \infty} \beta_n(1 - \mu_n) = \infty; \quad (b) \quad \lim_{n \rightarrow \infty} \left(\frac{1 - \mu_{n+\beta_n}}{1 - \mu_n} \right) = 1; \quad \text{and}$$

$$(c) \quad \lim_{n \rightarrow \infty} \left(\frac{\alpha_n - \alpha_{n+\beta_n}}{1 - \mu_n} \right) = 0.$$

Lemma 2.5 (cf. [13]). *Let K be a nonempty convex and bounded subset of a normed linear space E with $0 \in K$. Let (i) $S_n : K \mapsto K$ be such that $\|S_n(x) - S_n(y)\| \leq k_n \|x - y\| \forall n \in \mathbb{N}, \forall x, y \in K$ where $k_n \in [1, \infty)$; (ii) $\lambda_n \in (0, 1)$, $\alpha_n \in (0, \infty)$ be such that $(\{\alpha_n\}, \{\mu_n\})$ has property (A) and $(1 - \mu_n)(1 - \lambda_n)^{-1}$ is bounded, where $\mu_n := \lambda_n k_n^{-1} \forall n \in \mathbb{N}$; (iii) $\{x_n\} \subseteq K$ be such that $x_n = \mu_n S_n(x_n) \forall n \in \mathbb{N}$ and there exists a constant $M > 0$ such that $\|S_i(x) - S_j(x)\| \leq M |\alpha_i - \alpha_j| \forall i, j \in \mathbb{N}, \forall x \in K$. Assume that $\lim x_n := x^*$ exists. Define the sequence $\{z_n\}$ by $z_{n+1} := \mu_{n+1} S_n(z_n) \forall n \in \mathbb{N}$, where $z_0 \in K$ is arbitrary. Then, $\lim_{n \rightarrow \infty} z_n = x^*$.*

3. MAIN RESULTS

We prove the following theorems.

Theorem 3.1. *Let K be a nonempty subset of a q -uniformly smooth Banach space E possessing a weakly sequentially continuous duality map. Suppose that (i) $T : K \mapsto K$ is pseudocontractive and Lipschitzian (with Lipschitz constant $L \geq 0$);*

- (ii) $\alpha_n \in (0, \infty)$, $\mu_n \in (0, 1)$, $k_n := [1 + C_q \alpha_n^q (L + 1)^q]^{\frac{1}{q}}$ and $S_n := (1 - \alpha_n)I + \alpha_n T$;
- (iii) $\{x_n\} \subseteq K$ is such that for all positive integers n ,

$$x_n = \left[\frac{\mu_n \alpha_n}{1 - \mu_n (1 - \alpha_n)} \right] T x_n.$$

Then, $\forall n \in N$, the following results hold:

- (a) $\|S_n(x) - S_n(y)\| \leq k_n \|x - y\|$, $\forall x, y \in K$; and (b) $x_n = \mu_n S_n(x_n)$.

Proof. Define $G := (I - T)$ and observe that for all $x, y \in K$, $\langle Gx - Gy, j_q(x - y) \rangle \geq 0$. Moreover, $S_n = I - \alpha_n G$ so that for all $x, y \in K$,

$$\begin{aligned} \|S_n(x) - S_n(y)\|^q &\leq \|x - y\|^q - q \alpha_n \langle Gx - Gy, j_q(x - y) \rangle \\ &\quad + C_q \alpha_n^q (L + 1)^q \|x - y\|^q, \\ &\leq [1 + C_q \alpha_n^q (L + 1)^q] \|x - y\|^q, \end{aligned}$$

since $\|Gx - Gy\| \leq (L + 1) \|x - y\|$, and (a) follows. (b) is obvious. □

Theorem 3.2. *Let K be a nonempty closed convex and bounded subset of a q -uniformly smooth Banach space possessing a weakly sequentially continuous duality map. Suppose that (i) $T : K \mapsto K$ is pseudocontractive and Lipschitzian (with Lipschitz constant $L \geq 0$); (ii) $\lambda_n \in (0, 1)$ with $\lim \lambda_n = 1$; $\alpha_n \in (0, 1)$ with $\lim \alpha_n = 0$; $(\{\alpha_n\}, \{\mu_n\})$ has property (A); $(1 - \mu_n)(1 - \lambda_n)^{-1}$ is bounded and $\lim \alpha_n^{-1}(1 - \mu_n) = 0$, where $\mu_n := \lambda_n k_n^{-1}$ and $k_n := [1 + C_q \alpha_n^q (L + 1)^q]^{\frac{1}{q}}$. Fix $z_0 \in K$ arbitrary and for each positive integer n , define the sequence $\{z_n\}$, for arbitrary $\omega \in K$, by*

$$(4) \quad z_{n+1} = (1 - \mu_{n+1})\omega + \mu_{n+1}y_n; \quad y_n = (1 - \alpha_n)z_n + \alpha_n T z_n.$$

Then, $\{z_n\}$ converges strongly to a fixed point of T .

Proof. Without loss of generality, we may assume $\omega = 0$. For any positive integer n , define

$$\gamma_n := \left[1 + \frac{1}{\mu_n} \left(\frac{1 - \mu_n}{\alpha_n} \right) \right]^{-1}.$$

Clearly, $\gamma_n \in (0, 1)$. Since $\lim \mu_n = 1$ and $\lim \alpha_n^{-1}(1 - \mu_n) = 0$ it follows that $\lim \gamma_n = 1$. Hence, by Lemma 2.2, for each positive integer n , there exists a unique $x_n \in K$ such that $x_n = \gamma_n T x_n$ and $x_n \rightarrow x^*$, for some $x^* \in F(T)$. Furthermore, since $\alpha_n \in (0, 1)$ we easily obtain that $S_n := (1 - \alpha_n)I + \alpha_n T$ maps K into itself for each $n \in N$. Consequently, by Theorem 3.1, for each $n \in N$, $x_n = \mu_n S_n(x_n)$ and S_n is Lipschitzian with constant k_n . Moreover, $z_{n+1} = \mu_{n+1} S_n(z_n)$ for all positive integers n , and $\|S_i(x) - S_j(x)\| \leq |\alpha_i - \alpha_j| \cdot \text{diam.}(K)$ for all $i, j \in N$, all $x \in K$. It then follows from Lemma 2.5 that $z_n \rightarrow x^*$ as $n \rightarrow \infty$. This completes the proof. □

Corollary 3.3. *Let K be a nonempty closed convex and bounded subset of ℓ_p , $1 < p < \infty$. Suppose $T : K \mapsto K$ is pseudocontractive and Lipschitzian (with Lipschitz constant $L \geq 0$). Suppose further that $\omega, \lambda_n, \alpha_n, \mu_n$ and k_n are as in (ii) of Theorem 3.2. Then, for arbitrary $z_0 \in K$, the sequence $\{z_n\}$ defined by (4) converges strongly to a fixed point of T .*

Proof. The Banach space ℓ_p , $1 < p < \infty$, is q -uniformly smooth and possesses a weakly sequentially continuous duality map. The result follows from Theorem 3.2. \square

As immediate consequences of Corollary 3.3, we obtain Corollaries 3.4 and 3.5 below which are the main results of Schu [14].

Corollary 3.4. *Let H be a Hilbert space and let (i) K be a nonempty closed convex and bounded subset of H , $\omega \in K$ arbitrary; (ii) $T : K \mapsto K$ be pseudocontractive and Lipschitzian with constant $L \geq 0$; (iii) $\lambda_n \in (0, 1)$ with $\lim \lambda_n = 1$; $\alpha_n \in (0, 1)$ with $\lim \alpha_n = 0$ such that $(\{\alpha_n\}, \{\mu_n\})$ has property (A), $(1 - \mu_n)(1 - \lambda_n)^{-1}$ is bounded and $\lim \alpha_n^{-1}(1 - \mu_n) = 0$, where $k_n := (1 + \alpha_n^2(L + 1)^2)^{\frac{1}{2}}$ and $\mu_n := \lambda_n k_n^{-1} \forall n \in N$. Fix $z_0 \in K$ arbitrary and define $\{z_n\}$ by (4) $\forall n \in N$. Then $\{z_n\}$ converges strongly to the unique fixed point of T closest to ω .*

Proof. Set $p = 2$ in Corollary 3.3. The result follows from Corollary 3.3 and Remark 2.3. \square

Corollary 3.5. *Let H be a Hilbert space, and let (i) K be a nonempty closed convex and bounded subset of H , $\omega \in K$ arbitrary; (ii) $T : K \mapsto K$ be pseudocontractive and Lipschitzian with constant $L \geq 0$, $z_0 \in K$ arbitrary; (iii) For $n \in N$, set $\mu_n := 1 - (n + 1)^{-\frac{1}{2}}$, $\alpha_n := (n + 1)^{-\frac{1}{4}}(L + 1)^{-1}$ and define $\{z_n\}$ by (4). Then $\{z_n\}$ converges strongly to the unique fixed point of T closest to ω .*

Proof. For each $n \in N$, define $k_n := (1 + \alpha_n^2(L + 1)^2)^{\frac{1}{2}}$, $\lambda_n := \mu_n k_n$ and $\beta_n = [(n + 1)^{\frac{5}{8}}]$, where $[t] := \max\{n \in N : n \leq t\}$. Then, it has been shown in [5] that conditions (iii), (a) and (b) of Definition 2.4 are already satisfied. Easy calculations show that all the other assumptions of Corollary 3.3 are satisfied. The result follows from Corollary 3.3. \square

Remark 3.6. Corollary 3.4 is, in several respects, an improvement on the theorem of Ishikawa [7]. In particular, the Corollary does not require that the subset K be compact as is the case in [7]. Moreover, since the fixed points of a Lipschitz pseudocontraction are generally not unique, Corollary 3.4 gives the additional information (unlike in [7]) as to which fixed point the sequence converges. However, in the theorem of Ishikawa [7], the parameters of the recursion formula α_n, β_n can easily be chosen (e.g., $\alpha_n \equiv \frac{1}{\sqrt{n+1}} \equiv \beta_n$) whereas, as is easily seen from Corollary 3.5, the choices of the iteration parameters α_n and μ_n are not so simple.

Remark 3.7. As has been noted in the introduction, Theorems 3.1, 3.2 and Corollary 3.3 are the first theorems on iterative methods for fixed points of Lipschitz pseudocontractions in Banach spaces more general than Hilbert spaces.

NOTE ADDED IN PROOF

The open question referred to in the introduction has recently been resolved: C. E. Chidume and S. Mutangadura, *An example on the Mann iteration method for Lipschitz pseudocontractions*, Proc. Amer. Math. Soc. (to appear), published electronically January 18, 2001. In this reference, an example of a Lipschitz pseudocontractive map with a unique fixed point is constructed for which the Mann iteration process fails to converge.

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