

AN ASYMMETRIC PUTNAM–FUGLEDE THEOREM FOR UNBOUNDED OPERATORS

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Dedicated to Professor F. H. Szafraniec on the occasion of his sixtieth birthday

ABSTRACT. The intertwining relations between cosubnormal and closed hyponormal (resp. cohyponormal and closed subnormal) operators are studied. In particular, an asymmetric Putnam–Fuglede theorem for unbounded operators is proved.

INTRODUCTION

The Putnam–Fuglede theorem [12] says that if a bounded linear mapping intertwines two normal operators, then it intertwines their adjoints. Unfortunately, the Putnam–Fuglede theorem is no longer true for subnormal operators. However, as noticed by Furuta (cf. [2, 3]), an asymmetric version of it remains true for bounded subnormals; furthermore, subnormals can be replaced by more general bounded operators (cf. [17, 18, 14, 15, 8, 21]). In particular, the following holds true: if a bounded linear mapping intertwines a cohyponormal operator with a hyponormal one (both are bounded), then it intertwines their adjoints. In this paper we prove a decomposition theorem (see Theorem 4.2) for intertwining relations between cosubnormal (resp. cohyponormal) and closed hyponormal (resp. subnormal) unbounded operators (cf. [17, 8] for the bounded case). Contrary to what has been shown before (cf. [11, 7, 10]), here the operators in question have arbitrary domains and their spectra may cover the whole complex plane. We also prove an asymmetric version of the Putnam–Fuglede theorem for couples of unbounded operators whose first component is cosubnormal (resp. cohyponormal), while the other one is closed hyponormal (resp. subnormal).

1. PRELIMINARIES

All linear spaces taken into consideration in this paper are assumed to be complex. From now on the symbols \mathcal{H} and \mathcal{K} stand for Hilbert spaces. Given a linear mapping $A : \mathcal{H} \supseteq \mathcal{D}(A) \rightarrow \mathcal{K}$, we denote by $\mathcal{N}(A)$, $\mathcal{R}(A)$, $\mathcal{G}(A)$ and A^* the kernel, the range, the graph and the adjoint of A , respectively. As usual $A|_{\mathcal{E}}$ stands for

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the *restriction* of A to a linear subspace \mathcal{E} of $\mathcal{D}(A)$; \mathcal{E} is said to be a *core* for A if $\overline{\mathcal{G}(A)} \subseteq \overline{\mathcal{G}(A|_{\mathcal{E}})}$. Denote by $\mathbf{B}(\mathcal{H}, \mathcal{K})$ the set of all bounded linear mappings from \mathcal{H} into \mathcal{K} ; for simplicity we write $\mathbf{B}(\mathcal{H})$ instead of $\mathbf{B}(\mathcal{H}, \mathcal{H})$. Set $|X| = (X^*X)^{1/2}$ for $X \in \mathbf{B}(\mathcal{H}, \mathcal{K})$.

Given an operator A in \mathcal{H} (read: a linear mapping $A : \mathcal{H} \supseteq \mathcal{D}(A) \rightarrow \mathcal{H}$), we denote by $\sigma(A)$ the spectrum of A . A is said to be *pure* if the only closed linear subspace of \mathcal{H} reducing A to a normal operator is $\{0\}$. If \mathcal{L} is a closed linear subspace of \mathcal{H} , then $A|_{\mathcal{L}}$ stands for the operator in \mathcal{L} defined by

$$\mathcal{D}(A|_{\mathcal{L}}) = \{f \in \mathcal{L} \cap \mathcal{D}(A) : Af \in \mathcal{L}\} \quad \text{and} \quad A|_{\mathcal{L}} f = Af, \quad f \in \mathcal{D}(A|_{\mathcal{L}}).$$

If \mathcal{L} reduces A to the operator B , then $B = A|_{\mathcal{L}}$. It may happen that $A|_{\mathcal{L}}$ is densely defined in \mathcal{L} though \mathcal{L} is not invariant for A , i.e. $A(\mathcal{L} \cap \mathcal{D}(A)) \not\subseteq \mathcal{L}$.

Example 1.1. Let f_1 be a normalized vector in \mathcal{H} . Set $\mathcal{L} = \{f_1\}^\perp$. Let A_0 be an unbounded closed densely defined operator in \mathcal{L} . Take a normalized vector $e_1 \in \mathcal{L} \setminus \mathcal{D}(A_0)$. Define the operator A in \mathcal{H} by: $\mathcal{D}(A) = \mathcal{D}(A_0) \dot{+} \mathbb{C} \cdot e_1 \dot{+} \mathbb{C} \cdot f_1$ and $A(h + \lambda e_1 + \mu f_1) = A_0 h + \lambda f_1$ for $h \in \mathcal{D}(A_0)$ and $\lambda, \mu \in \mathbb{C}$. It is clear that A is closed and densely defined. Moreover, one can check that $\mathcal{D}(A|_{\mathcal{L}}) = \mathcal{D}(A_0) \subsetneq \mathcal{D}(A_0) \dot{+} \mathbb{C} \cdot e_1 = \mathcal{L} \cap \mathcal{D}(A) \subsetneq \mathcal{D}(A)$, $e_1 \in \mathcal{L} \cap \mathcal{D}(A)$ and $Ae_1 = f_1 \notin \mathcal{L}$.

A densely defined operator A in \mathcal{H} is said to be *hyponormal* if $\mathcal{D}(A) \subseteq \mathcal{D}(A^*)$ and $\|A^*f\| \leq \|Af\|$ for $f \in \mathcal{D}(A)$. We say that a densely defined operator S in \mathcal{H} is *subnormal* if there is a Hilbert space $\mathcal{L} \supseteq \mathcal{H}$ and a normal operator N in \mathcal{L} such that $S \subseteq N$. It is well-known that each subnormal operator is hyponormal and that each hyponormal operator is closable (see [4, 11, 5, 6] for more details).

We begin our considerations by proving a fact which is well-known for bounded hyponormal operators (cf. [1]; see also [5, Proposition 1] for the reverse result).

Proposition 1.2. *If T is a hyponormal operator in \mathcal{H} , then there exists a contraction $C \in \mathbf{B}(\mathcal{H})$ such that $T \subseteq T^*C$.*

Proof. Since $\|T^*f\| \leq \|Tf\|$ for $f \in \mathcal{D}(T) \subseteq \mathcal{D}(T^*)$, there exists a contraction $D' \in \mathbf{B}(\overline{\mathcal{R}(T)}, \overline{\mathcal{R}(T^*)})$ such that $D'T \subseteq T^*$. Let $D \in \mathbf{B}(\mathcal{H})$ be any contraction which extends D' (e.g. set $Df = 0$ for $f \in \mathcal{H} \ominus \overline{\mathcal{R}(T)}$). Then $DT \subseteq T^*$. Taking adjoints in the last inclusion and exploiting the closability of T , we get $T \subseteq T^{**} \subseteq (DT)^* = T^*D^*$. This gives us the conclusion with $C = D^*$. \square

The next result offers a sufficient condition for a hyponormal operator to be reduced by a closed subspace.

Proposition 1.3. *Let \mathcal{L} be a closed linear subspace of \mathcal{H} and let T be a closed hyponormal operator in \mathcal{H} such that $T|_{\mathcal{L}}$ is densely defined in \mathcal{L} . Then:*

- (i) $T|_{\mathcal{L}}$ is a closed hyponormal operator in \mathcal{L} ,
- (ii) \mathcal{L} reduces T to $T|_{\mathcal{L}}$ provided $T|_{\mathcal{L}}$ is normal.

Proof. (i) Since T is closed, it is easily seen that the operator $S := T|_{\mathcal{L}}$ is closed as well. We now show that

$$(1.1) \quad PT^*|_{\mathcal{D}(S)} \subseteq S^*,$$

where P is the orthogonal projection of \mathcal{H} onto \mathcal{L} . Indeed, by the hyponormality of T , we have $\mathcal{D}(S) \subseteq \mathcal{D}(T^*)$, so $\langle f, Sg \rangle = \langle T^*f, g \rangle = \langle PT^*f, g \rangle$ for $f, g \in \mathcal{D}(S)$. This, in turn, implies (1.1).

Using (1.1) and the hyponormality of T we get

$$(1.2) \quad \|S^*f\| = \|PT^*f\| \leq \|T^*f\| \leq \|Tf\| = \|Sf\|, \quad f \in \mathcal{D}(S).$$

Hence S is hyponormal.

(ii) It follows from the normality of S and (1.2) that $\|PT^*f\| = \|T^*f\|$ for $f \in \mathcal{D}(S)$. Hence $T^*(\mathcal{D}(S)) \subseteq \mathcal{L}$ and, by (1.1), we have $S^*f = T^*f$ for $f \in \mathcal{D}(S)$. Applying the latter equality, we obtain

$$\langle Pf, S^*g \rangle = \langle f, S^*g \rangle = \langle f, T^*g \rangle = \langle Tf, g \rangle = \langle PTf, g \rangle, \quad f \in \mathcal{D}(T), g \in \mathcal{D}(S),$$

which implies $P(\mathcal{D}(T)) \subseteq \mathcal{D}(S^{**}) = \mathcal{D}(S) \subseteq \mathcal{D}(T)$ and $TPf = SPf = S^{**}Pf = PTf$ for $f \in \mathcal{D}(T)$. Thus $PT \subseteq TP$, which shows that \mathcal{L} reduces T to S . \square

Part (ii) of Proposition 1.3 is no longer true for arbitrary operators T .

Example 1.4. Let e_0 be a normalized vector in \mathcal{H} . Set $\mathcal{L} = \{e_0\}^\perp$. Take a normalized vector f_0 in \mathcal{L} . Define $T \in \mathbf{B}(\mathcal{H})$ by $Th = h - \langle h, e_0 \rangle(e_0 + f_0)$ for $h \in \mathcal{H}$. One can show that $\|T\| = \sqrt{2}$ (use the fact that $\mathcal{H} = \mathcal{L} \dot{+} \mathbb{C} \cdot (e_0 + f_0)$). Since $Th = h$ for $h \in \mathcal{L}$, we conclude that $T|_{\mathcal{L}}$ is normal. However \mathcal{L} does not reduce T as $Te_0 = -f_0$.

Our next goal is to characterize linear mappings T satisfying the condition $\mathcal{D}(T) \subseteq \mathcal{R}(T^*)$. Let us begin with

Lemma 1.5. *If $T : \mathcal{H} \supseteq \mathcal{D}(T) \rightarrow \mathcal{K}$ is a closed densely defined linear mapping, then $\mathcal{H} = \mathcal{D}(T) + \mathcal{R}(T^*)$.*

Proof. It is known that $\mathcal{H} \oplus \mathcal{K} = \mathcal{G}(T) \oplus W\mathcal{G}(T^*)$ (cf. [20, Theorem 4.16]), where $W : \mathcal{K} \oplus \mathcal{H} \rightarrow \mathcal{H} \oplus \mathcal{K}$ is given by $W(k, h) = (-h, k)$ for $h \in \mathcal{H}$ and $k \in \mathcal{K}$. If $f \in \mathcal{H}$, then there exist $h \in \mathcal{D}(T)$ and $k \in \mathcal{D}(T^*)$ such that

$$(f, 0) = (h, Th) \oplus W(k, T^*k) = (h - T^*k, Th + k),$$

so $f = h - T^*k \in \mathcal{D}(T) + \mathcal{R}(T^*)$. \square

It is easily seen that Lemma 1.5 is no longer true for operators which are not closed (set $Th = 0$ for $h \in \mathcal{D}$, where \mathcal{D} is a dense proper linear subspace of \mathcal{H}).

Corollary 1.6 ([9, Theorem 3.3]). *If $T : \mathcal{H} \supseteq \mathcal{D}(T) \rightarrow \mathcal{K}$ is a closed densely defined linear mapping, then $\mathcal{R}(T) \subseteq \mathcal{D}(T^*)$ if and only if $T \in \mathbf{B}(\mathcal{H}, \mathcal{K})$.*

Proof. If $\mathcal{R}(T) \subseteq \mathcal{D}(T^*)$, then Lemma 1.5 yields $\mathcal{D}(T^*) = \mathcal{H}$, so by the closed graph theorem $T \in \mathbf{B}(\mathcal{H}, \mathcal{K})$. \square

Let $T : \mathcal{H} \supseteq \mathcal{D}(T) \rightarrow \mathcal{K}$ be a linear mapping. Define a new linear mapping $\tilde{T} : \mathcal{H} \supseteq \mathcal{D}(\tilde{T}) \rightarrow \overline{\mathcal{R}(T)}$ by $\mathcal{D}(\tilde{T}) = \mathcal{D}(T)$ and $\tilde{T}h = Th$ for $h \in \mathcal{D}(\tilde{T})$. It is clear that $\mathcal{N}(\tilde{T}) = \mathcal{N}(T)$, $\mathcal{R}(\tilde{T}) = \mathcal{R}(T)$, $\mathcal{G}(\tilde{T}) = \mathcal{G}(T)$ and $\tilde{T} = Q_T T$, where $Q_T : \mathcal{K} \rightarrow \overline{\mathcal{R}(T)}$ is the orthogonal projection of \mathcal{K} onto $\overline{\mathcal{R}(T)}$. If T is densely defined, then obviously $\mathcal{N}(\tilde{T}^*) = \{0\}$.

Theorem 1.7. *If $T : \mathcal{H} \supseteq \mathcal{D}(T) \rightarrow \mathcal{K}$ is a closed densely defined linear mapping and $\mathcal{H} \neq \{0\}$, then the following conditions are equivalent:*

- (i) $\mathcal{D}(T) \subseteq \mathcal{R}(T^*)$,
- (ii) $\mathcal{R}(T^*) = \mathcal{H}$,

- (iii) $\mathcal{N}(T) \subsetneq \mathcal{D}(T)$ and $\sup_{f \in \mathcal{D}(T) \setminus \mathcal{N}(T)} \frac{|\langle h, f \rangle|}{\|Tf\|} < \infty$ for every $h \in \mathcal{H}$,
- (iv) $\tilde{T}^{*-1} \in \mathbf{B}(\mathcal{H}, \overline{\mathcal{R}(T)})$,
- (v) $\mathcal{N}(T) = \{0\}$ and $\tilde{T}^{-1} \in \mathbf{B}(\overline{\mathcal{R}(T)}, \mathcal{H})$.

Proof. There is no loss of generality in assuming that $T = \tilde{T}$ and $\overline{\mathcal{R}(T)} = \mathcal{K}$ because, by [16, Theorem 1], $\mathcal{R}(T^*) = \mathcal{R}(\tilde{T}^*)$.

(i) \Rightarrow (ii). By Lemma 1.5, we have $\mathcal{H} = \mathcal{D}(T) + \mathcal{R}(T^*) \subseteq \mathcal{R}(T^*)$, so $\mathcal{R}(T^*) = \mathcal{H}$.

(ii) \Leftrightarrow (iii). One can show that T satisfies (iii) if and only if for every $h \in \mathcal{H}$ there exists $c_h \geq 0$ such that $|\langle h, f \rangle| \leq c_h \|Tf\|$ for all $f \in \mathcal{D}(T)$. By [16, Theorem 1] this is equivalent to $\mathcal{R}(T^*) = \mathcal{H}$.

(ii) \Rightarrow (iv). Since T^{*-1} is a closed linear mapping with $\mathcal{D}(T^{*-1}) = \mathcal{H}$, the closed graph theorem yields $T^{*-1} \in \mathbf{B}(\mathcal{H}, \mathcal{K})$.

The implications (iv) \Rightarrow (ii) and (ii) \Rightarrow (i) are obvious.

(i) \Leftrightarrow (v). According to (i) \Rightarrow (ii), $\mathcal{N}(T) = \{0\}$ whenever $\mathcal{D}(T) \subseteq \mathcal{R}(T^*)$. Thus there is no loss of generality in assuming that $\mathcal{N}(T) = \{0\}$. Then T^{-1} is a closed densely defined linear mapping and $(T^{-1})^* = T^{*-1}$. Hence $\mathcal{D}((T^{-1})^*) = \mathcal{R}(T^*)$ and $\mathcal{R}(T^{-1}) = \mathcal{D}(T)$. Applying Corollary 1.6 to T^{-1} we get (i) \Leftrightarrow (v). \square

Corollary 1.8. *Let $T : \mathcal{H} \supseteq \mathcal{D}(T) \rightarrow \mathcal{K}$ be a closed densely defined linear mapping. If $\overline{\mathcal{R}(T)} = \mathcal{K}$ (resp. $\mathcal{N}(T) = \{0\}$), then the following conditions are equivalent:*

- (i) $\mathcal{D}(T) \subseteq \mathcal{R}(T^*)$ (resp. $\mathcal{D}(T^*) \subseteq \mathcal{R}(T)$),
- (ii) $\mathcal{R}(T^*) = \mathcal{H}$ (resp. $\mathcal{R}(T) = \mathcal{K}$),
- (iii) $\mathcal{N}(T) = \{0\}$ and $\mathcal{R}(T) = \mathcal{K}$ or equivalently $\overline{\mathcal{R}(T)} = \mathcal{K}$ and $\mathcal{R}(T^*) = \mathcal{H}$,
- (iv) $\mathcal{N}(T) = \{0\}$ and $T^{-1} \in \mathbf{B}(\mathcal{K}, \mathcal{H})$.

Moreover, if $\mathcal{K} = \mathcal{H}$, $\mathcal{N}(T) = \{0\}$ and $\overline{\mathcal{R}(T)} = \mathcal{H}$, then $\mathcal{D}(T) \subseteq \mathcal{R}(T^*)$ if and only if $\mathcal{D}(T^*) \subseteq \mathcal{R}(T)$ if and only if $0 \notin \sigma(T)$.

2. INTERTWINING NORMAL WITH HYPONORMAL OPERATORS

In this section we investigate intertwining relations between normal and hyponormal operators. We begin with the question of preserving cores by selfadjoint intertwining operators (see [11, Theorem 2.5] for a related result).

Proposition 2.1. *Let \mathcal{E} be a core for a closed densely defined operator A in \mathcal{H} and let $X \in \mathbf{B}(\mathcal{H})$ be a selfadjoint operator with $\mathcal{N}(X) = \{0\}$. If $XA \subseteq AX$, then $X(\mathcal{E})$ is a core for A .*

Proof. Take $f \in \mathcal{D}(A)$ such that $f \oplus Af$ is orthogonal in $\mathcal{H} \oplus \mathcal{H}$ to $Xg \oplus AXg$ for every $g \in \mathcal{E}$. Then, according to $X = X^*$ and $XA \subseteq AX$, we see that

$$\begin{aligned} \langle Xf, g \rangle &= \langle f, Xg \rangle = -\langle Af, AXg \rangle \\ &= -\langle Af, XAg \rangle \\ &= -\langle XAf, Ag \rangle \\ &= -\langle AXf, Ag \rangle, \quad g \in \mathcal{E}. \end{aligned}$$

Since \mathcal{E} is a core for A , we conclude that $AXf \in \mathcal{D}(A^*)$ and $A^*AXf = -Xf$. However A^*A is positive and selfadjoint, so $Xf = 0$. Since $\mathcal{N}(X) = \{0\}$, we get $f = 0$. This completes the proof. \square

The next result can be extracted from [13, Theorem 1] (it is easily seen that [13, Theorem 1] remains true for unbounded normal operators).

Theorem 2.2. *Let E be the spectral measure of a normal operator N . If σ is a Borel subset of \mathbb{C} and $f \in \bigcap_{z \in \sigma} \mathcal{R}(z - N)$, then $E(\sigma)f = 0$.*

We are now in a position to prove the crucial result for our further investigations. The idea of its proof is taken from [17, Theorem 1]).

Theorem 2.3. *Assume that N is a normal operator in \mathcal{H} , T is a closed hyponormal operator in \mathcal{K} and $X \in \mathbf{B}(\mathcal{H}, \mathcal{K})$ is such that $XN \subseteq TX$. Then:*

- (i) $|X|N \subseteq N|X|$,
- (ii) $N = T$ provided $\mathcal{K} = \mathcal{H}$, $X \geq 0$ and $\mathcal{N}(X) = \{0\}$.

Proof. (i) Let E be the spectral measure of N . Take a compact subset σ of \mathbb{C} and $f \in \mathcal{H}_\sigma := \mathcal{R}(E(\sigma))$. The space \mathcal{H}_σ reduces $\lambda - N$ and $\mathcal{H}_\sigma \subseteq \mathcal{D}(\lambda - N)$ for $\lambda \in \mathbb{C}$. Define the function $\varphi : \mathbb{C} \setminus \sigma \rightarrow \mathcal{H}_\sigma$ by

$$\varphi(\lambda) = \int_\sigma \frac{1}{\lambda - z} E(dz)f = (\lambda - N|_{\mathcal{H}_\sigma})^{-1}f, \quad \lambda \notin \sigma.$$

Then

$$(2.1) \quad (\lambda - N)\varphi(\lambda) = f, \quad \lambda \notin \sigma.$$

According to $XN \subseteq TX$, we have

$$(2.2) \quad X(\lambda - N) \subseteq (\lambda - T)X, \quad \lambda \in \mathbb{C}.$$

Combining (2.1) and (2.2), we get $Xf = X(\lambda - N)\varphi(\lambda) = (\lambda - T)X\varphi(\lambda)$ for $\lambda \notin \sigma$. Hence

$$(2.3) \quad X^*Xf = X^*(\lambda - T)X\varphi(\lambda), \quad \lambda \notin \sigma.$$

Since $\lambda - T$ is hyponormal, we conclude from Proposition 1.2 that there exists a contraction $C_\lambda \in \mathbf{B}(\mathcal{K})$ such that

$$(2.4) \quad \lambda - T \subseteq (\lambda - T)^*C_\lambda, \quad \lambda \in \mathbb{C}.$$

Taking adjoints in (2.2), we get

$$(2.5) \quad X^*(\lambda - T)^* \subseteq (\lambda - N)^*X^*, \quad \lambda \in \mathbb{C}.$$

It follows from (2.3), (2.4) and (2.5) that

$$\begin{aligned} X^*Xf &= X^*(\lambda - T)X\varphi(\lambda) = X^*(\lambda - T)^*C_\lambda X\varphi(\lambda) \\ &= (\lambda - N)^*X^*C_\lambda X\varphi(\lambda), \quad \lambda \notin \sigma. \end{aligned}$$

Hence X^*Xf belongs to $\bigcap_{z \in \mathbb{C} \setminus \sigma^*} \mathcal{R}(z - N^*)$, where $\sigma^* = \{z : \bar{z} \in \sigma\}$. According to Theorem 2.2, we have $E(\mathbb{C} \setminus \sigma)X^*Xf = 0$ (because the spectral measure F of N^* is related to E via $F(\tau) = E(\tau^*)$ for every Borel subset τ of \mathbb{C}) or equivalently $X^*Xf = E(\sigma)X^*Xf$. Since f was an arbitrary element of \mathcal{H}_σ , we conclude that $X^*X(\mathcal{H}_\sigma) \subseteq \mathcal{H}_\sigma$. However $X^*X \geq 0$, so $X^*XE(\sigma) = E(\sigma)X^*X$. This, in turn, implies that $|X|E(\sigma) = E(\sigma)|X|$ for every compact subset σ of \mathbb{C} . Since E is a regular measure, the latter equality holds for every Borel subset σ of \mathbb{C} , which implies (i).

(ii) Since $X \geq 0$, we conclude from (i) that $NXf = XNf = TXf$ for $f \in \mathcal{D}(N)$. Thus

$$(2.6) \quad N|_{X\mathcal{D}(N)} \subseteq T.$$

However, according to Proposition 2.1, $X\mathcal{D}(N)$ is a core for N , so taking closures in (2.6) leads to $N \subseteq T$. Hence $T^* \subseteq N^*$ and, by the hyponormality of T , we have $\mathcal{D}(T) \subseteq \mathcal{D}(T^*) \subseteq \mathcal{D}(N^*) = \mathcal{D}(N)$. This gives us $N = T$. \square

3. MORE ON INTERTWINING RELATIONS

In this section we continue the study of intertwining relations. Let us first collect notations and definitions which play the basic role in the rest of the paper.

Given $X \in \mathbf{B}(\mathcal{H}, \mathcal{K})$, we set $\mathcal{I}(X) := \overline{\mathcal{R}(X^*)} = \overline{\mathcal{R}(|X|)}$ and $\mathcal{F}(X) := \overline{\mathcal{R}(X)}$. Let $X = i_x|X|$ be the polar decomposition of X , i.e. $i_x \in \mathbf{B}(\mathcal{H}, \mathcal{K})$ is the partial isometry with $\mathcal{N}(i_x) = \mathcal{N}(X)$, which maps isometrically $\mathcal{I}(X)$ onto $\mathcal{F}(X)$. Denote by u_x and \widehat{X} the bounded linear mappings from $\mathcal{I}(X)$ into $\mathcal{F}(X)$ defined by $u_x := i_x|_{\mathcal{I}(X)}$ and $\widehat{X} := X|_{\mathcal{I}(X)}$. Then u_x is a unique unitary isomorphism in $\mathbf{B}(\mathcal{I}(X), \mathcal{F}(X))$ such that

$$(3.1) \quad u_x|X|f = Xf, \quad f \in \mathcal{H}.$$

Lemma 3.1. *If $X \in \mathbf{B}(\mathcal{H}, \mathcal{K})$, then*

- (i) $\mathcal{N}(\widehat{X}) = \{0\}$,
- (ii) $\mathcal{R}(X) = \mathcal{R}(\widehat{X})$,
- (iii) $\widehat{X^*} = \widehat{X}^*$,
- (iv) $|\widehat{X}| = |\widehat{X}|$,
- (v) $i_{\widehat{X}} = u_{\widehat{X}} = u_x$.

Proof. The proof of (i), (ii) and (iii) is left to the reader.

(iv) Since $\mathcal{I}(X) = \mathcal{I}(|X|)$ reduces $|X|$ and $\widehat{X^*} = \widehat{X}^*$, we get $|\widehat{X}| = (\widehat{X^*}\widehat{X})^{1/2} = ((X^*X)|_{\mathcal{I}(X)})^{1/2} = |X|_{\mathcal{I}(X)} = |\widehat{X}|$, which proves (iv).

(v) It follows from (i) and (ii) that

$$(3.2) \quad \mathcal{I}(X) = \mathcal{I}(\widehat{X}) \quad \text{and} \quad \mathcal{F}(X) = \mathcal{F}(\widehat{X}).$$

On the other hand, by (iv) and (3.1), we have $u_x|\widehat{X}|f = u_x|\widehat{X}|f = u_x|X|f = Xf = \widehat{X}f$ for $f \in \mathcal{I}(X)$. This implies that $u_{\widehat{X}} = u_x$ due to the uniqueness of $u_{\widehat{X}}$. The equality $i_{\widehat{X}} = u_{\widehat{X}}$ is a consequence of (3.2). \square

If A and B are operators in \mathcal{H} and \mathcal{K} , respectively, and $X \in \mathbf{B}(\mathcal{H}, \mathcal{K})$, then we set $A_x := A|_{\mathcal{I}(X)}$ and $B^x := B|_{\mathcal{F}(X)}$. To shorten notation, we write A_x^* instead of $(A_x)^*$. In case X intertwines A^* and B , A_x and B^x have the following properties.

Theorem 3.2. *Let A and B be closed densely defined operators in \mathcal{H} and \mathcal{K} , respectively, and let $X \in \mathbf{B}(\mathcal{H}, \mathcal{K})$ be such that $XA^* \subseteq BX$.*

- (i) *If $\mathcal{I}(X)$ reduces A , then B^x is closed densely defined in $\mathcal{F}(X)$ and*

$$(3.3) \quad \widehat{X}A_x^* \subseteq B^x\widehat{X}.$$

- (ii) *If $\mathcal{I}(X)$ and $\mathcal{F}(X)$ reduce A and B to normal operators, respectively, then*

$$(3.4) \quad \begin{aligned} XA \subseteq B^*X, \quad |X|A \subseteq A|X|, \quad |X^*|B \subseteq B|X^*|, \\ A_x^* = u_x^*B^xu_x. \end{aligned}$$

Proof. (i) The closedness of B^x follows directly from that of B . We now show that

$$(3.5) \quad \widehat{X}(\mathcal{D}(A_x^*)) \subseteq X(\mathcal{D}(A^*)) \subseteq \mathcal{D}(B^x).$$

Indeed, since $A_x^* = A^*|_{\mathcal{I}(X)}$, we get $\widehat{X}(\mathcal{D}(A_x^*)) = X(\mathcal{D}(A^*) \cap \mathcal{I}(X)) \subseteq X(\mathcal{D}(A^*))$. According to $XA^* \subseteq BX$, we have $X(\mathcal{D}(A^*)) \subseteq \mathcal{D}(B) \cap \mathcal{F}(X)$ and $BXf = XA^*f \in \mathcal{F}(X)$ for $f \in \mathcal{D}(A^*)$. Hence $X(\mathcal{D}(A^*)) \subseteq \mathcal{D}(B^x)$, which proves (3.5).

Since the operator A^* is densely defined in \mathcal{H} , we conclude from (3.5) that $\mathcal{F}(X) = \overline{X(\mathcal{D}(A^*))} = \overline{X(\mathcal{D}(A^*))} \subseteq \overline{\mathcal{D}(B^x)}$, which shows that B^x is densely defined in $\mathcal{F}(X)$. Applying once more the equality $A_x^* = A^*|_{\mathcal{I}(X)}$ and (3.5), we obtain $\widehat{X}A_x^*f = XA^*f = BXf = B^x\widehat{X}f$ for $f \in \mathcal{D}(A_x^*)$. This completes the proof of (i).

(ii) Applying the Putnam–Fuglede theorem (cf. [12, Lemma]) to (3.3), we obtain

$$(3.6) \quad \widehat{X}A_x \subseteq (B^x)^* \widehat{X}.$$

Therefore, by Lemma 3.1 (v), we have $\mathbf{u}_x|\widehat{X}|A_x \subseteq (B^x)^* \mathbf{u}_x|\widehat{X}|$ or, equivalently,

$$(3.7) \quad |\widehat{X}|A_x \subseteq (\mathbf{u}_x^*(B^x)^* \mathbf{u}_x)|\widehat{X}|.$$

Since, according to Lemma 3.1 (i), (iv), $\mathcal{N}(|\widehat{X}|) = \mathcal{N}(|\widehat{X}|) = \{0\}$, we conclude from (3.7) via Theorem 2.3 that $A_x = \mathbf{u}_x^*(B^x)^* \mathbf{u}_x$ (which implies (3.4)) and

$$(3.8) \quad |\widehat{X}|A_x \subseteq A_x|\widehat{X}|.$$

As $X = \widehat{X} \oplus 0$, $A = A_x \oplus A'$ and $B = B^x \oplus B'$ with $A' = A|_{\mathcal{N}(X)}$ and $B' = B|_{\mathcal{N}(X^*)}$, we deduce from (3.6) that

$$XA \subseteq \widehat{X}A_x \oplus 0 \subseteq (B^x)^* \widehat{X} \oplus 0 = B^*X.$$

It follows from Lemma 3.1 (iv) that $|X| = |\widehat{X}| \oplus 0$. Hence (3.8) leads to

$$(3.9) \quad |X|A \subseteq |\widehat{X}|A_x \oplus 0 \subseteq A_x|\widehat{X}| \oplus 0 = A|X|.$$

Notice that $XA^* \subseteq BX$ implies $X^*B^* \subseteq AX^*$. Since $\mathcal{I}(X^*)$ and $\mathcal{F}(X^*)$ reduce B and A to normal operators, respectively, we can replace the linear mappings X and A in (3.9) by X^* and B , respectively, to get $|X^*|B \subseteq B|X^*|$. This completes the proof. \square

4. INTERTWINING COHYPNORMAL WITH HYPONORMAL OPERATORS

In this section we describe the structure of intertwining relations between co-subnormal and hyponormal (as well as between cohyponormal and subnormal) operators. In particular, we obtain an asymmetric Putnam–Fuglede theorem for unbounded operators. First we consider a special case of intertwining relations.

Proposition 4.1. *Assume that N is a normal operator in \mathcal{H} , T is a closed hyponormal operator in \mathcal{K} and $X \in \mathbf{B}(\mathcal{H}, \mathcal{K})$ is such that $XN \subseteq TX$. Then $\mathcal{I}(X)$ and $\mathcal{F}(X)$ reduce N and T to normal operators, respectively.*

Proof. It follows from Theorem 2.3 (i) that $|X|E(\sigma) = E(\sigma)|X|$ for every Borel subset σ of \mathbb{C} , E being the spectral measure of N . Since $\mathcal{I}(X) = \overline{\mathcal{R}(|X|)}$, we conclude that $\mathcal{I}(X)$ reduces E and, consequently, N as well. According to Theorem 3.2 (i), the operator T^x is densely defined in $\mathcal{F}(X)$ and

$$(4.1) \quad \widehat{X}N_x \subseteq T^x\widehat{X}.$$

By Proposition 1.3 (i), T^x is a closed hyponormal operator in $\mathcal{F}(X)$. Applying Lemma 3.1 (v) to (4.1), we get $|\widehat{X}|N_x \subseteq (\mathbf{u}_x^*T^x\mathbf{u}_x)|\widehat{X}|$. Since $\mathcal{N}(|\widehat{X}|) = \{0\}$ and $\mathbf{u}_x^*T^x\mathbf{u}_x$ is a closed hyponormal operator, Theorem 2.3 (ii) yields $N_x = \mathbf{u}_x^*T^x\mathbf{u}_x$. Hence T^x , being unitarily equivalent to the normal operator N_x , is normal as well.

By Proposition 1.3 (ii), $\mathcal{F}(X)$ reduces T to a normal operator. This completes the proof. \square

We are now in a position to prove the main result of the paper.

Theorem 4.2. *Assume that A is a closed subnormal (resp. a closed hyponormal) operator in \mathcal{H} , B is a closed hyponormal (resp. a closed subnormal) operator in \mathcal{K} and $X \in \mathbf{B}(\mathcal{H}, \mathcal{K})$ is such that $XA^* \subseteq BX$. Then:*

- (i) $XA \subseteq B^*X$,
- (ii) $|X|A \subseteq A|X|$,
- (iii) $|X^*|B \subseteq B|X^*|$,
- (iv) $\mathcal{I}(X)$ reduces A to the normal operator A_x ,
- (v) $\mathcal{F}(X)$ reduces B to the normal operator B^x ,
- (vi) $A_x^* = u_x^* B^x u_x$.

Proof. Case 1. A is subnormal and B is hyponormal.

Let N be a normal extension of A acting in a Hilbert space $\mathcal{L} \supseteq \mathcal{H}$. Define $Y \in \mathbf{B}(\mathcal{K}, \mathcal{L})$ by $Yf = X^*f$ for $f \in \mathcal{K}$. We show that

$$(4.2) \quad \mathcal{F}(X) = \mathcal{F}(Y^*).$$

Indeed, since $Y = JX^*$, where $J \in \mathbf{B}(\mathcal{H}, \mathcal{L})$ is defined by $Jh = h$ for $h \in \mathcal{H}$, we get $Y^* = XP$, where $P \in \mathbf{B}(\mathcal{L}, \mathcal{H})$ is the orthogonal projection of \mathcal{L} onto \mathcal{H} . The latter equality implies (4.2).

(v) Taking adjoints in $XA^* \subseteq BX$, we obtain

$$(4.3) \quad X^*B^* \subseteq AX^*,$$

which implies $YB^* \subseteq NY$. This, in turn, forces $Y^*N^* \subseteq BY^*$. Hence, by (4.2) and Proposition 4.1, we come to the conclusion that $\mathcal{F}(X) = \mathcal{F}(Y^*)$ reduces B to the normal operator B^x .

(iv) Applying Theorem 3.2 (i) to (4.3) and Lemma 3.1 (iii), we obtain that $A_x = A^{(x^*)}$ is a closed densely defined operator in $\mathcal{I}(X)$ and

$$(4.4) \quad \widehat{X}^*(B^x)^* \subseteq A_x \widehat{X}^*.$$

Since $A_x \subseteq A \subseteq N$, we see that A_x is subnormal and, in consequence, hyponormal. It follows from Proposition 4.1 applied to (4.4) that $\mathcal{I}(X) = \mathcal{F}(\widehat{X}^*)$ reduces A_x to a normal operator. Hence A_x is normal itself. By Proposition 1.3 (ii), $\mathcal{I}(X)$ reduces A to the normal operator A_x .

The rest of the conclusion follows directly from Theorem 3.2 (ii) applied to the intertwining relation $XA^* \subseteq BX$.

Case 2. A is hyponormal and B is subnormal.

It follows from $XA^* \subseteq BX$ that $X^*B^* \subseteq AX^*$. According to Case 1, $\mathcal{I}(X) = \mathcal{F}(X^*)$ reduces A to the normal operator A_x and $\mathcal{F}(X) = \mathcal{I}(X^*)$ reduces B to the normal operator B^x . The rest of the conclusion follows once more from Theorem 3.2 (ii) applied to the relation $XA^* \subseteq BX$. \square

5. CONCLUDING REMARKS AND COMMENTS

We begin with a result which can be inferred from Theorem 4.2 via the implications:

- a) $\mathcal{N}(X) = \{0\} \Rightarrow \mathcal{I}(X) = \mathcal{H}$,
- b) $\mathcal{N}(X^*) = \{0\} \Rightarrow \mathcal{F}(X) = \mathcal{K}$,

- c) $(\mathcal{K} = \mathcal{H} \wedge X \geq 0 \wedge \mathcal{N}(X) = \{0\}) \Rightarrow u_x$ is the identity operator on \mathcal{H} ,
- d) $X \neq 0 \Rightarrow (\mathcal{I}(X) \neq \{0\} \wedge \mathcal{F}(X) \neq \{0\})$.

Corollary 5.1. *Assume that A is a closed subnormal (resp. a closed hyponormal) operator in \mathcal{H} , B is a closed hyponormal (resp. a closed subnormal) operator in \mathcal{K} and $X \in \mathbf{B}(\mathcal{H}, \mathcal{K})$ is such that $XA^* \subseteq BX$. Then:*

- (i) A is normal provided $\mathcal{N}(X) = \{0\}$,
- (ii) B is normal provided $\mathcal{N}(X^*) = \{0\}$,
- (iii) B and A^* are normal and unitarily equivalent provided $\mathcal{N}(X) = \{0\}$ and $\mathcal{N}(X^*) = \{0\}$,
- (iv) $B = A^*$ and A is normal provided $\mathcal{K} = \mathcal{H}$, $X \geq 0$ and $\mathcal{N}(X) = \{0\}$,
- (v) neither A nor B is pure provided $X \neq 0$.

Let us make some comments concerning Corollary 5.1: part (ii) of it extends [7, Theorem 3] (the assumption $\sigma(T) \neq \mathbb{C}$ can be dropped); part (iii) generalizes [11, Theorem 4.1], [11, Proposition 4.2], [11, Corollaries 4.4 and 4.6], [7, Corollary 4] (the assumption $\sigma(N) \neq \mathbb{C}$ can be dropped), [7, Corollary 9] (the assumption $\mathcal{D}(S_j) = \mathcal{D}(S_j^*)$, $j = 1, 2$, can be dropped) and [10, Theorem 3.3] (the operator C does coincides with \overline{A}); part (iv) generalizes [11, Theorem 4.5], [7, Theorem 8] (the assumptions $\mathcal{D}(S_j) = \mathcal{D}(S_j^*)$, $j = 1, 2$, and $\sigma(S_1) \neq \mathbb{C}$ can be dropped) and [10, Proposition 3.2] (the operator B does coincides with \overline{A}); finally, part (v) has been applied in [19, Lemma 8.5] to prove that the strong symmetric commutant of a closed hyponormal operator is equal to the orthogonal sum of strong symmetric commutants of its normal and pure parts.

In particular part (iii) of Corollary 5.1 implies that a closed hyponormal operator A which is quasi-similar to a selfadjoint operator B is selfadjoint itself and unitarily equivalent to B (see [11] for the definition of quasi-similarity and Corollary 4.6 therein for a related result).

The last result of the paper is related to Remark 2.6 following [11, Theorem 2.5]. We show that the assumption imposed on the operator B in [11, Remark 2.6] is not weaker than that in [11, Theorem 2.5].

Theorem 5.2. *Let A and B be closed densely defined operators in \mathcal{H} and \mathcal{K} , respectively, and let $X \in \mathbf{B}(\mathcal{H}, \mathcal{K})$ be such that $XA \subseteq BX$. Assume that $\overline{\mathcal{R}(A)} = \mathcal{H}$ and $\overline{\mathcal{R}(X)} = \mathcal{K}$. Then:*

- (i) $\overline{\mathcal{R}(B)} = \mathcal{K}$,
- (ii) $\mathcal{D}(B) \subseteq \mathcal{R}(B^*)$ if and only if $0 \notin \sigma(B)$,
- (iii) $0 \notin \sigma(A)$ provided $0 \notin \sigma(B)$ and $X^{-1} \in \mathbf{B}(\mathcal{K}, \mathcal{H})$.

Proof. Since $\overline{\mathcal{R}(X)} = \mathcal{K}$ and $\overline{\mathcal{R}(A)} = \mathcal{H}$, we conclude that $\overline{\mathcal{R}(XA)} = \mathcal{K}$. This and $XA \subseteq BX$ lead to $\overline{\mathcal{R}(B)} = \mathcal{K}$. Hence (ii) follows from Corollary 1.8.

Assume that $0 \notin \sigma(B)$ and $X^{-1} \in \mathbf{B}(\mathcal{K}, \mathcal{H})$. By $XA \subseteq BX$, we have $Af = X^{-1}XAf = X^{-1}BXf$ for $f \in \mathcal{D}(A)$, so

$$\begin{aligned} \|Af\| &= \|X^{-1}BXf\| \geq \|X\|^{-1}\|BXf\| \\ &\geq (\|X\|\|B^{-1}\|)^{-1}\|Xf\| \\ &\geq (\|X\|\|B^{-1}\|\|X^{-1}\|)^{-1}\|f\|, \quad f \in \mathcal{D}(A). \end{aligned}$$

This means that A is bounded from below and $\overline{\mathcal{R}(A)} = \mathcal{H}$. Therefore $A^{-1} \in \mathbf{B}(\mathcal{H})$, which completes the proof of (iii). □

We close the paper with an example which shows that the assumption $X^{-1} \in \mathbf{B}(\mathcal{K}, \mathcal{H})$ could not be removed from part (iii) of Theorem 5.2.

Example 5.3. Let \mathcal{H} be a separable Hilbert space with an orthonormal basis $\{e_n\}_{n \in \mathbb{Z}}$ and let \mathcal{E} be the linear span of $\{e_n\}_{n \in \mathbb{Z}}$ ($\mathbb{Z} = \{\dots, -1, 0, 1, \dots\}$). Define the sequence $\lambda = \{\lambda_n\}_{n \in \mathbb{Z}}$ by

$$\lambda_n = \begin{cases} 1, & n \leq 0, \\ 1/n, & n \in 2 \cdot \mathbb{Z}, \quad n \geq 1, \\ 1/n^2, & n \in 2 \cdot \mathbb{Z} + 1, \quad n \geq 1. \end{cases}$$

Let B and D_λ be the members of $\mathbf{B}(\mathcal{H})$ determined by the conditions $Be_n = e_{n+1}$ and $D_\lambda e_n = \lambda_n e_n$ ($n \in \mathbb{Z}$), and let $X := B^2 D_\lambda$ ($\in \mathbf{B}(\mathcal{H})$). Then $\mathcal{N}(B) = \{0\}$, $B^{-1} \in \mathbf{B}(\mathcal{H})$ and $\mathcal{N}(X) = \{0\}$ because $\mathcal{N}(D_\lambda) = \{0\}$. Since $Xe_n = \lambda_n e_{n+2}$ for $n \in \mathbb{Z}$, we get $\mathcal{E} \subseteq \mathcal{R}(X)$ and consequently $\overline{\mathcal{R}(X)} = \mathcal{H}$. Set $A = BD_\mu$, where D_μ is a unique normal operator in \mathcal{H} determined by the condition $D_\mu e_n = \mu_n e_n$ with $\mu_n = \lambda_n / \lambda_{n+1}$ ($n \in \mathbb{Z}$). The operator A is closed densely defined and unbounded; moreover, $\mathcal{N}(A) = \{0\}$. Since $Ae_n = \mu_n e_{n+1}$ for $n \in \mathbb{Z}$, we get $\mathcal{E} \subseteq \mathcal{R}(A)$ and $A^{-1}e_n = \mu_{n-1}^{-1} e_{n-1}$ for $n \in \mathbb{Z}$. Therefore $\overline{\mathcal{R}(A)} = \mathcal{H}$ and the operator A^{-1} is unbounded (because the sequence $\{\mu_n^{-1}\}_{n \in \mathbb{Z}}$ is unbounded). One can check that $X Ae_n = B X e_n$ for $n \in \mathbb{Z}$, hence that $X(A|_{\mathcal{E}}) \subseteq BX$ and finally that $XA \subseteq BX$ (because \mathcal{E} is a core for A and B is closed). Clearly, the operator X^{-1} is unbounded.

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