PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 129, Number 9, Pages 2647–2654
S 0002-9939(00)05891-3
Article electronically published on December 7, 2000

ON STATISTICAL LIMIT POINTS

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(Communicated by David Preiss)

Abstract. The set of all statistical limit points of a given sequence x_n is characterized as an F_σ-set. It is also characterized in terms of discontinuity points of distribution functions of x_n.

Introduction

Following the concept of a statistically convergent sequence, J. A. Fridy [Fr93] introduced the notion of a *statistical limit point* of a given sequence x_n, $n = 1, 2, \dots,$ of real numbers: A real number x is said to be a statistical limit point of the sequence x_n if there exists a subsequence x_{k_n}, $n = 1, 2, \dots$, such that $\lim_{n\to\infty} x_{k_n} = x$ and the set of indices k_n has a positive upper asymptotic density (see Definitions and Notations). Fridy studied the set $\Lambda(x_n)$ of all such points. He has shown that this set need not be closed or open in $\mathbb{R}$.

In section 1 of the paper we prove that the set $\Lambda(x_n)$ is an F_σ-set in $\mathbb{R}$ for an arbitrary sequence x_n and vice-versa for any given F_σ-set X there exists a sequence x_n such that $X = \Lambda(x_n)$.

In section 2 we prove that the set $\Lambda(x_n)$ coincides with the set of all discontinuity points of distribution functions of x_n. This leads to some sufficient conditions for $\Lambda(x_n) = \emptyset$.

Finally, in section 3 we compute $\Lambda(x_n)$ using an explicit form of the set $G(x_n)$ of all distribution functions of x_n for some special x_n.

Definitions and notations

If $A \subset \mathbb{N}$, then we denote by $|A|$ the cardinality of A and $A(n) = |\{k \le n; k \in A\}|$. The numbers

$$\underline{d}(A) = \liminf_{n\to\infty} \frac{A(n)}{n}, \quad \overline{d}(A) = \limsup_{n\to\infty} \frac{A(n)}{n}$$

are called the lower and upper asymptotic density of A, respectively. If there exists the limit $\lim_{n\to\infty} A(n)/n$, then $d(A) = \underline{d}(A) = \overline{d}(A)$ is said to be the asymptotic density of A (cf. [HR66, xix]).

Received by the editors December 12, 1998 and, in revised form, January 10, 2000.
2000 *Mathematics Subject Classification.* Primary 40A05, 11K31, 11B05.
Key words and phrases. Statistically convergent sequence, statistical limit point, asymptotic density, distribution function.
This work was supported in part by grants from VEGA No. 2/5123/98 and No. 1/4323/97.

It is easy to see that if A is ordered to the increasing sequence $k_1 < k_2 < \dots$, then

$$\underline{d}(A) = \liminf_{n\to\infty} \frac{n}{k_n}, \quad \overline{d}(A) = \limsup_{n\to\infty} \frac{n}{k_n}$$

and we write $\underline{d}(A) = \underline{d}(k_n)$ and $\overline{d}(A) = \overline{d}(k_n)$.

In the following x_n, $n = 1, 2, \dots$, is an infinite sequence of real numbers.

The main object of the paper is the set $\Lambda(x_n)$ defined as

$$\Lambda(x_n) = \{\alpha \in \mathbb{R}; \exists (k_1 < k_2 < \dots) \lim_{n\to\infty} x_{k_n} = \alpha, \overline{d}(k_n) > 0\}.$$

In section 2 we use the set $G(x_n)$ of all distribution functions of the sequence x_n, where $g : (-\infty, \infty) \to [0, 1]$ is said to be a distribution function of the sequence x_n if there is a sequence $N_1 < N_2 < \dots$ of positive integers such that

$$\lim_{k\to\infty} \frac{1}{N_k} \sum_{n=1}^{N_k} c_{(-\infty,x)}(x_n) = g(x)$$

for every continuity point x of g. Here $c_{(-\infty,x)}(t)$ is the characteristic function of the interval $(-\infty, x)$.

x mod 1 is the fractional part of x and for reduced sequence x_n mod 1 we take the restrictions of distribution functions $g(x) \in G(x_n \bmod 1)$ to the interval $[0, 1]$.

For singleton $G(x_n) = \{g\}$, g is called the asymptotic distribution function of x_n.

The sequence x_n statistically converges to α if $d(\{n \in \mathbb{N}; |x_n - \alpha| \geq \varepsilon\}) = 0$ for any $\varepsilon > 0$.

Finally, $X \bmod 1 = \{x \bmod 1; x \in X\}$ and $X + k = \{x + k; x \in X\}$ for a set X of real numbers.

1. Topological structure of $\Lambda(x_n)$

As we already mentioned the set $\Lambda(x_n)$ need not belong to the zero Borel class in $\mathbb{R}$. Thus, the following result is the best possible in this direction.

Theorem 1.1. *For every sequence x_n, the set $\Lambda(x_n)$ is an F_σ-set in $\mathbb{R}$.*

Proof. For any $0 < t \leq 1$ we denote

$$\Lambda(x_n, t) = \{\alpha \in \mathbb{R}; \exists (k_1 < k_2 < \dots) \lim_{n\to\infty} x_{k_n} = \alpha, \overline{d}(k_n) \geq t\}.$$

We prove that for every $0 < t \leq 1$, the set $\Lambda(x_n, t)$ is closed in $\mathbb{R}$. This gives the theorem directly, since $\Lambda(x_n) = \bigcup_{j=1}^{\infty} \Lambda(x_n, 1/j)$.

Assume that $\alpha_i \in \Lambda(x_n, t)$ and $\lim_{i\to\infty} \alpha_i = \alpha$, for $i = 1, 2, \dots$. For every α_i we can select a subsequence of x_n which converges to α_i having the set of indices $K^{(i)}$ with $\overline{d}(K^{(i)}) \geq t$. Moreover, for an arbitrary given sequence ε_i of positive numbers we can select $N_1 < N_2 < \dots$ such that

$$\frac{|K^{(i)} \cap (N_{i-1}, N_i]|}{N_i} \geq t - \varepsilon_i$$

for $i = 1, 2, \dots$. Putting $K = \bigcup_{i=1}^{\infty} (K^{(i)} \cap (N_{i-1}, N_i])$ and assuming $\lim_{i\to\infty} \varepsilon_i = 0$ we have $\overline{d}(K) \geq t$. Furthermore, if K is ordered to increasing k_n, then $\lim_{k\to\infty} x_{k_n} = \alpha$, because $K \subset \bigcup_{i=j}^{\infty} K^{(i)} \subset \{n \in \mathbb{N}; |x_n - \alpha| < \varepsilon\}$ with the exception of finitely many terms for any $\varepsilon > 0$ and sufficiently large $j = j(\varepsilon)$. □

In connection with Theorem 1.1 the question arises as to whether an arbitrary F_σ-set coincides with a $\Lambda(x_n)$ for a suitable x_n. We give the positive answer to this question.

Theorem 1.2. *If $X \subset \mathbb{R}$ is an arbitrary F_σ-set in $\mathbb{R}$, then there exists a sequence x_n such that $\Lambda(x_n) = X$.*

Proof. Let $X = \bigcup_{k=1}^{\infty} X_k$, where X_k are nonempty closed sets. For every $k = 1, 2, \ldots$ select $x_{k,i} \in X_k$, $i = 1, 2, \ldots$, such that the set of all usual limit points of the sequence $x_{k,i}$ coincide with X_k. Decompose $\mathbb{N} = \bigcup_{k=1}^{\infty} A_k$ on pairwise disjoint sets A_k having positive asymptotic density $d(A_k)$ and such that $d(\mathbb{N} \setminus \bigcup_{i=1}^{k} A_i) \to 0$ for $k \to \infty$ (cf. [Fr93, Ex. 3]). Finally, for every $k = 1, 2, \ldots$, decompose $A_k = \bigcup_{i=1}^{\infty} B_{k,i}$ on pairwise disjoint sets $B_{k,i}$ having $\overline{d}(B_{k,i}) = d(A_k)$ for every $i = 1, 2, \ldots$. (First, decompose $A_k = B_{k,1} \cup B_{k,1}^*$ such that $d(A_k) = \overline{d}(B_{k,1}) = \overline{d}(B_{k,1}^*)$. Then decompose $B_{k,1}^* = B_{k,2} \cup B_{k,2}^*$ such that $\overline{d}(B_{k,1}^*) = \overline{d}(B_{k,2}) = \overline{d}(B_{k,2}^*)$, etc.)

For a searching sequence x_n, for every $n \in B_{k,i}$, we define x_n as $x_n = \text{constant} = x_{k,i}$. Then we have:

1°. If $x_{n_j} \to \alpha$ for $j \to \infty$ and $\alpha \notin X$, then for every k the set $\bigcup_{i=1}^{k} A_i$ contains only finitely many n_j and thus $d(n_j) = 0$.

2°. If $\alpha \in X_k$, $0 < \varepsilon < d(A_k)$, and $x_{k,i_j} \to \alpha$ for $j \to \infty$, we can select $N_1 < N_2 < \ldots$ such that

$$\frac{|B_{k,i_j} \cap (N_{j-1}, N_j]|}{N_j} \geq d(A_k) - \varepsilon.$$

Then x_n over $n \in \bigcup_{j=1}^{\infty} B_{k,i_j} \cap (N_{j-1}, N_j]$ converges to α and

$$d\left(\bigcup_{j=1}^{\infty} B_{k,i_j} \cap (N_{j-1}, N_j]\right) \geq d(A_k) - \varepsilon.$$

□

2. Points of $\Lambda(x_n)$ as discontinuity points of $g \in G(x_n)$

I. J. Schoenberg [Sch59] noted that the sequence x_n is statistically convergent to α if and only if it admits asymptotic distribution function $c_{[\alpha,\infty)}(x)$. This motivates the following theorem.

Theorem 2.1. *For every sequence x_n we have*

$$\Lambda(x_n) = \{\alpha \in \mathbb{R}; \exists (g(x) \in G(x_n))\ g(x) \text{ is discontinuous at } \alpha\}.$$

Proof. 1°. Let α be a discontinuity point of $g(x) \in G(x_n)$ with a jump of size h. Let α_i and β_i be two sequences satisfying $\beta_i - \alpha_i \to 0$ as $i \to \infty$, $\alpha_i < \alpha < \beta_i$ and α_i and β_i are continuity points of $g(x)$ for every i. From N_k (using the definition of $g(x)$) we can select a subsequence N_{k_i} such that

$$\frac{1}{N_{k_i}} \sum_{n=N_{k_{i-1}}+1}^{N_{k_i}} c_{[\alpha_i,\beta_i)}(x_n) > h - \varepsilon$$

for some $\varepsilon \in (0, h)$. Ordering elements of $\bigcup_{i=1}^{\infty} \{n \in \mathbb{N}, N_{k_{i-1}} < n \leq N_{k_i}\}$ to an increasing sequence n_i, we then have $x_{n_i} \to \alpha$ and $\overline{d}(n_i) \geq h - \varepsilon$.

2°. Assume that $x_{n_i} \to \alpha$ for $i \to \infty$, $\overline{d}(n_i) = h > 0$, and $\varepsilon \in (0, h)$. Then there exists a sequence N_k such that, for every $k = 1, 2, \dots$,

$$\frac{|\{i \in \mathbb{N}; n_i \le N_k\}|}{N_k} \ge h - \varepsilon.$$

By Helly selection principle from N_k we can select a subsequence N_{k_i} such that

$$\lim_{i\to\infty} \frac{1}{N_{k_i}} \sum_{n=1}^{N_{k_i}} c_{(-\infty,x)}(x_n) = g(x)$$

for every point x of continuity of $g(x)$. Clearly, $g(x)$ has at α a jump of size $\ge h - \varepsilon > 0$ and $g(x) \in G(x_n)$. □

Using Theorem 2.1, some results from the theory of uniform distribution can be transcribed for $\Lambda(x_n)$. We can now state an analogue of the well-known Wiener-Schoenberg theorem and Van der Corput difference theorem, but first we begin with a technical result.

Proposition 2.2. (i) *For any sequence* x_n, $\Lambda(x_n \bmod 1) = \emptyset \Longrightarrow \Lambda(x_n) = \emptyset$.

(ii) *For bounded sequence* x_n, *the sets* $\Lambda(x_n \bmod 1)$ *and* $\Lambda(x_n) \bmod 1$ *can differ only in* 0 *or* 1 *and if* $0 \in \Lambda(x_n) \bmod 1$, *then* $0 \in \Lambda(x_n \bmod 1)$ *or* $1 \in \Lambda(x_n \bmod 1)$.

Proof. The implication $x_{k_n} \to \alpha \Longrightarrow x_{k_n} \bmod 1 \to \alpha \bmod 1$ does not hold only for integer α and $x_{k_n} < \alpha$. □

Theorem 2.3. *Let* x_n *be a sequence of real numbers, and let*

$$\omega_h = \limsup_{N\to\infty} \left| \frac{1}{N} \sum_{n=1}^{N} e^{2\pi i h x_n} \right|^2$$

for $h = 1, 2, \dots$. *Then*

$$\lim_{H\to\infty} \frac{1}{H} \sum_{h=1}^{H} \omega_h = 0 \Longrightarrow \Lambda(x_n) = \emptyset.$$

Proof. It suffices to prove

$$\lim_{H\to\infty} \frac{1}{H} \sum_{h=1}^{H} \omega_h = 0 \Longrightarrow \forall (g(x) \in G(x_n \bmod 1)) g(x) \text{ is continuous}$$

since (by Theorem 2.1) the right-hand side implies $\Lambda(x_n \bmod 1) = \emptyset$ which (by Proposition 2.2 (i)) gives $\Lambda(x_n) = \emptyset$. For a singleton $G(x_n \bmod 1) = \{g(x)\}$ the Wiener-Schoenberg theorem states

$$\lim_{H\to\infty} \frac{1}{H} \sum_{h=1}^{H} \omega_h = 0 \Longleftrightarrow g(x) \text{ is continuous.}$$

Next, we adapt the proof which appeared in [KN74, pp. 55-56].

For $g(x) \in G(x_n \bmod 1)$ we select N_k, $k = 1, 2, \dots$, such that

$$\lim_{k\to\infty} \frac{1}{N_k} \sum_{n=1}^{N_k} c_{[\alpha,x)}(x_n \bmod 1) = g(x)$$

for every point $x \in [0,1]$ of continuity of $g(x)$. Applying Helly-Bray lemma we compute $\omega_h(g)$ as

$$\omega_h(g) = \lim_{k\to\infty}\left|\frac{1}{N_k}\sum_{n=1}^{N_k} e^{2\pi i h x_n}\right|^2 = \lim_{k\to\infty}\frac{1}{N_k^2}\sum_{m,n=1}^{N_k} e^{2\pi i h(x_m-x_n)}$$
$$= \int_0^1\int_0^1 e^{2\pi i h(x-y)}\mathrm{d}g(x)\mathrm{d}g(y).$$

Next, for every $(x,y)\in[0,1]^2$, the sequence $\frac{1}{H}\sum_{h=1}^{H} e^{2\pi i h(x-y)}$, $H = 1,2,\dots$, converges and

$$\lim_{H\to\infty}\frac{1}{H}\sum_{h=1}^{H} e^{2\pi i h(x-y)} = \begin{cases} 1 & \text{if } x-y\in\mathbb{Z},\\ 0 & \text{otherwise.}\end{cases}$$

Thus by Lebesgue theorem of dominated convergence we have

$$\lim_{H\to\infty}\frac{1}{H}\sum_{h=1}^{H}\omega_h(g) = \int_0^1\int_0^1\left(\lim_{H\to\infty}\frac{1}{H}\sum_{h=1}^{H} e^{2\pi i h(x-y)}\right)\mathrm{d}g(x)\mathrm{d}g(y)$$
$$= \iint_X 1\cdot\mathrm{d}g(x)\mathrm{d}g(y), \tag{1}$$

where

$$X = \{(x,y)\in[0,1]^2; x-y\in\mathbb{Z}\}.$$

Since X is a null set, then for continuous $g(x)$, the integral (1) equals zero. If $g(x)$ has at $x_0\in[\alpha,\beta]$ a jump, then (1) has a lower bound $(g(x_0+0)-g(x_0-0))^2$. In all cases we have $\omega_h \geq \omega_h(g)$ and thus the proof is finished. □

Note that, for every $h = 1,2,\dots$, there exists $g_h(x)\in G(x_n)$ such that $\omega_h = \omega_h(g_h)$, but the $\limsup_{H\to\infty}\frac{1}{H}\sum_{h=1}^{H}\omega_h(g_h) > 0$ need not imply

$$\lim_{H\to\infty}\frac{1}{H}\sum_{h=1}^{H}\omega_h(g) > 0$$

for some $g(x)\in G(x_n)$.

Theorem 2.4. *Let x_n be a sequence of real numbers. If for every $k = 1,2,\dots$ the difference sequence $x_{n+k}-x_n$, $n = 1,2,\dots$, has $\Lambda(x_{n+k}-x_n) = \emptyset$, then $\Lambda(x_n) = \emptyset$.*

Proof. Van der Corput difference theorem (cf. [KN74, Th. 3.1, p. 26]) in the form [St97, Th. 8] shows that if $G((x_{n+k}-x_n) \bmod 1)$, $k = 1,2,\dots$, contains only continuous distribution functions, then the same holds for $G(x_n \bmod 1)$. Thus we have the implication $\Lambda((x_{n+k}-x_n) \bmod 1) = \emptyset, k = 1,2,\dots \Longrightarrow \Lambda(x_n \bmod 1) = \emptyset$.

For the general case the referee suggested the following simple proof: Assume that $x_n(n\in A)\to\alpha$ and $\overline{d}(A) > 0$. Then there exists k such that $\overline{d}(A\cap A+k) > 0$ and moreover $x_{n+k}-x_n(n+k\in A\cap A+k)\to 0$. □

In the proof of Theorem 1.2 we have constructed, for an arbitrary F_σ-set X, a sequence x_n such that $\Lambda(x_n) = X$, but this x_n is not one-to-one. The implication $\lim_{n\to\infty} x_n - y_n = 0 \Longrightarrow \Lambda(x_n) = \Lambda(y_n)$ gives change x_n to a one-to-one y_n. Inspired by [KN74, Ex. 2.11, .23] we can obtain the following generalization.

Theorem 2.5. *For any two sequences x_n and y_n we have*

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}|x_n-y_n|=0\Longrightarrow\Lambda(x_n)=\Lambda(y_n).$$

Proof. For $x_n,y_n\in[0,1]$ [St97, Th. 7] gives $\lim_{N\to\infty}N^{-1}\sum_{n=1}^{N}|x_n-y_n|=0\Longrightarrow G(x_n)=G(y_n)$ and thus $\Lambda(x_n)=\Lambda(y_n)$ applying Theorem 2.1.

For general x_n, y_n the referee suggested: The limit $N^{-1}\sum_{n=1}^{N}|x_n-y_n|\to 0$ implies $\overline{d}(\{n\in\mathbb{N};|x_n-y_n|\geq\varepsilon\})=0$ for every $\varepsilon>0$. Thus there exists $B\subset\mathbb{N}$ such that $d(B)=1$ and $|x_n-y_n|(n\in B)\to 0$. Now, let $x_n(n\in A)\to\alpha$ and $\overline{d}(A)>0$. Then $\overline{d}(A\cap B)=\overline{d}(A)>0$ and $y_n(n\in A\cap B)\to\alpha$. □

3. Computations of $\Lambda(x_n)$

Example 3.1. If $G(x_n \bmod 1)=\{g(x)\}$ and $g(x)=x$, then the sequence x_n is called uniformly distributed mod 1. Applying Theorem 2.1 and the continuity of $g(x)=x$ we find (as in [Fr93, Ex. 4]) that

$$\Lambda(x_n \bmod 1)=\emptyset$$

for every uniformly distributed sequence x_n mod 1. E.g. Λ-set is empty for

$n\theta$ mod 1 with irrational θ;
$n^2\theta+\sin(2\pi\sqrt{n})$ mod 1 with irrational θ (cf. [KN74, p. 31]);
$\log F_n$ mod 1 with Fibonacci numbers $F_{n+1}=F_n+F_{n-1}$, $F_1=F_2=1$ (cf. [KN74, p. 31]);
$n\log\log\dots\log n$ mod 1 (cf. [KN74, p. 24]), etc.

Example 3.2. It is well known that (cf. [KN74, p. 58])

$$G(\log n \bmod 1)=\left\{e^{-\alpha}\frac{e^x-1}{e-1}+e^{-\alpha}\big(e^{\min(x,\alpha)}-1\big);\alpha\in[0,1]\right\}.$$

Since all distribution functions in $G(\log n \bmod 1)$ are continuous, we have

$$\Lambda(\log n \bmod 1)=\emptyset.$$

Another proof follows from Theorem 2.3, because, for $x_n=\log n$ mod 1, we have

$$\omega_h=\frac{1}{4\pi^2h^2+1}.$$

More generally, for $x_n=t\log n$ mod 1, $t\neq 0$, we have

$$\omega_h=\frac{1}{4\pi^2h^2t^2+1}$$

which implies $\lim_{H\to\infty}\frac{1}{H}\sum_{h=1}^{H}\omega_h=0$ and thus $\Lambda(t\log n \bmod 1)=\emptyset$ for any $t\neq 0$. For computing ω_h we have used a method described in [Pa84, Solution 5.18, p. 281].

Example 3.3. It is proved in [St95] that starting with $\log\log n$ mod 1 all the sequences $\log\log\dots\log n$ mod 1 have

$$G(\log\log\dots\log n \bmod 1)=\{c_\alpha(x);\alpha\in[0,1]\}\cup\{h_\alpha(x);\alpha\in[0,1]\}.$$

Here $c_\alpha:[0,1]\to[0,1]$ is a one-jump distribution function

$$c_\alpha(x)=\begin{cases}0, & \text{if } x\in[0,\alpha),\\ 1, & \text{if } x\in(\alpha,1];\end{cases}$$

$c_\alpha(0)=0$, $c_\alpha(1)=1$, and $c_\alpha(\alpha)=0$ if $0<\alpha<1$; $h_\alpha:[0,1]\to[0,1]$ is constant distribution function, where $h_\alpha(0)=0$, $h_\alpha(1)=1$, and $h_\alpha(x)=\alpha$ if $x\in(0,1)$.

Applying Theorem 2.1 we have

$$\Lambda(\log\log\dots\log n \bmod 1)=[0,1].$$

Example 3.4. Let $\alpha=\frac{p}{q}\pi$, where p and q are positive integers and g.c.d. $(p,q)=1$. It is proved in [BBK95] that the sequence

$$x_n=n\cos(n\cos n\alpha) \bmod 1,\quad n=1,2,\dots,$$

has $G(x_n)=\{g(x)\}$, where

$$g(x)=\begin{cases} x & \text{if } q \text{ is odd},\\ \left(1-\frac{1}{q}\right)x+\frac{1}{q}c_0(x) & \text{if } q \text{ is even},\end{cases}$$

and $c_0(x)$ is introduced in Example 3.3 for $\alpha=0$. Theorem 2.1 implies

$$\Lambda(x_n)=\begin{cases}\emptyset & \text{if } q \text{ is odd},\\ \{0\} & \text{if } q \text{ is even}.\end{cases}$$

4. Concluding remarks

In [Fr93], Fridy has also introduced the concept of *statistical cluster points* of a sequence x_n and studied the set $\Gamma(x_n)$ of all such points. A number α is called the statistical cluster point of the sequence x_n, $n=1,2,\dots$, provided that for every $\varepsilon>0$, $\overline{d}(\{n\in\mathbb{N};|x_n-\alpha|<\varepsilon\})>0$. Fridy proved that $\Gamma(x_n)$ is a closed point set. Evidently, $\alpha\notin\Gamma(x_n)$ if and only if there exist an open interval I for which $\alpha\in I$ and $g(x)=$ constant for every $x\in I$ and any $g(x)\in G(x_n)$. Now, let X be a given nonempty closed subset of $[0,1]$ with component intervals $\bigcup_{k\in K}I_k=[0,1]\setminus X$. Denote $g(x)=\sum_{I_k\subset[0,x)}\lambda(I_k)+\lambda(X\cap[0,x))$, where λ is the Lebesgue measure. We can see that $g(x)$ is a distribution function with constant value on every interval I_k and it increases in any neighbourhood of $x\in X$. For such $g(x)$ (cf. [KN74, Th. 4.3, p. 138] there exists a sequence $x_n\in[0,1]$ such that $g(x)$ is an asymptotic distribution function of x_n. Thus $\Gamma(x_n)=X$.

G. Myerson [My93] calls a sequence $x_n\in[0,1]$ maldistributed if for any nondegenerate interval $I\subset[0,1]$, $\overline{d}(\{n\in\mathbb{N};x_n\in I\})=1$. In [St95] the maldistribution is characterized by $G(x_n)\supset\{c_\alpha(x);\alpha\in[0,1]\}$. Thus, by Theorem 2.1, $\Lambda(x_n)=[0,1]$ and from the definition of cluster points also, $\Gamma(x_n)=[0,1]$, for any maldistributed x_n. As an example of maldistributed sequences see Example 3.3.

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