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ON ALGEBRAIC POLYNOMIALS WITH RANDOM COEFFICIENTS

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ABSTRACT. The expected number of real zeros and maxima of the curve representing algebraic polynomial of the form $a_0\binom{n-1}{0}^{1/2} + a_1\binom{n-1}{1}^{1/2}x + a_2\binom{n-1}{2}^{1/2}x^2 + \cdots + a_{n-1}\binom{n-1}{n-1}^{1/2}x^{n-1}$ where $a_j, j = 0, 1, 2, \ldots, n-1$, are independent standard normal random variables, are known. In this paper we provide the asymptotic value for the expected number of maxima which occur below a given level. We also show that most of the zero crossings of the curve representing the polynomial are perpendicular to the x axis. The results show a significant difference in mathematical behaviour between our polynomial and the random algebraic polynomial of the form $a_0 + a_1x + a_2x^2 + \cdots + a_{n-1}x^{n-1}$ which was previously the most studied.

1. Introduction

The random algebraic polynomial is commonly defined as

$$Q(x) \equiv Q_n(x, \omega) = \sum_{j=0}^{n-1} a_j(\omega)x^j,$$

where $(\Omega, \mathcal{A}, \Pr)$ is a fixed probability space, and $\{a_j(\omega)\}_{j=0}^{n-1}$ is a sequence of independent random variables defined on Ω. The previous works on $Q(x)$ mainly assume identical distribution for the coefficients a_j's. They include the pioneer works of Littlewood and Offord [7] and [8] and recent works of Wilkins [9] and Farahmand [5]. It is known that for identical standard normally distributed coefficients and n sufficiently large the expected number of real zeros of $Q(x)$ is asymptotic to $(2/\pi)\log n$. However, there is little known about random polynomials with non-identical coefficients. Motivated by their close relation with physics, reported by Edelman and Kostland [1], as well as their mathematical interest, we assume that the coefficients a_j, $j = 0, 1, 2, \cdots, n-1$, have means zero and non-identical variances $\binom{n-1}{j}$. This is the same as considering polynomials of the form

$$P(x) \equiv P_n(x, \omega) = \sum_{j=0}^{n-1} a_j(\omega)\binom{n-1}{j}^{1/2} x^j$$

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with the same assumption of identically normal standard distribution for a_j's as above. Let $N(a,b)$ be the number of real zeros and $M_u(a,b)$ the number of maxima of $P(x)$ which occurs below a level u, in the interval (a,b). In [6] it is shown that for n sufficiently large $EN(-\infty,\infty) \sim EM_\infty(-\infty,\infty) \sim \sqrt{n}$. This is interesting as it shows that the curve representing $P(x)$ has a significantly larger expected number of real zero crossings and therefore oscillates more frequently than $Q(x)$. Also, unlike $Q(x)$, since $EN(-\infty,\infty) \sim EM_u(-\infty,\infty)$ asymptotically all the oscillations of $P(x)$ are between two zero crossings. Therefore, in order to obtain a better understanding of the mathematical behaviour of $P(x)$, it is of special interest to study the number of local maxima which occurs below a given level as well as the slope or the nature of crossings. To this end, as in [4] and [5], we define $S_u(a,b)$ as the number of up-crossings of $P(x)$ which possess a slope greater than u or down-crossings with a slope less than $-u$. We define these crossings as u-sharp. Theorem 1 shows that there is no significant number of maxima which occurs below the x-axis while Theorem 2 gives a high level below which asymptotically all the maxima occur. By letting $u \to \infty$ as $n \to \infty$, Theorem 3 shows that asymptotically all the crossings are sharp. We prove the following theorems:

Theorem 1. *For all sufficiently large n, the expected number of maxima which occurs below the x-axis is*

$$EM_0(-\infty,\infty) \sim O(1).$$

Theorem 2. *For $f_n \equiv f$ as any function of n such that $f \to \infty$ as $n \to \infty$ and u such that $u/f^{2n}\sqrt{n} \to \infty$ as $n \to \infty$*

$$EM_u(-\infty,\infty) \sim \frac{\sqrt{n}}{2}.$$

Theorem 3. *For any u such that $u/\sqrt{n} \to 0$ as $n \to \infty$*

$$ES_u(-\infty,\infty) \sim \frac{\sqrt{n}}{2}.$$

2. Moments

In order to be able to obtain estimates for the expected number of maxima below a level and the expected number of sharp crossings we need the following identities ((2.1) is well known and (2.2)-(2.6) are easily derived by consecutive differentiation of (2.1)):

$$A^2 = \operatorname{var}\{P(x)\} = \sum_{j=0}^{n-1} \binom{n-1}{j} x^{2j} = (x^2+1)^{n-1}, \tag{2.1}$$

$$B^2 = \operatorname{var}\{P'(x)\} = \sum_{j=0}^{n-1} j^2 \binom{n-1}{j} x^{2j-2}$$

$$= (n-1)(x^2+1)^{n-3}(nx^2 - x^2 + 1), \tag{2.2}$$

$$\begin{aligned}
C^2 &= \operatorname{var}\{P''(x)\} = \sum_{j=0}^{n-1} j^2(j-1)^2 \binom{n-1}{j} x^{2j-4} \\
(2.3) \qquad &= (n-1)(n-2)(x^2+1)^{n-5}\{(n-1)(n-2)x^4 + 4(n-2)x^2 + 2\}, \\
(2.4)\ D &= \operatorname{cov}\{P(x), P'(x)\} = \sum_{j=0}^{n-1} j \binom{n-1}{j} x^{2j-1} = (n-1)x(x^2+1)^{n-2}, \\
E &= \operatorname{cov}\{P(x), P''(x)\} = \sum_{j=0}^{n-1} j(j-1) \binom{n-1}{j} x^{2j-2} \\
(2.5) \qquad &= (n-1)(n-2)x^2(x^2+1)^{n-3},
\end{aligned}$$

and

$$\begin{aligned}
F &= \operatorname{cov}\{P'(x), P''(x)\} = \sum_{j=0}^{n-1} j^2(j-1) \binom{n-1}{j} x^{2j-3} \\
(2.6) \qquad &= (n-1)(n-2)x(x^2+1)^{n-4}\{(n-1)x^2 + 2\}.
\end{aligned}$$

Then as is seen in [3] and [2]

$$EM_u(a,b) = \int_a^b \int_{-\infty}^u \int_{-\infty}^0 |z| p_x(t,0,z)\ dz\ dt\ dx,$$

where $p_x(t,y,z)$ denotes the three-dimensional density function for $P(x)$, $P'(x)$ and $P''(x)$. Since from (2.1)-(2.6) the determinant of the covariance matrix of the above density function is

$$\begin{aligned}
|\Sigma| &= A^2B^2C^2 - A^2F^2 - B^2E^2 - C^2D^2 + 2DEF \\
(2.7) \qquad &= 2(n-1)^2(n-2)(x^2+1)^{3n-9}
\end{aligned}$$

from [3] and [2] we have

(2.8)

$$EM_u(a,b) = \frac{1}{\pi} \int_a^b \frac{d}{2e\sqrt{2a}} \left\{ \Phi\left(u\sqrt{2a}\right) - \frac{b}{2\sqrt{ac}} \Phi\left(\frac{ub}{\sqrt{2c}}\right) \exp\left(-\frac{aeu^2}{c}\right) \right\} du$$

where from (2.1)-(2.7)

$$\begin{aligned}
(2.9)\ a &= \frac{B^2C^2 - F^2}{2|\Sigma|} = \frac{2nx^2 - 2x^2 + n^2x^4 - 3nx^4 + 2x^4 + 2}{4(x^2+1)^{n-1}}, \\
(2.10)\ b &= \frac{DF - B^2E}{|\Sigma|} = \frac{x^2}{2(x^2+1)^{n-3}}, \\
(2.11)\ c &= \frac{A^2B^2 - D^2}{2|\Sigma|} = \frac{1}{4(n-1)(n-2)(x^2+1)^{n-5}}, \\
e &= c - \frac{b^2}{4a} \\
(2.12) \qquad &= \frac{nx^2 - x^2 + 1}{2(n-1)(n-2)(x^2+1)^{n-5}(2nx^2 - 2x^2 + n^2x^4 - 3nx^4 + 2x^4 + 2)},
\end{aligned}$$

and

$$(2.13)\ d = \frac{1}{\sqrt{|\Sigma|}}.$$

In the next section we will use (2.8) together with (2.9)-(2.13) to find the asymptotic estimate for $EM_u(-\infty,\infty)$. As far as the expected number of sharp crossings is concerned, from [4] we have

$$ES_u(a,b) = \frac{1}{\pi}\int_a^b \frac{\Delta}{A^2}\exp\left(-\frac{u^2A^2}{2\Delta^2}\right)\,dx \tag{2.14}$$

where from (2.1)-(2.3)

$$\Delta^2 = A^2B^2 - D^2 = (n-1)(x^2+1)^{2n-4}. \tag{2.15}$$

We give the asymptotic value for $ES_u(-\infty,\infty)$ in the final part of the paper. Without loss of generality we only consider the interval $(0,\infty)$. However, since in all above identities the power of x's are even we obtain our results for the entire real line.

3. Maxima below a level

(i). Level zero

Here we obtain the expected number of maxima below the x-axis. From (2.8) we can easily show

$$\begin{aligned} EM_0(0,\infty) &= \frac{\sqrt{n-2}}{4\pi} \\ &\times\int_0^\infty \frac{\sqrt{2nx^2-2x^2+n^2x^4-3nx^4+2x^4+2}-\sqrt{(n-1)(n-2)}x^2}{(x^2+1)(nx^2-x^2+1)}dx. \end{aligned}\tag{3.1}$$

First we assume $x>\epsilon$ where $\epsilon = g_n/\sqrt{n}$ and g_n is any function of n such that $g_n\to\infty$ as $n\to\infty$. Then since with the above assumption $nx^2\to\infty$ as $n\to\infty$ the first term that appears in the integrand of (3.1) is

$$\begin{aligned}\sqrt{2nx^2-2x^2+n^2x^4-3nx^4+2x^4+2} &= \sqrt{(n-1)(n-2)x^4+2(nx^2-x^2+1)} \\ &= \sqrt{(n-1)(n-2)}x^2+Y\end{aligned}$$

where Y is a function of x and n satisfies the relation $2(nx^2-x^2+1) = Y^2 + 2Y\sqrt{(n-1)(n-2)}x^2$ and therefore tends to unity as $n\to\infty$. Hence from (3.1) letting $x=\tan\theta$ and since $\cot\arctan\epsilon = 1/\epsilon$ we obtain

$$\begin{aligned} EM_0(\epsilon,\infty) &\sim \frac{\sqrt{n-2}}{4\pi}\int_\epsilon^\infty\frac{dx}{(x^2+1)(nx^2-x^2+1)} \\ &\sim \frac{1}{4\pi\sqrt{n}}\int_\epsilon^\infty\frac{dx}{x^2(x^2+1)} \\ &= \frac{1}{4\pi\sqrt{n}}\int_{\arctan\epsilon}^{\pi/2}\cot^2\theta\,d\theta \\ &\sim \frac{1}{4\pi\sqrt{n}\epsilon}\sim\frac{1}{4\pi g_n}. \end{aligned}\tag{3.2}$$

Now we find the expected number of maxima in the interval $(0,\epsilon)$. To this end for all sufficiently large n we have

$$\begin{aligned} EM_0(0,\epsilon) &< \frac{\sqrt{n-2}}{4\pi}\int_0^\epsilon\frac{\sqrt{nx^4+2nx^2+2}-x^2\sqrt{(n-1)(n-2)}}{(x^2+1)(nx^2-x^2+1)}\,dx \\ &= \frac{\sqrt{n-2}}{4\pi}\int_0^\epsilon\frac{\sqrt{(nx^2+1)^2+1}-x^2\sqrt{(n-3/2)^2-1/4}}{(x^2+1)(nx^2-x^2+1)}\,dx. \end{aligned}\tag{3.3}$$

Now we can show that both terms appearing in the numerator of the integrand in (3.3) can be written as

$$\sqrt{(nx^2+1)^2+1} = (nx^2+1)+X_1$$

and

$$\sqrt{\left(n-\frac{3}{2}\right)^2-\frac{1}{4}} = \left(n-\frac{3}{2}\right)+X_2$$

where X_1 and X_2 satisfy relations

$$X_1^2+2X_1(nx^2+1)-1=0 \quad \text{and} \quad X_2^2+2X_2(n-3/2)+1/4=0.$$

Therefore $X_1 < 1$ and $X_2 > -1/n$ are functions of n and x. Hence from (3.3) letting $(\sqrt{n-1})x = \tan\theta$ we obtain

$$\begin{aligned} EM_0(0,\epsilon) &< \frac{\sqrt{n-2}}{4\pi}\int_0^{\epsilon}\frac{2\,dx}{(nx^2-x^2+1)(x^2+1)} \\ &< \frac{\sqrt{n-2}}{2\pi}\int_0^{\epsilon}\frac{dx}{(n-1)x^2+1} \\ &\sim \frac{1}{2\pi}\int_0^{\arctan\epsilon\sqrt{n-1}} d\theta \\ &= O(1). \end{aligned}$$

This together with (3.2) completes the proof of Theorem 1.

(ii). High level

Since the second integrand that appears in (2.8) is positive we can obtain an upper limit for $EM_u(0,\infty)$ as

$$\begin{aligned} EM_u(0,\infty) &< \int_0^{\infty}\frac{\sqrt{(n-2)(2nx^2-2x^2+n^2x^4-3nx^4+2x^4+2)}}{2\pi(1+x^2)(nx^2-x^2+1)} \\ &\quad \times\Phi\left\{\frac{unx^2}{\sqrt{n}(x^2+1)^{(n+1)/2}}\right\}\,dx. \end{aligned} \tag{3.4}$$

Now in order to evaluate the above integral we divide the positive line into two subintervals $(0,\epsilon)$ and (ϵ,∞) where $\epsilon=\sqrt{f_n/n}$ and f_n is any function of n smaller than $o(\sqrt{n})$ such that $f_n\to\infty$ as $n\to\infty$. Then since in (ϵ,∞)

$$\frac{nx^2}{(1+x^2)^{n-1}} \sim nf^{-2n},$$

by the assumptions of Theorem 2 the term inside Φ function tends to infinity as $n\to\infty$. Hence from (3.4) we have

$$\begin{aligned} EM_u(\epsilon,\infty) &< \frac{\sqrt{n-2}}{2\pi}\int_{\epsilon}^{\infty}\frac{\sqrt{2nx^2-2x^2+n^2x^4-3nx^4+2x^4+2}}{(1+x^2)(nx^2-x^2+1)}\,dx \\ &\sim \frac{\sqrt{n-2}}{2\pi}\int_{\epsilon}^{\infty}\frac{dx}{1+x^2} \\ &\sim \frac{\sqrt{n-2}}{4}. \end{aligned} \tag{3.5}$$

Also from (2.8) we obtain

$$
\begin{aligned}
EM_u(0,\epsilon) &< \frac{\sqrt{n-2}}{2\pi}\int_0^\epsilon \frac{\sqrt{(nx^2-x^2+1)^2+x^4+1}}{(1+x^2)(nx^2-x^2+1)}\,dx \\
&= \int_0^\epsilon \frac{\sqrt{n-2}}{2\pi(1+x^2)}\sqrt{1+\frac{1+x^4}{nx^2-x^2+1}}\,dx \\
&< \frac{\sqrt{n-2}}{\pi}\int_0^\epsilon \frac{dx}{1+x^2} = \sqrt{n-2}\,\epsilon = o(f_n). \qquad (3.6)
\end{aligned}
$$

Therefore from (3.5) and (3.6) we have $\sqrt{n}/4$ as an upper limit for $EM_u(0,\infty)$. In order to obtain a lower limit again from (2.8) and for our assumption of u, for all sufficiently large n we can say

$$
\begin{aligned}
EM_u(0,\infty) &> EM_u(\epsilon,\infty) \\
&\sim \int_\epsilon^\infty \frac{\sqrt{(n-2)(2nx^2-2x^2+n^2x^4-3nx^4+2x^4+2)}}{(1+x^2)(nx^2-x^2+1)}\,dx \\
&\sim \frac{\sqrt{n-2}}{2\pi}\int_\epsilon^\infty \frac{dx}{1+x^2} \\
&\sim \frac{\sqrt{n-2}}{4}. \qquad (3.7)
\end{aligned}
$$

Hence from (3.7) and the above upper limit we have the proof of Theorem 2.

4. Sharp crossings

Since from (2.1), (2.2), (2.4) and (2.5) $\Delta^2/A^2 = (n-1)(x^2+1)^{n-3}$ and $\Delta/A^2 = \sqrt{n-1}/(x^2+1)$, then from (2.14) we have

$$
\begin{aligned}
ES_u(0,\infty) &= \frac{1}{\pi}\int_0^\infty \frac{\Delta}{A^2}\exp\left(-\frac{u^2A^2}{2\Delta^2}\right)\,dx \\
&= \frac{\sqrt{n-1}}{\pi}\int_0^\infty \frac{\exp\{-u^2/(n-1)(x^2+1)^{n-3}\}}{(x^2+1)}\,dx \\
&= \frac{\sqrt{n-1}}{\pi}\int_0^{\pi/2}\exp\left(-\frac{u^2\cos^{2n-6}\theta}{n-1}\right)\,d\theta \qquad (4.1)
\end{aligned}
$$

where $x=\tan\theta$. Now we divide the interval $(0,\pi/2)$ into two subintervals $(0,\epsilon)$ and $(\epsilon,\pi/2)$ where $\epsilon = 1/\sqrt{n}$. Then we can evalute the integral which appears in (4.1) as

$$
\begin{aligned}
&\int_0^{\pi/2}\exp\left(-\frac{u^2\cos^{2n-6}\theta}{n-1}\right)\,d\theta \\
&> \int_0^\epsilon \exp\left(-\frac{u^2}{n-1}\right)\,d\theta + \int_\epsilon^{\pi/2}\exp\left(-\frac{u^2\cos^{2n-6}\epsilon}{n-1}\right)\,d\theta \\
&\sim \epsilon\exp\left(-\frac{u^2}{n-1}\right) + \left(\frac{\pi}{2}-\epsilon\right)\exp\left\{-\frac{u^2}{n-1}(1-\epsilon^2)^{n-3}\right\} \\
&\sim \epsilon\exp\left(-\frac{u^2}{n-1}\right) + \frac{\pi}{2}\exp\left\{-\frac{u^2}{n-1}e^{-(n-3)/n}\right\} \\
&\sim \frac{\pi}{2}.
\end{aligned}
$$

Hence from (4.1) we obtain $(\sqrt{n-1})/2$ as a lower limit for $ES_u(0,\infty)$. Therefore, since $ES_u(0,\infty) < ES_0(0,\infty) \sim (\sqrt{n-1})/2$, from [6] we have the proof of Theorem 3.

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