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# SHAPE ASPHERICAL COMPACTA–APPLICATIONS OF A THEOREM OF KAN AND THURSTON TO COHOMOLOGICAL DIMENSION AND SHAPE THEORIES

TAKAHISA MIYATA

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Abstract. Dydak and Yokoi introduced the notion of shape aspherical compactum. In this paper, we use this notion to obtain a generalization of Kan and Thurston theorem for compacta and pro-homology. As an application, we obtain a characterization of cohomological dimension with coefficients in $\mathbb{Z}$ and $\mathbb{Z}/p$ ($p$ prime) in terms of acyclic maps from a shape aspherical compactum, which improves the theorems of Edwards and Dranishnikov. Furthermore, we obtain the shape version of the theorem and as a consequence we show that every compactum has the stable shape type of a shape aspherical compactum.

## 1. Introduction

First recall

**Theorem 1.1** (Kan and Thurston [KT]). *For each path-connected space $X$, there exist a space $TX$ and a map $t : TX \to X$, natural for maps on $X$, with the following properties:*

- **(KT1):** $t_* : H_*(TX; t^*A) \to H_*(X; A)$ *and* $t^* : H^*(X; A) \to H^*(TX; t^*A)$ *are isomorphisms of singular homologies and cohomologies with local coefficients; and*
- **(KT2):** $t_* : \pi_1(TX) \to \pi_1(X)$ *is onto, and* $\pi_i(TX) \cong 0$ *for* $i \neq 1$.

Maunder gave a simpler proof to the theorem and obtained the following variation:

**Theorem 1.2** (Maunder [Ma]). *For each finite connected simplicial complex $K$, there exist a finite simplicial complex $TK$ of the same dimension, and a map $t_K : TK \to K$, natural for simplicial maps on $K$, with properties* (KT1) *and* (KT2).

Throughout the paper, a compactum means a compact metric space, and a continuum means a connected compactum.

The paper consists of three parts. In the first part, we generalize those results as follows:

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**Theorem A.** *For each continuum $X$ (resp., continuum with $\dim X < \infty$), there exist an approximately aspherical compactum $Y$ (resp., approximately aspherical compactum $Y$ with $\dim Y = \dim X$) and a surjective map $\varphi : Y \to X$ with the following properties:*

- **(S1):** *$\varphi$ induces isomorphisms of Čech homologies and cohomologies;*
- **(S2):** *$\varphi_* : \text{pro-}\pi_1^S(Y) \to \text{pro-}\pi_1^S(X)$ is an epimorphism; and*
- **(S3):** *For each connected closed subset $A$ of $X$, $\varphi^{-1}(A)$ is an aproximately aspherical compactum, and $\varphi|\varphi^{-1}(A) : \varphi^{-1}(A) \to A$ satisfies properties* (S1) *and* (S2).

Here, for each compactum $X$, $\dim X$ denotes the covering dimension of $X$. A compactum $X$ is said to be *approximately aspherical* if every map of $X$ into a polyhedron factors up to homotopy through a finite aspherical CW complex. Note that our definition is slightly stronger than the original definition of shape asphericity of Dydak and Yokoi [DY] by requiring the finiteness of the factoring CW complex. Asphericity of compacta in the study of cell-like maps was first considered by Daverman [Da] and continued by Daverman and Dranishnikov [DD].

As an application of Theorem A, in the second part of the paper we obtain a characterization of cohomological dimension with coefficients in $\mathbb{Z}$ and $\mathbb{Z}/p$ for any prime $p$, which improves the well-known characterizations by Edwards and Dranishnikov in the theorems below.

For each compactum $X$ and abelian group $G$, the *cohomological dimension* $\operatorname{cdim}_G X \leq n$ if $X \tau K(G,n)$, where for any ANR $P$, $X \tau P$ denotes the property that every map of any closed subset of $X$ into $P$ extends over $X$.

**Theorem 1.3** (Edwards [E, W])**.** *For each compactum $X$, $\operatorname{cdim}_{\mathbb{Z}} X \leq n$ if and only if there exists a cell-like map $f : Y \to X$ from a compactum $Y$ of $\dim Y \leq n$.*

**Theorem 1.4** (Dranishnikov [Dr])**.** *For each compactum $X$ and for each prime $p$, $\operatorname{cdim}_{\mathbb{Z}/p} X \leq n$ if and only if there exists a surjective map $f : Y \to X$ from a compactum $Y$ of $\dim Y \leq n$ such that each fibre is acyclic modulo $p$.*

Koyama [K] and Koyama and Yokoi [KY] extended those results to approximable dimensions with arbitrary coefficient groups. Note the approximable dimension with a finitely generated coefficient group coincides with the cohomological dimension.

We obtain the following:

**Theorem B.** *For each continuum $X$ and for each prime $p$, $\operatorname{cdim}_{\mathbb{Z}} X \leq n$ (resp., $\operatorname{cdim}_{\mathbb{Z}/p} X \leq n$) if and only if there exist an approximately aspherical compactum $Y$ with $\dim Y \leq n$ and a surjective map $f : Y \to X$ such that each fibre is acyclic (resp., acyclic modulo $p$).*

In the third and final part of the paper we give applications to shape theory. Let $\operatorname{sd} X$ denote the shape dimension of $X$ (see [MaS, p. 95]).

**Theorem C.** *For each continuum $X$ of $\operatorname{sd} X < \infty$, there exist an approximately aspherical compactum $Y$ of $\dim Y = \operatorname{sd} X$ and a shape morphism $\varphi : Y \to X$ with properties* (S1) *and* (S2).

**Theorem D.** 1. *Every continuum has the weak stable shape type of an approximately aspherical compactum.*

2. *Every continuum $X$ of* $\operatorname{sd} X < \infty$ *has the stable shape type of an approximately aspherical compactum $Y$ of* $\dim Y = \operatorname{sd} X$.

See [MiS1] for the definitions in stable shape theory.

## 2. Characterizations of approximately aspherical compacta

**Theorem 2.1.** *For every compactum $X$, the following are equivalent:*

i) *$X$ is an approximately aspherical compactum;*

ii) *$X$ admits an expansion of $X$, $\mathbf{p} = (p_i) : X \to \mathbf{X} = (X_i, p_{ii+1}, \mathbb{N})$, such that each $X_i$ is a finite aspherical polyhedron (here, the expansion is in the sense of* [MaS, p. 19]*); and*

iii) *Every polyhedral expansion of $X$, $\mathbf{p} = (p_i) : X \to \mathbf{X} = (X_i, p_{ii+1}, \mathbb{N})$ has the property that every $i$ admits $i' \geq i$ such that $p_{ii'}$ factors through a finite aspherical polyhedron.*

*Proof.* ii) $\Rightarrow$ i) is obvious. We wish to verify i) $\Rightarrow$ iii) $\Rightarrow$ ii). For i) $\Rightarrow$ iii), let $\mathbf{p} = (p_i)\colon X \to \mathbf{X} = (X_i, p_{ii+1}, \mathbb{N})$ be a polyhedral expansion of $X$. For each $i$, there exist a finite aspherical polyhedron $P$ and homotopy maps $g : X \to P$ and $h : P \to X_i$ such that $p_i = hg$. Then for some $i'' \geq i$ there exists a homotopy map $g' : X_{i''} \to P$ such that $g = g' p_{i''}$. So, $p_{ii''} p_{i''} = p_i = hg' g_{i''}$, and hence there exists $i' \geq i''$ such that $p_{ii'} = hg' p_{i''i'}$ as desired. For iii) $\Rightarrow$ ii), let $\mathbf{p} = (p_i)\colon X \to \mathbf{X} = (X_i, p_{ii+1}, \mathbb{N})$ be any polyhedral expansion of $X$. Then by iii), there is an increasing sequence $1 = i_1 \leq i_2 \leq \cdots \leq i_k \leq \cdots$ and finite aspherical polyhedra $Y_k$, $k = 1, 2, \ldots$, such that $p_{i_k i_{k+1}} = h_k g_k$ for some $g_k : X_{i_{k+1}} \to Y_k$ and $h_k : Y_k \to X_{i_k}$. For each $k = 1, 2, \ldots$, let $q_k = g_k p_{i_{k+1}} : X \to Y_k$ and $q_{kk+1} = g_k h_{k+1} : Y_{k+1} \to Y_k$. Then it is a routine to check that $\mathbf{q} = (q_k) : X \to \mathbf{Y} = (Y_k, q_{kk+1}, \mathbb{N})$ forms an expansion of $X$. □

*Remark* 2.2. Analogous characterization for Dydak and Yokoi's definition holds without the finiteness conditions on the aspherical polyhedra in ii) and iii).

## 3. Proof of Theorem A

Before we prove the theorem, we observe the following properties for the map $t_K : TK \to K$ in Maunder's Theorem, which were obtained in the original proof [Ma]:

- **(M1):** For each connected subcomplex $M$ of $K$, $\dim t_K^{-1}(M) = \dim M$, and $t_K|t_K^{-1}(M) : t_K^{-1}(M) \to M$ satisfies properties (KT1) and (KT2) and is natural in the following sense: For any simplicial map $f : K \to K'$, if $L$ and $L'$ are subcomplexes of $K$ and $K'$, respectively, such that $f(L) \subseteq L'$, then the following diagram commutes:

$$\begin{array}{ccc} L & \xrightarrow{f|L} & L' \\ {\scriptstyle t_K|t_K^{-1}(L)}\uparrow & & \uparrow{\scriptstyle t_{K'}|t_{K'}^{-1}(L')} \\ t_K^{-1}(L) & \xrightarrow{Tf|t_K^{-1}(L)} & t_{K'}^{-1}(L') \end{array}$$

  where $Tf : TK \to TK'$ is the induced simplicial map;
- **(M2):** Each fibre of $t_K$ is either a point or an acyclic and aspherical subcomplex of $TK$; and
- **(M3):** $t_K$ is onto.

Let $X$ be a continuum, and let $\mathfrak{U}_i$, $i = 1, 2, \dots$, be a sequence of finite open coverings of $X$ which form a base for the toplogy on $X$. For each $i$, let $K_i$ be the nerve of $\mathfrak{U}_i$ with realization $X_i$, let $p_{ii+1} : K_{i+1} \to K_i$ be a connecting simplicial map and let $p_i : X \to X_i$ be a canonical map. Then the map $\mathbf{p} = (p_i)\colon X \to \mathbf{X} = (X_i, p_{ii+1}, \mathbb{N})$ forms an inverse limit. By Theorem 1.2 and the above observation, for each $i$, there exist a complex $TK_i$ and a map $\varphi_i = t_{K_i} : TK_i \to K_i$ with properties (KT1), (KT2), (M1), (M2) and (M3) and a simplicial map $q_{ii+1} = Tp_{ii+1} : TK_{i+1} \to TK_i$ which makes the following diagram commute:

$$\begin{array}{ccc} K_i & \xleftarrow{p_{ii+1}} & K_{i+1} \\ \uparrow{\scriptstyle \varphi_i} & & \uparrow{\scriptstyle \varphi_{i+1}} \\ TK_i & \xleftarrow{Tp_{ii+1}} & TK_{i+1} \end{array}$$

For each $i$, let $Y_i$ be the realization of $TK_i$, and let $Y$ be the limit of the inverse sequence $\mathbf{Y} = (Y_i, q_{ii+1}, \mathbb{N})$ with the projections $q_i : Y \to Y_i$. Then the level morphism $\boldsymbol{\varphi} = (\varphi_i) : \mathbf{Y} \to \mathbf{X}$ induces the limit map $\varphi : Y \to X$, which is surjective. By properties (KT1) and (KT2) for each $\varphi_i : Y_i \to X_i$, $\varphi$ satisfies properties (S1) and (S2). To verify property (S3), let $A$ be a closed subset of $X$. For each $i$, let $L_i$ be the nerve of the open covering $\mathfrak{U}_i|A = \{U \cap A : U \in \mathfrak{U}_i\}$ of $A$. Then $L_i$ is a subcomplex of $K_i$. So, for each $i$, if $A_i$ is the realization of $L_i$, then $A_i$ is a subpolyhedron of $X_i$, and by property (M1) we have the following commutative diagram:

$$\begin{array}{ccc} A_i & \xleftarrow{p_{ii+1}|A_{i+1}} & A_{i+1} \\ \uparrow{\scriptstyle \varphi_i|\varphi_i^{-1}(A_i)} & & \uparrow{\scriptstyle \varphi_{i+1}|\varphi_{i+1}^{-1}(A_{i+1})} \\ \varphi_i^{-1}(A_i) & \xleftarrow{q_{ii+1}|\varphi_{i+1}^{-1}(A_{i+1})} & \varphi_{i+1}^{-1}(A_{i+1}) \end{array}$$

and each $\varphi_i|\varphi_i^{-1}(A_i) : \varphi_i^{-1}(A_i) \to A_i$ satisfies properties (KT1) and (KT2). Note that the restricted maps $\mathbf{p}|A = (p_i|A) : A \to \mathbf{A} = (A_i, p_{ii+1}|A_i, \mathbb{N})$ and $\mathbf{q}|\varphi^{-1}(A) = (\mathbf{q}_i|\varphi^{-1}(A)) : \varphi^{-1}(A) \to \varphi^{-1}(\mathbf{A}) = (\varphi^{-1}(A_i), \mathbf{q}_{ii+1}|\varphi_{i+1}^{-1}(A_{i+1}), \mathbb{N})$ form the inverse limits of $A$ and $\varphi^{-1}(A)$, respectively. So the map $\varphi|\varphi^{-1}(A) : \varphi^{-1}(A) \to A$ which is the limit map of the level morphism $\boldsymbol{\varphi}|\varphi^{-1}(\mathbf{A}) = (\varphi_i|\varphi_i^{-1}(A_i)) : \varphi^{-1}(\mathbf{A}) \to \mathbf{A}$ has properties (S1) and (S2). Since each $\varphi_i^{-1}(A_i)$ is aspherical, $\varphi^{-1}(A)$ is approximately aspherical by Proposition 2.1. Hence property (S3) is fulfilled.

Now suppose $\dim X = n < \infty$. Then we can take the base $\mathfrak{U}_i$, $i = 1, 2, \dots$, so that the nerves of $\mathfrak{U}_i$ have dimension at most $n$. So, for each $i$, $\dim Y_i = \dim TK_i = \dim K_i \le n$, and hence $\dim Y \le n$. On the other hand, the commutative diagram for each closed subset $A$ of $X$

$$\begin{array}{ccc} \check{H}^q(\varphi^{-1}(A); \mathbb{Z}) & \xleftarrow{(\varphi|\varphi^{-1}(A))^*} & \check{H}^q(A; \mathbb{Z}) \\ \uparrow{\scriptstyle j_A^*} & & \uparrow{\scriptstyle i_A^*} \\ \check{H}^q(Y; \mathbb{Z}) & \xleftarrow{\varphi^*} & \check{H}^q(X; \mathbb{Z}) \end{array}$$

and property (S3) imply $\operatorname{cdim}_{\mathbb{Z}} X \le \operatorname{cdim}_{\mathbb{Z}} Y$, and by Alexandroff theorem, $\dim X = \operatorname{cdim}_{\mathbb{Z}} X$ and $\operatorname{cdim}_{\mathbb{Z}} Y = \dim Y$. Hence $\dim Y = n$, as required.

## 4. Proof of Theorem B

Assume there is a surjective map $f : Y \to X$ from an approximately aspherical compactum $Y$ with $\dim Y \le n$ such that $\check{H}^*(f^{-1}(x);\mathbb{Z}) = 0$ for all $x \in X$. Using Vietoris-Begle theorem, we can obtain $\operatorname{cdim}_{\mathbb{Z}} X \le \operatorname{cdim}_{\mathbb{Z}} Y = \dim Y \le n$. Hence $\operatorname{cdim}_{\mathbb{Z}} X \le n$.

Conversely, suppose $\operatorname{cdim}_{\mathbb{Z}} X \le n$. Then Edwards theorem (Theorem 1.3) implies that there exists a cell-like map $g : X' \to X$ from a compactum $X'$ with $\dim X' \le n$. By taking each component of $X'$, without loss we can assume $X'$ is connected. Theorem A implies that there exists a surjective map $h : Y \to X'$ from a shape aspherical map $Y$ of $\dim Y = \dim X'$ such that for each closed subset $B$ of $X'$, the restricted map $h|h^{-1}(B) : h^{-1}(B) \to B$ induces an isomorphism $(h|h^{-1}(B))^* : \check{H}^q(B;\mathbb{Z}) \to \check{H}^q(h^{-1}(B);\mathbb{Z})$ for each $q$. So, if we let $f = gh : Y \to X$, then $\check{H}^q(f^{-1}(x);\mathbb{Z}) \cong \check{H}^q(g^{-1}(x);\mathbb{Z}) \cong 0$ for each $q$. The case for $\mathbb{Z}/p$ is proved similarly, using Dranishnikov theorem (Theorem 1.4).

## 5. Proofs of Theorems C and D

*Proof of Theorem* C. If $\operatorname{sd} X \le n < \infty$, then there is a polyhedral expansion $\mathbf{p} = (p_i)\colon X \to \mathbf{X} = (X_i, p_{ii+1}, \mathbb{N})$ of $X$ such that $X_i$ are finite polyhedra with $\dim X_i \le n$. Then choose a triangulation $K_1$ of $X_1$, and using the simplicial approximation theorem, we can inductively choose triangulations $K_i$ of $X_i$ and simplicial maps $a_{ii+1} : K_{i+1} \to K_i$ that represent the corresponding homotopy classes $p_{ii+1} : X_{i+1} \to X_i$. As in the proof of Theorem A, for each $i$, there exist a simplicial complex $TK_i$ with $\dim TK_i = \dim K_i$ and maps $\varphi_i : TK_i \to K_i$ with properties (KT1) and (KT2) and $q_{ii+1} : TK_{i+1} \to TK_i$. Let $Y$ be the limit of the inverse sequence $\mathbf{Y} = (Y_i, q_{ii+1}, \mathbb{N})$ where $Y_i$ are the realizations of $TK_i$, and let $q_i : Y \to Y_i$ be the projection maps. Then $\mathbf{q} = (q_i) : Y \to \mathbf{Y}$ induces a polyhedral expansion of $Y$, so the maps $\varphi_i : Y_i \to X_i$ form a level morphism $\boldsymbol{\varphi} = (\varphi_i) : \mathbf{Y} \to \mathbf{X}$ which represents a shape morphism $\varphi : Y \to X$ with properties (S1) and (S2). Since $\dim Y_i = \dim X_i \le n$, then $\dim Y \le n$.

Thus $\dim Y \le \operatorname{sd} X$. On the other hand, by [L], property (S3) and Alexandroff Theorem, $\operatorname{sd} X \le \operatorname{cdim}_{\mathbb{Z}} X \le \operatorname{cdim}_{\mathbb{Z}} Y = \dim Y$. Hence $\operatorname{sd} X = \dim Y$. □

**Corollary 5.1.** 1. *For each continuum $X$ (resp., continuum $X$ with $\dim X < \infty$), there exists an approximately aspherical compactum $Y$ (resp., approximately aspherical compactum $Y$ with $dimY = \dim X$) and a surjective map $\varphi : Y \to X$ such that the induced map $\mathrm{SP}^\infty(\varphi) : \mathrm{SP}^\infty Y \to \mathrm{SP}^\infty X$ is a weak shape equivalence.*

2. *For each continuum $X$ with $\operatorname{sd} X < \infty$, there exists an approximately aspherical compactum $Y$ with $\dim Y = \operatorname{sd} X$ and a shape morphism $\varphi : Y \to X$ such that the induced map $\mathrm{SP}^\infty(\varphi) : \mathrm{SP}^\infty Y \to \mathrm{SP}^\infty X$ is a weak shape equivalence.*

*Proof.* This easily follows from Theorems A and C and [DT]. □

*Proof of Theorem* D. Let $X$ be a continuum. Then by Theorem A, there exists a map $\varphi : Y \to X$ from an approximately aspherical compactum $Y$ onto $X$ such that $\varphi_* : \text{pro-}H_q(Y;\mathbb{Z}) \to \text{pro-}H_q(X;\mathbb{Z})$ is an isomorphism for each $q$, which implies by [MiS2, Corollary 7.8] that $\varphi_* : \text{pro-}\pi_q^S(Y) \to \text{pro-}\pi_q^S(X)$ is an isomorphism for each $q$ as required. If $\operatorname{sd} X < \infty$, then Theorem C implies that there exists a shape morphism $\varphi : Y \to X$ from an approximately aspherical compactum $Y$ of

$\dim Y = \operatorname{sd} X$ such that $\varphi_* : \text{pro-}H_q(Y;\mathbb{Z}) \to \text{pro-}H_q(X;\mathbb{Z})$ is an isomorphism for each $q$, so $\varphi_* : \text{pro-}\pi_q^S(Y) \to \text{pro-}\pi_q^S(X)$ is an isomorphism for each $q$. Now by [MiS1, Theorem 6.1], $\varphi$ is an equivalence in the stable shape category. □

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Department of Computer Science, Shizuoka Institute of Science and Technology, 2200-2 Toyosawa, Fukuroi, Shizuoka-Pref., 437-8555 Japan

*E-mail address*: miyata@mb.sist.ac.jp