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AN ADDITION THEOREM FOR THE COLOR NUMBER

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Abstract. There is a close relation between the color number of a continuous map $f\colon X \to X$ without fixed points and the topological dimension. If f is an involution, the color number is also related to the co-index. An addition theorem for the color number is established thus underscoring the interrelations between color number, dimension and co-index.

1. Introduction

Our main result is an addition theorem for the color number of maps that is similar to the addition theorem of dimension theory. All topological spaces under consideration are metrizable. All mappings are assumed be continuous.

Definition 1. Suppose that $f\colon X \to X$ is a map of a topological space to itself. A subset B of X is called a *color* of the map f if $f^{-1}(B) \cap B = \emptyset$ or, equivalently, $f(B) \cap B = \emptyset$. A *coloring* of f is a cover of X consisting of colors of f.

Note that if B is a color of f, then so is $f^{-1}(B)$. It is known that every map has a coloring consisting of three colors; see [8] and [6]. The situation is totally different if we consider colorings consisting of open or closed sets only. As usual, an *open coloring* is a coloring consisting of open colors.

Definition 2. Suppose that $f\colon X \to X$ is a fixed-point free map. The *color number* $C(f)$ is the minimal cardinality of an open coloring of f.

Note that the color number $C(f)$ may not be finite. We then write $C(f) = \infty$. If, for example, α is the antipodal map on the unit sphere in Hilbert space, then $C(\alpha) = \infty$. A less elementary example, due to Mazur, is related to the Galvin-Prikry lemma from infinite combinatorics [8, Theorem 3.4]. By contrast, closed maps with bounded fibers on a finite-dimensional space have a finite color number [4]. For a fixed-point free homeomorphism f of an n-dimensional space we have $C(f) \leq n+3$. If moreover f is an involution, then $C(f) \leq n+2$; see [1]. The color number of the antipodal map of the n-dimensional sphere is equal to $n+2$.

Suppose that $f\colon X \to X$ is a fixed-point free map. We say that a subset A of X is *invariant* if $f(A) \subset A$. For an invariant subset A of X the restriction $f\colon A \to A$ is denoted by f_A. It is to be noted that a color of f_A is a subset of A. The following addition theorem is our main result.

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Theorem 1. *Suppose that X is a metric space. Let $f\colon X \to X$ be a fixed-point free map such that $X = A \cup B$ for invariant subsets A and B of X. Then $C(f) \leq C(f_A) + C(f_B) - 1$.*

We will prove this theorem in the next section. The relation between the color number and the co-index will be discussed in section 3.

2. Proof of the main result

The following simple lemma will be very useful.

Lemma 1. *Let $f\colon X \to X$ be a fixed-point free map and suppose that A is an invariant subspace. Suppose that the subset C of the subspace A is a closed color of f_A. Then there exists an open subset W of X such that $f_A^{-1}(C) \subset W$ and W is a color of f.*

Proof. The colors C and $f_A^{-1}(C)$ are disjoint closed subsets of the subspace A. Hence, these colors are separated subsets of X. It follows that there are disjoint open subsets U and V of X such that $f_A^{-1}(C) \subset U$ and $C \subset V$. Define $W = U \cap f^{-1}(V)$. One readily verifies that W is an open color which contains $f_A^{-1}(C)$. ☐

Corollary 1. *Suppose that $f\colon X \to X$ is a fixed-point free map and that $\mathcal{C}$ is a collection of n colors of f each of which is open or closed. Then there exists an open coloring of f with cardinality at most n.*

Proof. Taking $X = A$ in Lemma 1 we can replace each closed color B in the coloring $f^{-1}(\mathcal{C})$ of f by an open color U such that $B \subset U$. ☐

From the corollary it follows that the color number of a map $f\colon X \to X$ is the minimal cardinality of a cover of X consisting of colors that are open or closed. The following lemma relates an open coloring of an invariant subspace to a collection of open colors of the space.

Lemma 2. *Suppose that $f\colon X \to X$ is a fixed-point free map and that the subset A of X is an invariant subspace. Let $\mathcal{U}$ be an open coloring of f_A consisting of m colors, $m \in \boldsymbol{N}$. Then there exists a collection $\mathcal{V}$ consisting of at most m open colors of f such that $A \subset \bigcup \mathcal{V}$.*

Proof. In a standard fashion, in the subspace A there is a closed shrinking $\mathcal{G}$ of $\mathcal{U}$. Then $\mathcal{G}$ is a closed coloring of f_A and so is the collection $\mathcal{H} = f_A^{-1}(\mathcal{G})$. By Lemma 1 for each member H of $\mathcal{H}$ there is an open color V of f such that $H \subset V$. Let $\mathcal{V}$ be the collection of all V obtained in this way. ☐

Proof of the main theorem. We may assume that both f_A and f_B have a finite color number, say $n = C(f_A)$ and $m = C(f_B)$. By Lemma 2, there exist families $\mathcal{U} = \{\, U_i \mid i = 1, ..., n \,\}$ and $\mathcal{V} = \{\, V_j \mid j = 1, ..., m \,\}$ of open colors of f such that $\mathcal{U}$ covers A and $\mathcal{V}$ covers B. The union $\mathcal{V} \cup \mathcal{W}$ is a coloring of f with $n + m$ colors.

To complete the proof we remove one of the colors so as to obtain the required coloring of f. Consider the collection of open sets $\mathcal{W} = \{\, U_i \mid i = 1, ..., n-1 \,\} \cup \{\, V_j \mid j = 1, ..., m-1 \,\}$. Define $R = X \setminus \bigcup \mathcal{W}$. We claim that R is a closed color of f. To show that R is a color we prove $f^{-1}(R) \cap R = \emptyset$. Let $y \in f^{-1}(R)$. Without loss of generality we may assume that $y \in A$. Since A is invariant, $f(y) \in A$. As $f(y) \in R$, we have $f(y) \in U_n$. Since U_n is a color, $y \notin U_n$, whence $y \in U_i$ for some $i < n$. It follows that $y \notin R$, whence $f^{-1}(R) \cap R = \emptyset$. This proves the claim.

The collection $\mathcal{W} \cup \{ R \}$ is a coloring of f, each element of which is open or closed. According to Corollary 1 the color number of f is at most $n + m - 1$. □

3. Color number and co-index

The antipodal map of the n-dimensional sphere S_n is denoted by α_n. The Borsuk-Ulam theorem states that there is no equivariant map from α_n to α_m if $n > m$. This fact has been employed by Conner and Floyd to define the co-index of a fixed-point free involution [3].

Definition 3. The co-index $c(\tau)$ of a fixed-point free involution τ on a topological space X is the minimal number n such that there exists an equivariant map from τ to α_n. If there exists no such map, $c(\tau) = \infty$.

Surprisingly, the color number and the co-index of a fixed-point free involution are intimately related.

Theorem 2. *If $\tau\colon X \to X$ is a fixed-point free involution, then $C(\tau) = c(\tau) + 2$.*

Proof. Let $\tau\colon X \to X$ be a fixed-point free involution. We first show that $C(\tau) \leq c(\tau) + 2$. We may assume that $c(\tau)$ is finite. Thus there exists an equivariant map from τ to an antipodal map α_n on S^n for some n, that is, there exists a map $f\colon X \to S^n$ such that $f \circ \tau = \alpha_n \circ f$. A coloring of α_n with $n+2$ colors is pulled back onto a coloring of τ to obtain a coloring of τ. It follows that $C(\tau) \leq c(\tau) + 2$.

To show that the opposite inequality $C(\tau) \geq c(\tau) + 2$ also holds, we may assume that $C(\tau)$ is finite, say $C(\tau) = n$. Let $\{ U_i \mid i = 1, ..., n \}$ be an open coloring of τ and let $\{ f_i \mid i = 1, ..., n \}$ be a partition of unity subordinate to the open coloring. For $i = 1, \ldots, n$, let $g_i\colon X \to \mathbf{R}$ be the map defined by $g_i(x) = f_i(x) - f_i(\tau(x))$. Using the g_i as coordinate functions we define the map $g\colon X \to \mathbf{R}^n$. Note that $g(\tau(x)) = -g(x)$. For every x in X, the sum over the coordinates of $g(x)$ is 0, so $g(X)$ is contained in an $(n-1)$-dimensional hyperspace V of $\mathbf{R}^n$. Observe that $f_i(\tau(x)) = 0$ whenever $f_i(x) > 0$, $x \in X$. It follows that the image $g(X)$ does not contain the origin. By dividing each $g(x)$ by its norm we get an equivariant map from τ to the antipodal map on S^{n-2} (the unit sphere in V). It follows that $c(\tau) \leq C(\tau) - 2$. □

Conner and Floyd used the co-index to generalize the Borsuk-Ulam theorem for involutions on closed manifolds. The following addition property of the co-index now readily follows from the addition theorem of color numbers.

Corollary 2. *Let $\tau\colon X \to X$ be a fixed-point free involution such that $X = A \cup B$ for invariant subsets A and B of X. Then $c(\tau) \leq c(\tau_A) + c(\tau_B) + 1$.*

This addition theorem is a slight generalization of the addition property discussed in [3], as the invariant subsets are not necessarily closed. As an application of the addition property, we show that the co-index has a decomposition property which is similar to the decomposition property of topological dimension functions [5].

Corollary 3. *Suppose that $\tau\colon X \to X$ is a fixed-point free involution. For every n in $\boldsymbol{N}$ the following holds: $c(\tau) \leq n$ if and only if there exists a decomposition $X = X_0 \cup X_1 \cup \cdots \cup X_n$ into invariant subspaces X_i such that the co-index of τ_{X_i} is zero for each i.*

Proof. If X admits such a decomposition, then by repeated application of the addition property it follows that $c(\tau) \leq n$. We show that the opposite implication holds. Let $\{ U_i \mid i = 1, ..., n+2 \}$ be an open coloring of τ. Start with $X_{-1} = \emptyset$ and inductively define $X_i = (U_i \cup \alpha(U_i)) \setminus X_{i-1}$. Obviously $C(X_i) = 2$, whence $c(X_i) = 0$ for every i. This gives the required decomposition. ☐

Our final corollary is known [1], but we include it here, since it is a straightforward application of the addition property.

Corollary 4. *Let* $n \in \boldsymbol{N}$. *If* $\tau\colon X \to X$ *is a fixed-point free involution on a space of dimension* $\dim X \leq n$, *then* $C(\tau) \leq n+2$.

Proof. By induction. An involution on a zero-dimensional space can be colored by 2 colors [8]. A space of $\dim X \leq n$ admits a decomposition $X = F \cup G$ into a zero-dimensional F_σ-subset F and a set of $\dim G \leq n-1$. Then $F \cup \tau(F)$ is an invariant zero-dimensional subset. Its complement C is invariant and at most $(n-1)$-dimensional, since it is contained in G. By the induction hypothesis the color number of τ_C is at most $n+1$. Now apply the addition theorem for the color number. ☐

4. Remarks

The proof of Corollary 4 can also be applied to a fixed-point free homeomorphism. If $h\colon X \to X$ is a homeomorphism on a n-dimensional space, then X admits a decomposition into $n+1$ zero-dimensional, invariant sets. If in addition h is fixed-point free, then the color number of an invariant zero-dimensional subspace is at most 3. Repeated applications of the addition property then show that the color number is bounded by $2n+3$, but this upper bound is not sharp. The sharp bound of $n+3$ is derived in [1].

Conner and Floyd's addition property for the co-index is sharp; hence so is the addition property for the color number.

A relation between the color number and the genus $g(\tau)$ of a map [7] is suggested by the proof of Corollary 3. For an involution τ we have: $c(\tau)+1 = g(\tau) = C(\tau)-1$.

The example of [8, Theorem 3.4] shows that there exist maps on finite-dimensional spaces with color number ∞. It is natural to ask whether for every map with color number ∞ there is a zero-dimensional invariant subspace such that the restriction to that subspace has color number ∞. We are unable to resolve this question.

If a map has a finite coloring consisting of Borel sets, then the map can be colored by 3 Borel sets (cf. [8, Lemma 2.1]). The antipodal map α of the unit sphere S in Hilbert space ℓ_2 has $C(\alpha) = \infty$ but admits a coloring by 2 Borel sets, namely,

$$A_u = \{ \langle x_i \rangle_i \in S \mid x_1 = x_2 = \ldots = x_k < x_{k+1},\ k \in \mathbf{N} \}$$

and $A_l = \alpha(A_u)$.

A recent survey of the co-index and other invariants as well as some new results are given in [6].

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