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Lipα HARMONIC APPROXIMATION ON CLOSED SETS

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Abstract. In this paper the *Lipα* harmonic approximation ($0 < \alpha < \frac{1}{2}$) on relatively closed subsets of a domain in the complex plane is characterized under the same conditions given by S. Gardiner for the uniform case. Thus, the result of P. Paramonov on *Lipα* harmonic polynomial approximation for compact subsets is extended to closed sets. Moreover, the problem of uniform approximation with continuous extension to the boundary for harmonic functions and similar questions in *Lipα* harmonic approximation are also studied.

1. Introduction

Let F be a relatively closed subset of a domain G of the complex plane $\mathbb{C}$. A. Roth proved in [25] that if f is a uniform limit on F of holomorphic or meromorphic functions, then it is possible to select the approximating function m in such a way that the difference function $f - m$ can be extended continuously to the closure of F, including the points of $\partial F \cap \partial G$ for which f itself has no continuous extension. Furthermore if f is a Lipα limit of holomorphic or meromorphic functions ($0 < \alpha < 1$), it is proved in [7] that it is possible to choose the approximating function m such that $f - m$ belongs to $lip(\alpha, F)$.

Roth's result was extended in [9] to solutions of certain partial differential equations. Also, the main result obtained by M. Goldstein and W. Ow in [16], concerning the problem of uniform approximation with continuous extension to the boundary by harmonic functions, was improved in [9] by removing most of the unnecessary conditions assumed in their work. However a mild restriction on G remains, namely that G is not dense in $\mathbb{C}$. In this paper, this restriction is eliminated and we obtain analogous results to those in [7] and [9] for *Lipα* harmonic approximation. Another improvement of [16] was done by Gardiner ([14]) considering that the function to be approximated extends continuously to the closure of F in $\mathbb{R}^n \cup \{\infty\}$.

On the other hand, the characterization of the *Lipα* harmonic approximation on relatively closed subsets of a domain in the complex plane for $0 < \alpha < \frac{1}{2}$ is perhaps the main question dealt with in this paper. We prove that, under the same conditions given by S. Gardiner [12] for the uniform case, not only is this approximation always possible, but also, as in the holomorphic and meromorphic cases, we can choose the approximating function m such that $f - m$ belongs to $lip(\alpha, F)$. Thus, we extend to closed sets a theorem of P. Paramonov about harmonic polynomial

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approximation for compact subsets [22] and we obtain a result of decomposition of $Lip\alpha$ approximable functions on F by harmonic functions on G. This result is in the spirit of the papers of A. Stray ([26]) and S. Gardiner ([15]) dealing with the holomorphic and harmonic case respectively.

2. Preliminaries

Denote by $h(G)$ the set of harmonic functions on G and by $h(F)$ the class of harmonic functions on a neighbourhood of F relative to G. Following [10], a function u is said to be Δ-meromorphic on G if u is harmonic on G, except for isolated singularities and if in a neighborhood of any such singularity y, u can be represented in the form

$$u(x) = s_y(x) + u_y(x)$$

where u_y is harmonic at y and $s_y(x)$ is a finite linear combination of $E(x-y)$ and its derivatives, E being a fundamental solution of the Laplacian Δ. Such a singularity is named a pole. We denote by $m(G)$ the set of Δ-meromorphic functions on G, by $m(F)$ the set of Δ-meromorphic functions on a neighbourhood of F, and by $m_F(G)$ the set of Δ-meromorphic functions on G and having no poles on F.

Let f be a bounded complex function on F. As usual, the modulus of continuity w_f of f is given by

$$w_f(r) = \sup\{|f(z) - f(w)| : z, w \in F, \ \mid z - w \mid \leq r\}.$$

For $0 < \alpha < 1$, consider the seminorm

$$\|f\|_{\alpha,F} = \sup\left\{\frac{w_f(r)}{r^\alpha} : r > 0\right\}$$

and the function spaces

$$Lip(\alpha, F) = \{f : F \to \mathbb{C} : \|f\|_{\alpha,F} < \infty\}$$

and

$$lip(\alpha, F) = \{f \in Lip(\alpha, F) : r^{-\alpha} w_f(r) \to 0, \text{ as } r \to 0^+\}.$$

If f is defined on F, we will say that f belongs to $lip_{loc}(\alpha, F)$ if $f \in lip(\alpha, K)$ for every compact subset K of F. The $Lip\alpha$ norm is defined by

$$\|f\|^*_{\alpha,F} = \|f\|_{\alpha,F} + \|f\|_{\infty,F}$$

where $\|f\|_{\infty,F}$ is the supremum norm.

We define

$$a_\alpha(F) = \{f \in lip_{loc}(\alpha, F) : f \in h(F^o)\}$$

and

$$a_{\alpha u}(F) = \{f \in a_\alpha(F) : \exists g \in lip(\alpha, \bar{F}) \text{ and } g_{|F} = f\}$$

where $\overline{F}$ denotes the closure of F in $\mathbb{C}$.

If A is a class of complex functions on F 'we will denote $[A]^*_{\alpha,F}$ as the set of all functions which are limits in $Lip\alpha$ norm on F of functions belonging to A, i.e., $f \in [A]^*_{\alpha,F}$ if and only if, for each $\varepsilon > 0$, there exists $h \in A$ such that $f - h \in Lip(\alpha, F)$ and $\|f - h\|^*_{\alpha,F} < \varepsilon$. Also, we use the notation $G^\infty = G \cup \{\infty\}$ for the one point compactification of G.

Definition 1. The function $f : F \to \mathbb{C}^\infty$ is said to be LE-approximable on F by functions from $m(G)$ (respectively $h(G)$) if, given $\varepsilon > 0$, there are functions m and e with the following properties:

i) $m \in m(G)$ (respectively $h(G)$), $e \in h(F^o) \cap lip(\alpha, \bar{F})$,

ii) $f(z) - m(z) = e(z)$ $(z \in F)$,

iii) $\|e\|^*_{\alpha,\overline{F}} < \varepsilon$.

The letters LE stand for "$\underline{l}ip\alpha$ $\underline{e}$xtension". We also introduce the notation

$$\|f\|_{Lip1,E} = \sup\left\{\frac{|f(z) - f(w)|}{|z - w|} : z, w \in E\right\}.$$

Finally, to unify the notation, we understand the $Lip0$ approximation with $lip0$ extension to the boundary as the uniform approximation with continuous extension to the boundary of F referred to $\mathbb{C}^\infty$ and we will denote from now on the uniform norm on F by $\|.\|_{0,F}$.

3. LE-approximation by Δ-meromorphic and harmonic functions

In this section we prove that the approximation in $Lip\alpha$ norm by functions from $m_F(G)$ is equivalent to the LE-approximation by Δ-meromorphic functions. More precisely, we shall show that both kinds of approximations are equivalent to the approximation on compact subsets in $Lip\alpha$ norm. For this purpose, we need the following version of the Runge's theorem for closed sets in this norm and a fusion lemma for harmonic functions in the $Lip\alpha$ norm.

Proposition 1. *Let F be a closed subset of $\mathbb{C}$, and let g be any function Δ-meromorphic in a neighbourhood of F and $0 \le \alpha < 1$. Then, given any $\varepsilon > 0$, there exists a Δ-meromorphic function r in $\mathbb{C}$ such that*

$$\|r - g\|^*_{\alpha,F} < \varepsilon$$

and

$$\|r - g\|_{Lip1,F} < \varepsilon.$$

Moreover, for fixed $u \in K$, we can choose r such that $r(u) = g(u)$.

Proof. This follows from [5, Theorem 1] considering the space $V(F) = C^1(F)$ and the operator L as the Laplacian Δ. To be more precise and with the same notation as in [5], given $\varepsilon > 0$ and a Δ-meromorphic function g in a neighbourhood of F, we can obtain a Δ-meromorphic function r in $\mathbb{C}$ such that $r - g$ coincides with $f \in C^1(\mathbb{C})$ on F and $\|f\|_{C^1(\mathbb{C})} < \varepsilon$. Then, by [6, Proposition 3] we get the $Lip1$ and $Lip\alpha$ approximation on F. Note that r can be chosen such that, for a fixed point $u \in K$, $(r - g)(u) = 0$ just by replacing r with $r + g(u) - r(u)$.

Proposition 2 (Fusion Lemma)**.** *Let $0 \le \alpha < 1$. Suppose that K_1 and K are compact subsets of $\mathbb{C}$ and K_2 is a closed subset of $\mathbb{C}$ such that $K_1 \cap K_2 = \emptyset$ and $K_1 \cup K_2 \cup K \ne \mathbb{C}$. If r_1 and r_2 are two Δ-meromorphic functions in $\mathbb{C}$ satisfying that $\|r_1 - r_2\|^*_{\alpha,K} < \varepsilon$, then there exists a constant $C = C(K_1, K_2)$ and a Δ-meromorphic function r in $\mathbb{C}$ such that*

$$\|r - r_i\|^*_{\alpha,K\cup K_i} < C\varepsilon \quad (i = 1, 2) \tag{1}$$

and

$$\|r - r_i\|_{Lip1,K_i} < C\varepsilon \quad (i = 1, 2). \tag{2}$$

Moreover, for fixed $u \in K_2$, we can choose r such that

$$(r - r_2)(u) = 0. \tag{3}$$

Proof. The proof follows ideas from [8], [9], [11] and [24], and as there, without loss of generality we may assume that $r_2 \equiv 0$, $K_1 \cap K \neq \emptyset$ and $K \cap K_2 \neq \emptyset$. Then we can choose neighborhoods U_1 and U_2 of K_1 and K_2 respectively, such that U_1 and U_2 have C^1 boundary, $\bar{U}_1 \cap \bar{U}_2 = \emptyset$ and U_2 contains the exterior of a ball.

Let $M = \mathbb{C} \setminus (U_1 \cup U_2)$ and H be an infinitely differentiable function with compact support such that $0 \leq H \leq 1, H_{|U_1} \equiv 1$ and $H_{|U_2} \equiv 0$.

By assumption, there exists a neighborhood U of K with C^1 boundary such that $\|r_1\|_{\infty,\bar{U}} < C\varepsilon$. We set $h = r_1$ on $\bar{U} \cap M$ and extend this function to M verifying $\|h\|^*_{\alpha,M} \leq C\varepsilon$.

Now, we define

$$f(z) = \begin{cases} h(z) & \text{if } z \in M, \\ r_1(z) & \text{if } z \in \mathbb{C} \setminus M, \end{cases}$$

and

$$F(z) = V_H f(z) = \frac{1}{2\pi} E * H \Delta f = Hf + g$$

where

$$g = \sum_{\substack{|\gamma| \neq 0 \\ |\gamma+\beta|=2}} C_{\gamma\beta}(D^\beta E) * (f D^\gamma H)$$

for certain constants $C_{\gamma,\beta}$. Then, except for finitely many singularities in U_1, the function F is Δ-meromorphic in $U_1 \cup U_2 \cup U$. Moreover, since $K_1 \cup K_2 \cup K \neq \mathbb{C}$, there exists a ball $D = D(a, \delta)$ contained in $\mathbb{C} \setminus (K_1 \cup K_2 \cup K)$ and we can choose $\psi \in C^\infty(\mathbb{C})$, $\psi \equiv 1$ outside D and $\psi \equiv 0$ on $\frac{1}{2}D$. Next, take $S = c_0 \psi E(x-a)$, where $c_0 = \frac{1}{2\pi} \int h \Delta H dm$ that verifies $|c_0| \leq C\|h\|_{\alpha,D}$. Thus, if B is a ball containing the support of H, it follows that

$$\|S\|^*_{\alpha,3\overline{B}} \leq C\|h\|^*_{\alpha,K} < C\varepsilon$$

and, since $g - S$ satisfies the hypothesis of [21, Lemma 2] and $\|g\|^*_{\alpha,3B(0,R)} \leq C\|f\|^*_{\alpha,K}$ (see [20, Lemma 2.4]),

$$\|g - S\|^*_{\alpha,\mathbb{C}} < C\varepsilon.$$

Now, if we define $F_1 = F - S = fH + g - S$, arguing as in [11, Theorem 3] we get that

$$\|F_1 - r_i\|^*_{\alpha,K_i \cup K} < C\varepsilon, \quad i = 1, 2.$$

Although we can use F_1 to prove (1) and (2), in order to prove (3) we may consider $F_2 = F_1 - F_1(u)$, for u a fixed point of K_2, which is harmonic in $U_1 \cup U_2 \cup U$ except for finitely many singularities in U_1. Applying Proposition 1 to F_2 we get a Δ-meromorphic function r such that

$$\|F_2 - r\|^*_{\alpha,K_1 \cup K_2 \cup K} < \varepsilon,$$

$$\|F_2 - r\|_{Lip1,K_1 \cup K_2 \cup K} < \varepsilon$$

and $r(u) = F_2(u) = 0$. Thus we obtain

$$\|r - r_i\|^*_{\alpha,K_i \cup K} < C\varepsilon \ \ (i = 1, 2)$$

and $r(u) = r_2(u)$ which prove (1) and (2). Finally, if we are able to show that

$$\|g - S\|_{Lip1,K_1\cup K_2} < C\varepsilon, \tag{4}$$

by observing that $r - r_1 = r - F_1 + g - S$ on K_1 and $r = r - F_1 + g - S$ on K_2, we will conclude that (3) is satisfied and the theorem will be completely proved.

We now proceed to verify (4). Since E is an infinitely differentiable function except at the origin, if z_1 and z_2 belong to $K_1 \cup K_2$, then

$$\left|\frac{g(z_1) - g(z_2)}{z_1 - z_2}\right| \leq \sum_{\substack{|\alpha|\neq 0 \\ |\alpha+\beta|=2}} |C_{\alpha\beta}| \|fD^\alpha H\|_{\infty,\mathbb{C}} \int_{suppDH} \left|\frac{D^\beta E(y - z_1) - D^\beta E(y - z_2)}{z_1 - z_2}\right| dm(y).$$

Since the support of DH is at positive distance from $K_1 \cup K_2$, there exists $\delta > 0$ such that $|y - z_i| > \delta$, for $i = 1, 2$ and $y \in suppDH$. Owing to the fact that $D^\beta E \in Lip(1, \mathbb{C} \setminus B(0, \delta))$ it is readily seen that

$$\|g\|_{Lip1,K_1\cup K_2} < C\varepsilon. \tag{5}$$

To finish, since S is defined as the product of E and a C^∞ function that is identically equal to 1 outside a disc, it follows that $\|S\|_{Lip1,K_1\cup K_2} < C\varepsilon$ which together with (5) gives (4). □

We are now in a position to state the main result of this section which establishes the equivalence of the LE-approximation and the $Lip\alpha$ approximation.

Theorem 1. *Let F be a relatively closed subset of a domain G in $\mathbb{C}$ and $0 \leq \alpha < 1$. Then the following statements are equivalent:*

(a) *f can be approximated in $Lip\alpha$-norm on F by functions in $m_F(G)$.*
(b) *If K is a compact subset of F, then $f_{|K} \in [m_K(\mathbb{C})]^*_{\alpha,K}$.*
(c) *f is LE-approximable on F by functions in $m_F(G)$.*

Proof. (a) ⇒ (b) and (c) ⇒ (a) are trivial.

(b) ⇒ (c) Analogously to the proof of [7, Theorem 3.3], we may suppose that F is bounded. Thus, without loss of generality we can assume that $\partial F \cap \partial G$ is bounded and $\overline{B}(0, \rho) \cap \overline{F} = \emptyset$ for some $\rho > 0$. For $n \geq 1$, let $\Omega_n = B(0, n) \setminus \overline{B}(0, \frac{1}{n})$. Note that $\overline{\Omega}_n \subset \Omega_{n+1}$ and $\bigcup \Omega_n = \mathbb{C} \setminus \{0\}$. Let G_n be a sequence of bounded domains such that $\overline{G}_n \subset G_{n+1}$, $\bigcup G_n = G$ and $dist(\partial G_n, \partial F \cap \partial G) = \frac{1}{n}$, where $\partial F \cap \partial G$ is a compact set. If we denote $V_n = G_n \cap \Omega_n$, then $\overline{V}_n \subset V_{n+1}$, $\bigcup V_n = G \setminus \{0\}$ and $V_n \subset G_n$.

For each $n = 1, 2, \ldots$, we now apply the fusion lemma (Proposition 2) with $K_1 = \overline{V}_n, K_2 = \mathbb{C} \setminus B(0, n+1)$ and $K = F_n = F \cap \overline{V}_{n+1}$. Then, there exist constants a_n that correspond to the constant C in Proposition 2 and we may assume that the a_n are increasing and $a_n > 1$.

Let $\varepsilon > 0$. By hypothesis $f_{|F_n} \in [m_{F_n}(\mathbb{C})]^*_{\alpha,F_n}$; hence we can find a Δ-meromorphic function q_n without poles on F_n such that

$$\|f - q_n\|^*_{\alpha,F_n} < \frac{\varepsilon}{2^{n+1} a_n (n + 1)}$$

for $n = 1, 2, \ldots$. Thus, since $F_n \subset F_{n+1}, n = 1, 2, \ldots$, we have

$$\|q_{n+1} - q_n\|^*_{\alpha,F_n} < \frac{\varepsilon}{2^n a_n (n + 1)}.$$

Now, for each n, there exists a Δ-meromorphic function r_n which satisfies Proposition 3 and, by following ideas of [7], this implies the $Lip\alpha$ convergence on $\mathbb{C}^\infty \setminus G$ of

$$g_n(z) = \sum_{k=1}^{n-1} (r_k(z) - q_{k+1}(z))$$

as $n \to \infty$. Besides on $\partial F \cap \partial G$, since $lip(\alpha, \partial F \cap \partial G)$ is a closed subalgebra, g_n converges to a function $\phi \in lip(\alpha, \partial F \cap \partial G)$.

Finally, we define

$$m(z) = \sum_{k=1}^{n-1} (r_k - q_{k+1}) + q_n + \sum_{k=n}^{\infty} (r_k - q_k)$$

and

$$e(z) = \begin{cases} f(z) - m(z) & \text{if } z \in F, \\ \phi(t) & \text{if } z \in \overline{F} \setminus F. \end{cases}$$

Then, in a similar way as [9, Theorem 3.3] if $\alpha = 0$ and as [7, Theorem 3.3] if $0 < \alpha < 1$, it follows that $m(z) \in m_F(G \setminus \{0\})$, $e \in lip(\alpha, \overline{F})$ and

$$\|f - m\|^*_{\alpha,F} < \varepsilon.$$

If $0 \notin G$, the proof is finished. Otherwise, let h be a harmonic function on $\mathbb{C} \setminus \{0\}$ such that $m - h$ has a removable singularity at $z = 0$. Since h is harmonic on the compact subset $\overline{F}$ of $\mathbb{C}$, there exists $s \in m_{\overline{F}}(\mathbb{C})$ such that

$$\|h - s\|^*_{\alpha,\overline{F}} < \frac{\varepsilon}{2}.$$

Let $p = m - h - s$. Then p is Δ-meromorphic on G and $f - p$ extends $lip\alpha$ to $\partial F \cap \partial G$. This completes the proof. □

The following corollaries are a direct consequence of the above theorem and the characterization of the $Lip\alpha$-approximation of $a_\alpha(F)$ by $m_F(G)$ given for $\alpha = 0$ in [2] and for $0 < \alpha < 1$ in [20], [21] and [10]:

Corollary 1. *All functions of $a_0(F)$ can be LE-approximated by functions in $m_F(G)$ if and only if $G \setminus F$ and $G \setminus F^o$ are thin at the same points.*

Corollary 2. *Let $0 < \alpha < 1$. All functions of $a_\alpha(F)$ can be LE-approximated by functions in $m_F(G)$ if and only if there exists a constant $C > 0$ such that*

$$M^\alpha_*(D \setminus F^0) \le C M^\alpha(D \setminus F)$$

for every disc $D \subset G$.

Now our goal is to study when the LE-approximation by functions in $h(G)$ is possible. In other words, we look for conditions on the domain G and the closed set F in order to apply a pushing poles method in $Lip\alpha$ norm for harmonic functions. The conditions that appear are the same as in Arakeljan's Theorem for the uniform approximation ([1]). We collect them in the following theorem.

Theorem 2. *If $G^\infty \setminus F$ is connected and locally connected at $\{\infty\}$, $0 \le \alpha < 1$ and m is a function in $m_F(G)$, the restriction $m_{|F}$ is LE-approximable on F by functions in $h(G)$.*

For a proof of this theorem, we refer the reader to [9, Lemma 3.4] if $\alpha = 0$. For the case $0 < \alpha < 1$, the proof follows the same line as in [7, Lemma 4.2], where the pushing poles method is obtained by using partial sums of the Laurent series (see also [10]).

4. Lipα harmonic approximation

Recently, S. Gardiner [12] has characterized the uniform harmonic approximation of $a_0(F)$ by functions of $h(G)$ (see also [3], [13] and [17] for an extensive account of harmonic approximation results). His result is given in terms of the "holes" of F and the "long islands condition". In this section we prove that the same conditions solve the problem for the *lip*α harmonic approximation of $a_\alpha(F)$ by $h(G)$ when $0 < \alpha < \frac{1}{2}$.

Definition 2. Let G be an open subset of $\mathbb{C}$. A subset H of G is called G-bounded if $\overline{H}$ is a compact subset of G. A hole of F will be a connected component of $G \setminus F$ which is G-bounded. We will denote by $\hat{F}$ the union of F and all its holes.

Definition 3. We say that a family $\mathcal{A}$ of subsets of G satisfies the "long islands condition" provided that for each G-bounded subset $B \subset G$, the union of all members of the family $\mathcal{A}$ which intersect B is G-bounded.

We need a sort of maximum principle for harmonic function in *Lip*α-norm on compact subsets which allow us to "kill" the holes of F. The proof of our results follows the same ideas as [23, Lemma 1.8 and Lemma 2.2] (see also [19] and [22, Lemma 3.2]), with some modifications because now the compact $\hat{K}$ could have holes in $\mathbb{C}$. We include the proofs for the sake of completeness.

Theorem 3. *Let Ω be a complex domain, K a compact subset of Ω and $0 < \alpha < \frac{1}{2}$. Suppose that $f \in Lip(\alpha, \partial\hat{K})$ and let F be the solution of the Dirichlet problem in $\hat{K}$ with boundary values f. Then $F \in Lip(\alpha, \hat{K})$ and*

$$\|F\|_{\alpha,\hat{K}} \leq A(\alpha, K, \Omega)\|f\|_{\alpha,\partial\hat{K}}.$$

Proof. Without loss of generality we can suppose that $\hat{K}$ is connected and $\hat{K} = \hat{K}^o \cup \partial\hat{K}^o$. Let $\eta = dist(\hat{K}, \partial\Omega)$ and $\beta < \frac{\eta}{3}$. We need only to show that if $\|f\|_{\alpha,\partial\hat{K}} \leq 1$, then for any $x \neq a \in \hat{K}$ we have

$$|F(x) - F(a)| \leq A(\alpha)|x - a|^\alpha.$$

Fix x and a in $\hat{K}$, put $b = x - a$, and $G(y) = F(y + b) - F(y)$. Then $G(y)$ is harmonic on the set $\hat{K}_1^o = \{y \in \hat{K}^o : y + b \in \hat{K}^o\}$ and so attains its modulus maximun on $\partial\hat{K}_1$, that is, at some point y with $y \in \partial\hat{K}$ or $y + b \in \partial\hat{K}$. So it is enough to consider the case $a \in \partial\hat{K}$ and $x \in \hat{K}^o$.

Denote by $K' = \{x \in \hat{K} : dist(x, \partial\hat{K}) \geq \beta\}$ and consider a first case where $a \in \partial\hat{K}^o$ and $x \in K'$. In this case,

$$|F(x) - F(a)| \leq 2\frac{\|f\|_{\alpha,\partial\hat{K}}}{\beta^\alpha}|x - a|^\alpha.$$

If $x \notin K'$, using the triangle inequality, we can reduce the situation to the case that a is the nearest point to x on $\partial\hat{K}$. Write $\delta = |x - a| = dist(x, \partial\hat{K})$. For $j = 1, 2, ...$ consider the sets

$$\Gamma_j = \{y \in \partial\hat{K} : (j - 1)\delta \leq |y - a| < j\delta\}.$$

Note that $\Gamma_j \neq \emptyset$ only for $j \leq \frac{d}{\delta}+1$, where $d = diam\partial\hat{K}$.

If Ω is a domain in $\mathbb{C}$ and $E \subset \partial\Omega$ is a Borel set, write $\omega(y,\Omega,E)$ for the harmonic measure of E at the point y relative to Ω. Put $\omega_j = \omega(x,\hat{K}^o,\Gamma_j)$. Let k_0 be such that $k_0\delta < \eta \leq (k_0+1)\delta$. Then for $k = 1,2,...,k_0$ we claim that

$$\sum_{j>k} \omega_j \leq \frac{A}{\sqrt{k}} \tag{6}$$

with $A \geq 1$ an absolute constant.

For $k=1$, (6) is immediate because $\sum_{j>1}\omega_j \leq 1$. For $2 \leq k \leq k_0$ consider the connected component D_k of $\{y \in \hat{K}^o : |y-a| \leq k\delta\}$ which contains x. Note that by construction D_k is simply connected. From the properties of harmonic measure it follows that

$$\sum_{j>k} \omega_j \leq \omega(x, D_k, S_k) \tag{7}$$

where $S_k = \{y \in \partial D_k : |y-a| = k\delta\}$.

Consider a linear mapping from the disc $\{y \in \mathbb{R}^2 : |y-a| \leq k\delta\}$ onto $\{t \in \mathbb{R}^2 : |t| \leq 1\}$. Let D'_k be the image of D_k under this mapping, S'_k the image of S_k, and t the image of x. Then $|t| = \frac{1}{k}$ and according to the theorem of Milloux-Carleman ([18, pp. 347-350]), we have

$$\omega(x, D_k, S_k) = \omega(t, D'_k, S'_k) \leq \frac{2}{\pi}\left(\frac{\pi}{2} - arcsin\left(\frac{1-\frac{1}{k}}{1+\frac{1}{k}}\right)\right). \tag{8}$$

Hence, if $k \leq k_0$, then $\omega(x, D_k, S_k) \leq \frac{A}{\sqrt{k}}$ and thus (6) is proved.

Now from the definition of harmonic measure and the maximum principle we can get

$$|F(x) - F(a)| \leq \sum_{j=1}^{J} (j\delta)^\alpha \omega_j \tag{9}$$

where J is the integer part of $\frac{d}{\delta}+1$. Indeed, consider the function $G(y) = F(y) - F(a)$ which belongs to $a(\hat{K})$ and satisfies

$$|G(y)| = |f(y) - f(a)| \leq (j\delta)^\alpha$$

for $y \in \Gamma_j$. Then for all $y \in \hat{K}^o$

$$|G(y)| \leq \sum_{j=1}^{J} (j\delta)^\alpha \omega(y, \hat{K}^o, \Gamma_j),$$

and (9) follows from the last inequality by replacing y with x.

Let us maximize the sum in (9) as a function of $\omega = (\omega_1, \omega_2, ..., \omega_J)$ under the restrictions

$$\begin{aligned} \omega_1 &\le 1, \\ \omega_2 + \omega_3 + ... + \omega_J &\le \frac{A}{\sqrt{1}}, \\ \omega_3 + \omega_4 + ... + \omega_J &\le \frac{A}{\sqrt{2}}, \\ &\vdots \\ \omega_{k_0} + \omega_{k_0+1} + ... + \omega_J &\le \frac{A}{\sqrt{k_0 - 1}}, \\ \omega_{k_0+1} + \omega_{k_{0+2}} + ... + \omega_J &\le \frac{A}{\sqrt{k_0}}, \end{aligned}$$

and $\omega_j \ge 0$ $(j = 1, 2, ..., J)$. It is not difficult to see that the maximun is attained at $\omega_1 = 1$, $\omega_j = \frac{A}{\sqrt{j-1}} - \frac{A}{\sqrt{j}}$ for $j = 2, 3, ..., k_0$ and ω_j for $j = k_0, ..., J$ verifies the last restriction with the equality. Finally, from (9) we get

$$\begin{aligned} |F(x) - F(a)| &\le \sum_{j=1}^{J} (j\delta)^\alpha \omega_j \le \sum_{j=1}^{k_0} (j\delta)^\alpha \frac{A}{j^{\frac{3}{2}}} + \sum_{j=k_0+1}^{J} (j\delta)^\alpha \omega_j \\ &\le \sum_{j=1}^{k_0} (j\delta)^\alpha \frac{A}{j^{\frac{3}{2}}} + \frac{A(diam\hat{K})^\alpha}{\sqrt{k_0}} \le A(\alpha)\delta^\alpha + \frac{A(diam\hat{K})^\alpha}{\sqrt{\frac{\eta}{3\delta}}} \\ &\le \left(A(\alpha) + \frac{A(diam\hat{K})^\alpha}{\sqrt{\frac{\eta}{3}}} \delta^{\frac{1}{2}-\alpha} \right) \delta^\alpha = A(\alpha, K, \Omega)\delta^\alpha. \end{aligned}$$

□

Minor adjustments to the proof of above theorem prove the following result.

Theorem 4. *Suppose that under the conditions of the above theorem we also have* $f \in lip(\alpha, \partial\hat{K})$. *Then* F *belongs to* $lip(\alpha, \hat{K})$.

Proof. Without loss of generality we can suppose that there exists a function $\varepsilon(\delta)$ non-decreasing, such that $\varepsilon(\delta) \to 0$ as $\delta \to 0$ and

$$|f(x) - f(y)| \le \varepsilon(|x - y|)|x - y|^\alpha$$

for any $x, y \in \partial\hat{K}$. Now repeating the proof of the above theorem we get

$$\begin{aligned} |F(x) - F(a)| &\le \sum_{j=1}^{J} \varepsilon(j\delta)(j\delta)^\alpha \omega_j \le \sum_{j=1}^{k_0} \varepsilon(j\delta)(j\delta)^\alpha \frac{A}{j^{\frac{3}{2}}} + \sum_{j=k_0+1}^{J} \varepsilon(j\delta)(j\delta)^\alpha \omega_j \\ &\le \sum_{j=1}^{k_0} \varepsilon(j\delta)(j\delta)^\alpha \frac{A}{j^{\frac{3}{2}}} + \frac{A\varepsilon(diam\hat{K})(diam\hat{K})^\alpha}{\sqrt{k_0}} \\ &\le \sum_{j=1}^{k_0} A\delta^\alpha \varepsilon(j\delta) j^{(\alpha - \frac{3}{2})} + A\varepsilon(diam\hat{K})\delta^{\frac{1}{2}-\alpha}\delta^\alpha = \delta^\alpha \varepsilon_1(\delta) \end{aligned}$$

where $\varepsilon_1(\delta) \to 0$ as $\delta \to 0$. □

Our main result is the following:

Theorem 5. *Let F be a relatively closed subset of a domain $G \subset \mathbb{C}$ and $0 \leq \alpha < \frac{1}{2}$. Then the following statements are equivalent:*

a) *All functions of $a_\alpha(F)$ can be LE-approximated by functions in $h(G)$.*
b) *All functions of $a_\alpha(F)$ can be approximated in Lipα-norm on F by functions in $h(G)$.*
c) i) $\partial F = \partial \hat{F}$.
  ii) *The holes of F satisfy the long islands condition.*
  iii) $G^\infty \setminus \hat{F}$ *is locally connected at* $\{\infty\}$.

Proof. a) $\Rightarrow$ b) is trivial.

b) $\Rightarrow$ c) For $\alpha = 0$, see [12, Theorem 6] and [15, Corollary 1] (the conditions ii) and iii) together are equivalent to (G, F) satisfies the (K, L)-condition defined in [13]). We consider now the case $0 < \alpha < \frac{1}{2}$. If $a_\alpha(F) = [h(G)]^*_{\alpha,F}$, it is readily seen that $h(F) \subset [h(G)]^*_{\alpha,F}$ and therefore $h(F) \subset [h(G)]^*_{0,F}$. Thus (G, F) is a uniform Runge pair for harmonic functions, i.e. all functions of $h(F)$ can be uniformly approximated by functions from $h(G)$. Now by [12] and [4] the conditions i), ii) and iii) hold.

c) $\Rightarrow$ a) Suppose now that i), ii) and iii) hold. Since $G^\infty \setminus \hat{F}$ is connected, for each point $z \in \hat{F}$ we choose a disk $U_z \subset G$ such that the complement of $\overline{U}_z \cap \hat{F}$ is connected. Take an exhausting sequence $\{G_n\}$ of G, where each G_n is a bounded domain and ∂G_n consists of finitely many Jordan curves. Now suppose $z \in \hat{F}_n = \hat{F} \cap \overline{G_n}$ and choose a disk $V_z \subset U_z$ such that the complement of $(\overline{V}_z \cap \hat{F}_n)$ is connected and hence $f_{|\overline{V}_z \cap \hat{F}_n}$ is approximable in *lip*α-norm by harmonic functions in a neighborhood of $\overline{V}_z \cap \hat{F}_n$ ([20]). Besides, by taking into account the localization theorem given by P. Paramonov and J. Verdera [21], we have that f can be approximated in *Lip*α-norm by harmonic functions in a neighborhood of $\hat{F}_n$. Now, from Proposition 1 and Theorem 1, f can be approximated in *Lip*α-norm on $\hat{F}$ by functions in $m_{\hat{F}}(G)$. Thus, since $G^\infty \setminus \hat{F}$ is connected and locally connected at $\{\infty\}$, by Theorem 2, one has that every function in $a_\alpha(\hat{F})$ can be *LE*-approximated by functions in $h(G)$.

The proof will be complete if it is proved that under these conditions every function in $a_\alpha(F)$ can be extended to a function in $a_\alpha(\hat{F})$. But the conditions i), ii) and iii) imply that $a_0(F) = [h(G)]^*_{0,F}$ and consequently all functions in $a_0(F)$ can be extended to a function in $a_0(\hat{F})$ (see, [12], [4]). Hence all functions f in $a_\alpha(F)$ can be extended to a function in $a_0(\hat{F}) \cap a_\alpha(F)$. On the other hand, the condition ii) guarantees that there exists a sequence $\{K_n\}$ of compact sets verifying:

a) $\hat{F} = \bigcup K_n$, $K_n \subset K_{n+1}$,
b) $\partial K_n \cap (\hat{F} \setminus F) = \emptyset$.

Hence $f \in a_0(K_n) \cap a_\alpha(\partial K_n)$, and if $0 < \alpha < \frac{1}{2}$ from Theorem 4, we prove that $f \in a_\alpha(K_n)$ which implies $f \in a_\alpha(\hat{F})$. □

Remark that if $\frac{1}{2} < \alpha < 1$, then Theorem 5 is not true. It is still an open question if it holds for $\alpha = \frac{1}{2}$. Also observe that for $\mathbb{R}^n$, with $n \geq 3$ and $\alpha \in [0, 1]$, there is no purely geometric criterion in order for $a_\alpha(F)$ to coincide with the closure of harmonic functions on G in the norm $\|.\|^*_{\alpha,F}$ (see [22] for details).

Finally, an immediate consequence of the proof of Theorem 5 is the next result of decomposition for the class $[h(G)]^*_{\alpha,F}$.

Corollary 3. *Let F be a relatively closed subset of G such that $G^\infty \setminus F$ is connected and locally connected at $\{\infty\}$ and $0 \le \alpha < 1$. Then $[h(G)]^*_{\alpha,F} = a_{\alpha u}(F) + h(G)$, i.e. if $\varepsilon > 0$ and $g \in [h(G)]^*_{\alpha,F}$, there exist $g_1 \in a_{\alpha u}(F)$ and $g_2 \in h(G)$ such that $\|g_1\|^*_{\alpha,F} < \varepsilon$ and $g = g_1 + g_2$ on F.*

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