PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 129, Number 9, Pages 2591–2595
S 0002-9939(01)05876-2
Article electronically published on February 9, 2001

ON MINIMAL LENGTHS OF EXPRESSIONS OF COXETER GROUP ELEMENTS AS PRODUCTS OF REFLECTIONS

MATTHEW J. DYER

(Communicated by John R. Stembridge)

ABSTRACT. It is shown that the absolute length $l'(w)$ of a Coxeter group element w (i.e. the minimal length of an expression of w as a product of reflections) is equal to the minimal number of simple reflections that must be deleted from a fixed reduced expression of w so that the resulting product is equal to e, the identity element. Also, $l'(w)$ is the minimal length of a path in the (directed) Bruhat graph from the identity element e to w, and $l'(w)$ is determined by the polynomial $R_{e,w}$ of Kazhdan and Lusztig.

INTRODUCTION

Let (W, S) be a Coxeter system and let $T := \bigcup_{w \in W} wSw^{-1}$ denote the corresponding set of reflections. For $w \in W$, define $l(w)$ (resp., $l'(w)$) to be the minimal number of factors occuring amongst all expressions of w as a product of simple reflections S (resp., reflections). The function l (resp., l') is called the standard (resp., absolute) length function of (W, S). An expression $w = s_1 \cdots s_n$ with $s_i \in S$ and $n = l(w)$ is called a reduced expression for w.

The standard length function has been extensively studied (see e.g. [2], [11] as general references on Coxeter groups). In case W is a finite reflection group in its natural reflection representation, $l'(w)$ is the codimension of the 1-eigenspace of w (see [4, Lemmas 1–3], which do not use the crystallographic condition), and $\sum_{w \in W} X^{l'(w)} = \prod_{i=1}^{n}(1 + (d_i - 1)X)$ where $d_1, \ldots, d_n$ are the degrees of W ([13]; see also [1] for other natural interpretations of this polynomial). However, this geometric description of $l'(w)$ does not extend to arbitrary infinite W, for the function l' may be unbounded even if S is finite.

In this note, we show that $l'(w)$ is equal to the minimum number of simple reflections that must be deleted from a fixed reduced expression for w so that the resulting product of simple reflections is equal to the identity element e of W. Moreover, $l'(w)$ can also be characterized in terms of combinatorial data related to the well-known Chevalley-Bruhat order (namely the Bruhat graph [8] and the R-polynomial $R_{e,w}$ of [12]). The R-polynomials have been studied quite extensively (cf. [12], [5], [10], [3] for example) because of their close connections to Kazhdan-Lusztig polynomials, which are of interest for their significant applications to Lie representation theory and related geometry and combinatorics.

I wish to thank Alex Hahn for questions which stimulated this note.

Received by the editors August 23, 1999 and, in revised form, January 27, 2000.
2000 *Mathematics Subject Classification.* Primary 20F55, 22E47, 06A07.

1. Statement of results

Theorem 1.1. *Let $w = s_1 \dots s_n$ be a reduced expression for $w \in W$. Then $l'(w)$ is the minimum of the natural numbers p for which there exist $1 \le i_1 < \dots < i_p \le n$ such that $e = s_1 \cdots \widehat{s_{i_1}} \cdots \widehat{s_{i_p}} \cdots s_n$, where a hat over a factor indicates its omission.*

Write $t_i = s_1 \cdots s_{i-1} s_i s_{i-1} \cdots s_1$, and fix an expression for e as in the theorem, with $p = l'(w)$. Then $t_{i_1} \cdots t_{i_p} w = e$, or equivalently $w = t_{i_p} \cdots t_{i_1}$. Thus, $l'(w)$ and an expression of w as a product of $l'(w)$ reflections are effectively computable.

Define the "Bruhat graph" $\Omega = \Omega_{(W,S)}$ to be the directed graph with vertex set W and an edge from $x \in W$ to $y \in W$ iff $l(y) > l(x)$ and there is a reflection t such that $y = tx$ (see [8][1]). It is often convenient to regard Ω as an edge-labelled directed graph, by attaching the label $t \in T$ to an edge (x, tx). Let $\bar{\Omega}$ be the underlying undirected graph of Ω. A path of length n in Ω (resp., $\bar{\Omega}$) is a sequence $x_0, \dots, x_n$ in W such that for each i there is a directed (resp., undirected) edge of Ω (resp., $\bar{\Omega}$) from x_{i-1} to x_i. Note that $l'(x)$ is the minimal length of a path in $\bar{\Omega}$ from e to x, where e is the identity element of W.

The Chevalley-Bruhat order $\le$ on W may be defined by $x \le y$ if there is a path in Ω from x to y; equivalently, if $y = s_1 \cdots s_n$ is a reduced expression for y, then $x \le y$ iff $x = s_1 \cdots \widehat{s_{i_1}} \cdots \widehat{s_{i_p}} \cdots s_n$ for some $1 \le i_1 < \ldots < i_p \le n$.

Theorem 1.2. *For any $w \in W$, $l'(w)$ is the minimal length of a path in Ω from e to w.*

Let $\mathbb{Z}[\alpha]$ be the polynomial ring in an indeterminate α. It is known that there are polynomials $\tilde{R}_{x,y} = \tilde{R}_{x,y}(\alpha) \in \mathbb{Z}[\alpha]$ for $x, y \in W$ uniquely determined by the initial conditions $\tilde{R}_{x,y} = 0$ unless $x \le y$ and $\tilde{R}_{x,x} = 1$, and the following recurrence relation: for $x, y \in W$ and $s \in S$ with $l(sy) < l(y)$,

$$\tilde{R}_{x,y} = \begin{cases} \tilde{R}_{sx,sy} & \text{if } l(sx) < l(x), \\ \alpha \tilde{R}_{x,sy} + \tilde{R}_{sx,sy} & \text{if } l(sx) > l(x). \end{cases}$$

In fact, $q^{(l(y)-l(x))/2} \tilde{R}_{x,y}(q^{\frac{1}{2}} - q^{-\frac{1}{2}}) = R_{x,y}$ where $R_{x,y}$ is as defined in [12].

Theorem 1.3. *For $w \in W$, $l'(w)$ is the smallest integer m such that the coefficient of α^m in $\tilde{R}_{e,w}$ is non-zero.*

The subgroups of W generated by subsets of the simple reflections are called standard parabolic subgroups, and their conjugates in W are called parabolic subgroups. We record the following simple consequence of the theorems above.

Corollary 1.4. *If W' is a parabolic subgroup of W, then for all $w \in W'$, $l'(w)$ is equal to the minimal number of factors occuring amongst expressions for w as a product of elements of $W' \cap T$.*

Proof. If W' is a standard parabolic subgroup the corollary is clear from Theorem 1.1. The general case follows since the function l' is constant on conjugacy classes. ∎

[1] We take this opportunity to remove an unintended and unnecessary restriction in [8]; the initial sentence of [8, §1] should begin "Let (W, R) be a Coxeter system...".

2. Proof of the main results

We will deduce both Theorems 1.1 and 1.3 using 1.2. We begin by recalling from [7], [8] (cf. also [6], in which some of the proofs are simpler) certain properties of reflection subgroups of W which are used in the proofs here.

For a reflection subgroup W' of W (i.e. W' is generated by $W' \cap T$), let

$$\chi(W') := \{\, t \in W' \cap T \mid l(t't) > l(t) \text{ for all } t' \in W' \cap T \text{ with } t' \neq t \,\}.$$

Theorem 2.1. *For any reflection subgroup W' of W, $S' = \chi(W')$ is a set of Coxeter generators of W', and the set of reflections of (W', S') is $W' \cap T$. Moreover, for any $x \in W$, the coset $W'x$ contains a unique element x_0 with $l(x_0)$ minimal, and then the map $v \mapsto vx_0 \colon W' \mapsto W'x$ induces an isomorphism of directed labelled graphs between $\Omega_{(W',S')}$ and the full subgraph $\Omega_{(W,S)}(W'x)$ of $\Omega_{(W,S)}$ on vertex set $W'x$.*

We say $(W', \chi(W'))$ above is a dihedral reflection subsystem of (W, S) if W' can be generated by two distinct reflections, or, equivalently by [7], if $\chi(W')$ has exactly two elements.

The following lemma immediately establishes Theorem 1.2.

Lemma 2.2. *If there is a path from e to $x \in W$ in $\bar{\Omega}$ of length n, then there is a path from e to x in Ω of length n' for some $n' \leq n$.*

Proof. To prove the lemma, consider a path $e = x_0, x_1, \ldots, x_n = x$ in $\bar{\Omega}$ of length n from e to x. We proceed by induction on n, and then for $n > 0$ by induction on $m := l(x_{n-1})$. Replacing n by a smaller integer if necessary, we may assume by the inductive hypothesis that $n \geq 2$ and $x_0, \ldots, x_{n-1}$ is a path in Ω. We also assume without loss of generality that $x_n \neq x_{n-2}$ and $l(x_n) < l(x_{n-1})$.

There exist distinct reflections t and t' so $x_n = tx_{n-1}$ and $x_{n-1} = t'x_{n-2}$. Let W' be the dihedral reflection subgroup of W generated by t and t', $S' := \chi(W')$ and $\Omega' := \Omega_{(W,S)}(W'x_{n-1})$. By Theorem 2.1 and inspection of Bruhat graphs of dihedral Coxeter systems (cf. [8, (1.2)]) it follows that there is a path $x_0, \ldots, x_{n-2}, x'_{n-1}, x_n$ in $\bar{\Omega}$ with $x'_{n-1} \in W'x_{n-1}$ which is either a path in Ω (as required) or else such that there is an edge of Ω from x'_{n-1} to x_{n-2}. In the latter case, we have $l(x'_{n-1}) < l(x_{n-2}) < l(x_{n-1}) = m$ so we are done by induction on m. □

Proof of Theorem 1.1. Let $w \in W$ have reduced expression $s_1 \cdots s_n$, and define the reflection $t_i = s_1 \cdots s_i \cdots s_1$ for each i. If $e = s_1 \cdots \widehat{s_{i_1}} \cdots \widehat{s_{i_p}} \cdots s_n$, then $t_{i_1} \cdots t_{i_p} w = e$ so $w = t_{i_p} \cdots t_{i_1}$ and $p \geq l'(w)$. On the other hand, by Theorem 1.2 there is a directed path $x_0 = e, x_1, \ldots, x_p = w$ of length $p = l'(w)$ in Ω. The strong exchange condition [11] immediately implies that if x has an expression $x = r_1 \cdots r_m$ as a product of simple reflections and there is an edge in Ω from y to x, then $y = r_1 \ldots \widehat{r_i} \cdots r_m$ for some i. It follows immediately by descending induction on i that x_i has an expression obtained by deleting $p - i$ simple reflections from the reduced expression $s_1 \ldots s_n$ for w. For $i = 0$, this is what we had to show. □

Finally, Theorem 1.3 is a consequence of Theorem 1.2 and the following fact from [6]; we provide a more conceptual proof here.

Theorem 2.3. *The coefficient of α^n in the polynomial $\tilde{R}_{x,y}$ is non-zero iff there is a path of length n from x to y in Ω.*

Proof. Recall from [10] that a reflection order of a dihedral Coxeter system (W', S') is a total order $\preceq$ of its reflections such that if $S' = \{r, s\}$ with $r \prec s$, then $r \prec rsr \prec rsrsr \prec \ldots \prec srs \prec s$. A reflection order of an arbitrary Coxeter system (W, S) is a total order of the set T of its reflections which restricts to a reflection order on the set of reflections of each dihedral reflection subsystem $(W', \chi(W'))$ of (W, S); such reflection orders exist by [10, (2.3)].

We deduce Theorem 2.3 here from the formula [10, (3.4)] for $\tilde{R}_{x,y}$ as the generating function for the set of paths from x to y in Ω with increasing label. Fix a reflection order $\preceq$ of (W, S). Then

$$\tilde{R}_{x,y} = \sum_{n \in \mathbb{N}} \sum_{(t_1, \ldots, t_n)} \alpha^n \tag{2.1}$$

where the inner sum is over those $(t_1, \ldots, t_n) \in T^n$ such that $x, t_1 x, \ldots, t_n \cdots t_1 x = y$ is a path in Ω and $t_1 \prec t_2 \prec \cdots \prec t_n$.

Hence if the coefficient of α^n in $\tilde{R}_{x,y}$ is non-zero, there is a path of length n from x to y in Ω. Conversely, if there is a path $x = x_0, \ldots, x_n = y$ of length n from x to y in Ω, one may write $x_i = t_i \cdots t_1 x$ for some reflections $t_1, \ldots, t_n$ in W. One may suppose the "label" $(t_1, \ldots, t_n) \in T^n$ is chosen to be minimal (in the lexicographic order on T^n induced by $\preceq$) amongst the labels of all paths of length n from x to y in Ω. For each $i = 1, \ldots, n-1$, let W_i be the dihedral reflection subgroup of W generated by t_i and t_{i+1}, and let $S_i = \chi(W_i)$. Let y_i be the element of $W_i x_i$ with minimal length $l(y_i)$. The restriction of $\preceq$ to a total order on $W_i \cap T$ is a reflection order of (W_i, S_i). Now by Theorem 2.1, (t_i, t_{i+1}) is the lexicographically first label (in $(W_i \cap T)^2$) of length two paths in $\Omega_{(W_i, S_i)}$ from $x_{i-1} y_i^{-1}$ to $x_{i+1} y_i^{-1}$. An examination of Bruhat graphs for dihedral groups shows that $t_i \prec t_{i+1}$ for each i. Hence $t_1 \prec \cdots \prec t_n$ and the coefficient of α^n in $\tilde{R}_{x,y}$ is non-zero as required. □

Remark 2.4. By Theorem 2.3 and either [3, 6.1] or [5, 1.3 and 5.3], or by the proof of [8, (3.3)], it follows that if $x \leq y$, then there is an integer $m \equiv l(y) - l(x) \pmod 2$ such that there is a path of length n from x to y in Ω iff $m \leq n \leq l(y) - l(x)$ and $n \equiv l(y) - l(x) \pmod 2$. If $x = e$, then $m = l'(y)$.

Remark 2.5. The analogue of (2.1) holds for the R-functions of the "twisted Bruhat orders" of [9]; Theorem 2.3 and its proof, and Remark 2.4, extend to these orders mutatis mutandis.

References

[1] H. Barcelo and A. Goupil, *Combinatorial aspects of the Poincaré polynomial associated with a reflection group*, Jerusalem Combinatorics '93 (Providence, R.I.), Contemp. Math., vol. 178, Amer. Math. Soc, 1994, pp. 21–44. MR **96e:**20061

[2] N. Bourbaki, *Groupes ét algebres de Lie, Ch. 4–6*, Hermann, Paris, 1964.

[3] F. Brenti, *A combinatorial formula for Kazhdan-Lusztig polynomials*, Invent. Math **118** (1994), 371–394. MR **96c:**20074

[4] R. W. Carter, *Conjugacy classes in the Weyl group*, Comp. Math. **25** (1972), 1–59. MR **47:**6884

[5] V. V. Deodhar, *On some geometric aspects of Bruhat orderings. I. a finer decomposition of Bruhat cells*, Invent. Math. **79** (1985), 499–511. MR **86f:**20045

[6] M. J. Dyer, *Hecke algebras and reflections in Coxeter groups*, Ph.D. thesis, Univ. of Sydney, 1987.

[7] ———, *Reflection subgroups of Coxeter systems*, J. of Alg. **135** (1990), 57–73. MR **91j:**20100

[8] ———, *On the "Bruhat graph" of a Coxeter system*, Comp. Math. **78** (1991), 185–191. MR **92c:**20076

[9] ______, *Hecke algebras and shellings of Bruhat intervals II: twisted Bruhat orders*, Kazhdan-Lusztig theory and related topics (V. V. Deodhar, ed.), Contemp. Math., vol. 139, 1992, pp. 141–165. MR **94c:**20072

[10] ______, *Hecke algebras and shellings of Bruhat intervals*, Comp. Math. **89** (1993), 91–115. MR **95c:**20053

[11] J. E. Humphreys, *Reflection groups and Coxeter groups*, Cambridge Studies in Advanced Mathematics, no. 29, Camb. Univ. Press, Cambridge, 1990. MR **92h:**20002

[12] D. Kazhdan and G. Lusztig, *Representations of Coxeter groups and Hecke algebras*, Invent. Math. **53** (1979), 165–184. MR **81j:**20066

[13] L. Solomon, *Invariants of finite reflection groups*, Nagoya Math. J. **22** (1963), 57–64. MR **27:**4872

Department of Mathematics, University of Notre Dame, Room 370 CCMB, Notre Dame, Indiana 46556-5683

E-mail address: `Dyer.1@nd.edu`