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INTEGRAL CLOSURE OF A CUBIC EXTENSION AND APPLICATIONS

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Abstract. In this paper, we compute the integral closure of a cubic extension over a Noetherian unique factorization domain. We also present some applications to triple coverings and to rank two reflexive sheaves over an algebraic variety.

0. Introduction

In commutative algebra and algebraic geometry, it is a fundamental problem to compute the integral closure of a finite extension over a commutative ring R. The corresponding geometric problem is the normalization of a finite covering over an algebraic variety. The integral closure of a quadratic extension (or the normalization of a double covering) is well known. But for a cubic extension, the computation is non-trivial. In 1991, Shapiro and Sparer [ShS] computed the minimal integral bases for cubic extensions over integral ring $\mathbb{Z}$.

R. Miranda [Mir] establishes the fundamental theory of triple coverings. He proves that a flat triple covering is determined by a rank two vector bundle E and a morphism $\Phi : S^3E \to \Lambda^2 E$. In order to apply Miranda's theory as effectively as double coverings, we need to simplify these data from vector bundles to line bundles. It is easy to find this simplification (see Proposition 3.3), but the difficulty is shifted to the computation of normalization.

In the first two sections, we compute the integral closure of a cubic extension ring over a Noetherian unique factorization domain. As a direct application of the computations, we study triple coverings starting from some simple data as in double coverings. Roughly speaking, the new data are two global sections (a, b) of some invertible sheaves whose divisors satisfy $3\operatorname{div}(a) \equiv 2\operatorname{div}(b)$, or equivalently, 3 coprime global sections (A, B, C) of an invertible sheaf satisfying $A + B = C$. Any triple covering $\pi : Y \to X$ can be constructed from these data (a, b). In fact, Y is just the normalization of the cubic extension of X defined by $z^3 + az + b = 0$ (see Sect. 3 for the details). Then we try to find out some important invariants of Y directly from a and b (e.g., the sheaf $\pi_*(\mathcal{O}_Y)$, the branch locus). In the last section,

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we give an application to rank two reflexive sheaves. Based on the results of this paper, we find the "canonical resolution" of the singularities of a triple covering over a surface [Ta2] which is almost the same as in double coverings.

1. The statement of the main result

Let R be a Noetherian unique factorization domain (UFD) such that 2 and 3 are units, and let K be its fraction field. Let $p(x) = x^3 + ax + b$ be an irreducible polynomial over R, and let $A = R[x]/(p(x))$. Then $A = R[\alpha]$ is an integral domain, where

$$\alpha^3 + a\alpha + b = 0.$$

Our purpose in this section is to find the integral closure B of A in its fraction field $K[\alpha]$. Because $K[\alpha] \supseteq B$ is a cubic extension field of K, for any element $\beta \in K[\alpha]$, we can define its trace $\mathrm{tr}\,(\beta) \in K$. If $\beta \in B$, then $\mathrm{tr}\,(\beta) \in A$. We denote by B_0 the trace-free R-submodule of B. Then we have $B = A \oplus B_0$.

First we give some notation. We denote by $\delta = 4a^3 + 27b^2$ the discriminant of $p(x)$. If p is a prime in R and $x \in R$, then we shall denote by x_p the highest power of p in x, i.e., $x = x_0 \prod_p p^{x_p}$ for some unit x_0 of R. $[s/t]$ is the maximal integer $\leq s/t$.

Now we set

$$\varepsilon_p = 3a_p - 2b_p, \qquad e_p = \min\left\{ \left[\frac{a_p}{2}\right], \ \left[\frac{b_p}{3}\right] \right\}.$$

Let $e = \prod_p p^{e_p}$, and let

$$m_0 = \prod_{\substack{\varepsilon_p > 0 \\ \varepsilon_p \equiv 1\,(3)}} p, \quad m_1 = \prod_{\substack{\varepsilon_p < 0 \\ \varepsilon_p \equiv 1\,(2)}} p, \quad m_2 = \prod_{\substack{\varepsilon_p > 0 \\ \varepsilon_p \equiv 2\,(3)}} p, \quad m_3 = \prod_{\substack{\varepsilon_p = 0 \\ \delta_p \equiv 1\,(2)}} p. \tag{1.1}$$

Then we have the factorizations

$$a = m_0 m_1 m_2^2 e^2 \bar{a}, \quad b = m_0 m_1^2 m_2^2 e^3 \bar{b}, \quad (\bar{a}, \bar{b}) = 1, \tag{1.2}$$

$$4 m_0 m_2^2 \bar{a}^3 + 27 m_1 \bar{b}^2 = m_3 \bar{\delta}^2, \tag{1.3}$$

where $\delta = m_0^2 m_1^3 m_2^4 e^6 (4 m_0 m_2^2 \bar{a}^3 + 27 m_1 \bar{b}^2)$, so $\bar{\delta} = \delta_0 \prod_{\varepsilon_p = 0} p^{[\frac{\delta_p}{2}] - b_p}$ for some unit δ_0.

We define

$$f_1 = \frac{2}{3} m_2 \bar{a}, \quad f_2 = \bar{b}, \quad f_3 = \bar{\delta}. \tag{1.4}$$

It is easy to see that f_1, f_2 and f_3 are pairwise coprime.

Now we state our main result.

Theorem 1.1. *Assume that B is the integral closure of $R[\alpha]$ and B_0 is its trace-free R-submodule. Then $B = R \oplus B_0$. If $a \neq 0$, then*

$$B_0 = \left\{ \frac{u}{f_3}\gamma + \frac{v}{f_3}\beta \,\middle|\, u, v \in R \text{ and } f_3 \mid f_1 u + f_2 v \right\},$$

where $\gamma = \alpha/e$ and $\beta = (3\alpha^2 + 2a)/3m_1 m_2 e^2$.

If $a = 0$ and b has a factorization $b = e^3 m_1^2 m_0$, where m_0 and m_1 are square-free, then B has a base

$$1, \quad \frac{\alpha}{e}, \quad \frac{\alpha^2}{e^2 m_1}.$$

In fact, the second part of the previous theorem can be viewed as a special case of the first part ($m_2 = 1, \bar{a} = 0$ and f_3 is a unit). Because the proof of the second part is trivial, in what follows, we always assume that $a \neq 0$.

From this theorem, we obtain

Corollary 1.2. *B_0 is isomorphic to the syzygy module of (f_1, f_2, f_3); namely we have an exact sequence of R-modules*

$$0 \to B_0 \to R^{\oplus 3} \xrightarrow{f} R,$$

where f is defined by $f(u, v, w) = f_1 u + f_2 v + f_3 w$. Hence B_0 is reflexive, i.e., $B_0 \cong B_0^{**}$.

Proof. The first part is obvious. The second part is well known (cf. [Ha3]). □

If R is a principal ideal domain, then the computation of the syzygies is not difficult, and $B_0 \cong R \oplus R$. In fact, there exist $s, t \in R$ such that $f_1 s + f_2 t = 1$. Then B_0 is generated by the following two syzygies:

$$(1 + f_1(f_3 - 1)s,\ f_1(f_3 - 1)t,\ -f_1), \quad (f_2(f_3 - 1)s,\ 1 + f_2(f_3 - 1)t,\ -f_2).$$

If R is a two dimensional regular local ring, we have also $B_0 \cong R \oplus R$, because in this case reflexive implies locally free (cf. [Ha3]).

The following definition is analogous to the one given for elliptic curves over $\mathbb{Z}$ (cf. [Lan], §4).

Definition 1.3. The pair (a, b) or the polynomial $p(x) = x^3 + ax + b$ is called *minimal* if $e = 1$, i.e., there is no prime p such that $p^2 | a$ and $p^3 | b$.

Two minimal pairs (a', b') and (a, b) are said to be equivalent if there is an invertible element $e_0 \in R$ such that $a' = e_0^2 a$ and $b' = e_0^3 b$.

Remark 1.4. Assume that R is an algebra over an algebraically closed field k such that $\operatorname{char}(k) \neq 2, 3$. Then there is a one-to-one correspondence between the following two sets (up to equivalence):

$$\{\text{Minimal pairs } (a, b)\} \longleftrightarrow \{\text{Coprime triples } (A, B, C) \text{ with } A + B = C\}.$$

Two such triples (A', B', C') and (A, B, C) are said to be equivalent if there is an invertible element $e_0 \in R$ such that $A' = e_0 A$, $B' = e_0 B$ and $C' = e_0 C$.

For a given minimal (a, b), we define

$$A = 4 m_0 m_2^2 \bar{a}^3, \quad B = 27 m_1 \bar{b}^2, \quad C = m_3 \bar{\delta}^2. \tag{1.5}$$

Hence A, B and C are pairwise coprime and satisfy (see (1.2)) $A + B = C$.

Conversely, for a given coprime triple (A, B, C) with $A + B = C$, we define

$$m_0 = \prod_{A_p \equiv 1\,(3)} p, \quad m_1 = \prod_{B_p \equiv 1\,(2)} p, \quad m_2 = \prod_{A_p \equiv 2\,(3)} p, \quad m_3 = \prod_{C_p \equiv 1\,(2)} p,$$

$$\bar{a} = 2^{-\frac{2}{3}} \prod_p p^{[\frac{A_p}{3}]}, \quad \bar{b} = 3^{-\frac{3}{2}} \prod_p p^{[\frac{B_p}{2}]}, \quad \bar{\delta} = \prod_p p^{[\frac{C_p}{2}]}.$$

Therefore we get a minimal pair (a, b) by (1.2) (here $e = 1$). This is the correspondence we wanted.

Lemma 1.5. *Let $\tilde{\alpha} = \alpha/e \in K[\alpha]$, $\tilde{a} = a/e^2$ and $\tilde{b} = b/e^3$. Then $\tilde{\alpha}^3 + \tilde{a}\tilde{\alpha} + \tilde{b} = 0$, and $R[\bar{\alpha}]$ has the same integral closure as $R[\alpha]$.*

Proof. Trivial. □

Thus the minimal case is essential in cubic extensions.

2. The proof of Theorem 1.1

Due to Lemma 1.5, we assume in this section that (a, b) is minimal, i.e., $e = 1$.

We shall say that $A = R[\alpha]$ is non-normal (resp. singular) over a prime p of R if there is a prime ideal q of A over p such that A_q is non-normal (resp. singular). We also say that p is contained in the non-normal (resp. singular) locus of A. Miranda classified in ([Mir], Lemma 5.1) the codimension one singular locus of A:

Lemma 2.1. *Let p be a prime in R. Then A is singular over p if and only if*

1) *$p \mid a$ and $p^2 \mid b$, or*
2) *$p \nmid a$ but $p^2 \mid \delta$ (hence $p \nmid b$),*

where $\delta = 4a^3 + 27b^2$ is the discriminant of $p(x)$.

Proof. Reduce to the localization of R at the prime ideal (p) and then use ([Mir], Lemma 5.1). □

Note that the codimension one singular locus of A is exactly the non-normal locus of A, because A is flat over R. Now we see that the primes $p \in R$ contained in the non-normal locus of A can be divided into the following types I to III:

0 $a_p \geq 1$, $b_p = 1$, $(\delta_p = 2)$;
I $a_p = 1$, $b_p \geq 2$, $(\delta_p = 3)$;
II $a_p \geq 2$, $b_p = 2$, $(\delta_p = 4)$;
III $a_p = b_p = 0$, $\delta_p \geq 2$.

The primes of type **0** are not contained in the non-normal locus of A.

Let $\widetilde{A} \subset K$ be the R-submodule generated by 1 and the elements of the form

$$\frac{u}{f_3}\alpha + \frac{v}{f_3}\beta, \quad u, v \in R, \quad f_3 \mid uf_1 + vf_2, \tag{2.1}$$

where $\beta = (3\alpha^2 + 2a)/3m_1m_2$.

Lemma 2.2. *The R-module $\widetilde{A}$ is a reflexive Noetherian ring integral over A.*

Proof. Let $\widetilde{A}_0$ be the R-submodule of $\widetilde{A}$ generated by the elements of the form (2.1). Then we see easily that $\widetilde{A}_0$ is isomorphic to the syzygy module of $f = (f_1, f_2, f_3)$. Thus we have an exact sequence

$$0 \longrightarrow \widetilde{A}_0 \longrightarrow R^3 \xrightarrow{f} R .$$

By the Hilbert Basis Theorem, $\widetilde{A}_0$ is a finitely generated R-module since R is Noetherian, and thus also $\widetilde{A} = R \oplus \widetilde{A}_0$ is finitely generated over R and Noetherian. $\widetilde{A}_0$ is reflexive; so is $\widetilde{A}$. If we can prove that $\widetilde{A}$ is a subring of K, then it is well known that $\widetilde{A}$, containing A, is integral over A.

In order to prove that $\widetilde{A}$ is a ring, we only need to prove that the product of any two elements of the form (2.1) is still in $\widetilde{A}$.

By some computations, we have

$$
\begin{aligned}
\alpha^2 &= m_1 m_2 \beta - m_0 m_1 m_2 f_1, \\
\alpha\beta &= -\frac{1}{2} m_0 f_1 \alpha - m_0 m_1 m_2 f_2, \\
\beta^2 &= \frac{1}{2} m_0 f_1 \beta - m_0 f_2 \alpha + \frac{1}{2} m_0^2 f_1^2.
\end{aligned} \tag{2.2}
$$

Thus if $x_i = (u_i\alpha + v_i\beta)/f_3$, $i = 1, 2$, satisfy (2.1), i.e, there are two elements w_1 and w_2 in R such that

$$f_1 u_i + f_2 v_i + f_3 w_i = 0, \quad i = 1,\ 2, \tag{2.3}$$

then $x_1 x_2 = (u\alpha + v\beta + w)/f_3^2$, where

$$
\begin{aligned}
u &= -\frac{1}{2} m_0 v_2 (f_1 u_1 + f_2 v_1) - \frac{1}{2} m_0 v_1 (f_1 u_2 + f_2 v_2), \\
v &= m_1 m_2 u_1 u_2 + \frac{1}{2} m_0 f_1 v_1 v_2, \\
w &= -m_0 m_1 m_2 f_1 u_1 u_2 - m_0 m_1 m_2 f_2 (u_1 v_2 + v_1 u_2) + \frac{1}{2} m_0^2 f_1^2 v_1 v_2.
\end{aligned}
$$

First we have to prove that f_3 divides u and v, and f_3^2 divides w.

Indeed, from (2.3) u is obviously divided by f_3. From (2.3), we have

$$f_1 u_i = -f_2 v_i - f_3 w_i, \quad i = 1,\ 2. \tag{2.4}$$

(1.3) implies

$$m_1 m_2 f_2^2 + \frac{1}{2} m_0 f_1^3 \equiv 0 \pmod{f_3^2}.$$

Substituting $f_1 u_1$ and $f_1 u_2$ in $f_1^2 v$ by (2.4), we obtain

$$f_1^2 v \equiv \left(m_1 m_2 f_2^2 + \frac{1}{2} m_0 f_1^3 \right) v_1 v_2 \equiv 0 \pmod{f_3}.$$

This implies that f_3 divides v.

Similarly, we can prove that w is divided by f_3^2 by considering $f_1 w$.

Now we have to check that $uf_1 + vf_2$ is divided by f_3^2. In fact, this can also be proved similarly by considering $f_1(uf_1 + vf_2)$. This completes the proof. □

In order to prove Theorem 1.1, we only need to prove that $\widetilde{A}$ is the integral closure of A in its fraction field K. Then by Serre's criterion for normality, it is sufficient to prove that $\widetilde{A}$ satisfies the following two conditions:

(R_1) $\widetilde{A}$ is nonsingular in codimension one.

(S_2) Every ideal I of codimension two contains a regular sequence on $\widetilde{A}$ with two elements.

Lemma 2.3. *$\widetilde{A}$ satisfies Serre's condition S_2.*

Proof. Since $\widetilde{A}$ is reflexive, $\widetilde{A}$ satisfies S_2 (cf. [Vas]). □

Lemma 2.4. *$\widetilde{A}$ satisfies Serre's condition R_1.*

Proof. In order to prove that $\widetilde{A}$ is nonsingular in codimension one, we only need to prove that $\widetilde{A}$ is the codimension one normalization of A. Equivalently we have to show that for any p of type I, II and III, we can find an element z in $\widetilde{A}$ such that p

is not contained in the non-normal locus of $R[z]$ (cf. [Sto]). In fact, by Lemma 2.1, it is sufficient to find a z in $\widetilde{A}_0$ such that its discriminant $\delta(z)$ is not divided by p^2.

For this purpose, we consider a general element z in $\widetilde{A}_0$, i.e., $z = (u\alpha + v\beta)/f_3$, and there is a w in R such that $f_1u + f_2v + f_3w = 0$.

Note that if M is the representation matrix of $(1, z, z^2)$ under the base $(1, \alpha, \alpha^2)$, then $\delta(z) = (\det M)^2\delta$. By a straightforward computation, we have

$$\delta(z) = \frac{\delta}{m_1^2 m_2^2 f_3^6}\left(u^3 m_1 m_2 + \frac{3}{2}uv^2 m_0 f_1 + m_0 v^3 f_2\right)^2. \tag{2.5}$$

Now let p be a prime in R contained in the non-normal locus of A.

First, if p is of type I with $b_p \geq 3$, then we take $\gamma' = \alpha + \beta$, i.e., $u = v = f_3$. Note that p divides m_1 and f_2, but does not divide $m_0 f_1$. Then, from the above formula we have $\delta(\gamma')_p = 1$ since $\delta_p = 3$.

If p is of type II, or of type I with $b_p = 2$, we consider β $(u = 0, v = f_3)$. By the above formula, we see that $\delta(\beta)_p = 1$.

Now we assume that p is of type III and δ_p is even. Let $\tau = (f_2\alpha - f_1\beta)/f_3$. Then

$$\delta(\tau) = \frac{f_2^2 {m_3}^2 \delta}{m_1^2 m_2^2 f_3^2}.$$

Hence $\delta(\tau)_p = 0$.

Finally, we consider the remaining case when p is of type III and δ_p is odd. In this case, replacing v in (2.5) by $-(f_1u + f_3w)/f_2$, we get

$$\delta(z) = \frac{\delta}{m_1^2 m_2^2 f_3^2}\left(\frac{m_3 u^3}{f_2^2} - \frac{3m_0 f_1 u w^2}{2f_2^2} + \text{ terms containing } f_3\right)^2.$$

Note that in this case $m_{3p} = 1$. If p does not divide uw^2, then we can see that $\delta(z)_p = 1$. So we need to find a syzygy (u, v, w) of (f_1, f_2, f_3) such that $(p, uw) = 1$.

It is obvious that such a syzygy exists, because for generic $t_1, t_2, t_3 \in R$, we have a syzygy $(t_3f_2 - t_2f_3,\ t_1f_3 - t_3f_1,\ t_2f_1 - t_1f_2)$, which satisfies our requirement.

Up to now, we have completed the proof of this lemma ☐

Therefore Theorem 1.1 has been proved.

3. Application to triple coverings

In this section, we denote by X an algebraic variety defined over an algebraically closed field k whose characteristic $\operatorname{char} k \neq 2, 3$. We always assume that X is *factorial*, i.e., its local ring at any point is a UFD. A *section* means a global section of some invertible sheaf. So we can talk about the factorizations of sections into primes as in Sect. 1 and globalize all the notations, concepts and results. In particular, we use the capital letters $F_1, F_2, F_3, M_0, M_1, M_2, M_3, \bar{A}, \bar{B}, \bar{\Delta}, , ...$ to denote the corresponding divisors of $f_1, f_2, f_3, m_0, m_1, m_2, m_3, \bar{a}, \bar{b}, \bar{\delta}, ...$ A prime section p is a global section whose divisor is a reduced and irreducible hypersurface.

In what follows, *a triple covering* is defined as a surjective finite morphism of degree 3 between two normal varieties.

Definition 3.1. Let $\mathcal{L}$ be an invertible sheaf on X. If a and $b \neq 0$ are respectively global sections of $\mathcal{L}^2$ and $\mathcal{L}^3$, then $(a, b, \mathcal{L})$ will be called *triple covering data*. $(a, b, \mathcal{L})$ are called *minimal* if there is no prime section p such that $p^2 \mid a$ and $p^3 \mid b$.

Note that if $a \neq 0$, then $(a, b, \mathcal{L})$ is determined by (a, b) because

$$\mathcal{L} \cong \mathcal{O}(\operatorname{div}(b) - \operatorname{div}(a)),$$

where $\operatorname{div}(a)$ is the divisor of a. So the two sections (a, b) are triple covering data if and only if $3\operatorname{div}(a) \equiv 2\operatorname{div}(b)$, where $\equiv$ denotes the linear equivalence of divisors. Sometimes we simply call (a, b) the triple covering data.

Remark 3.2. The following data are equivalent (see Remark 1.4):
1) Minimal triple covering data (a, b) with $a \neq 0$;
2) Triples of coprime sections (A, B) with $\operatorname{div}(A) \equiv \operatorname{div}(B)$.

We recall now the well known construction of triple coverings from the data $(a, b, \mathcal{L})$. Denote by $V(\mathcal{L}) = \operatorname{Spec} S(\mathcal{L})$ the associated line bundle of $\mathcal{L}$, where $S(\mathcal{L})$ is the symmetric $\mathcal{O}_X$-algebra of $\mathcal{L}$. Let z be the global coordinate in the fibers of $V(\mathcal{L})$. Then z is a global section of $p^*\mathcal{L}$, where p is the bundle projection of $V(\mathcal{L})$. Thus we obtain a polynomial section $p(z) = z^3 + az + b$ of $p^*\mathcal{L}^3$, where a and b are viewed as sections of $p^*\mathcal{L}^2$ and $p^*\mathcal{L}^3$ respectively. Then the zero set of $p(z)$ defines a subvariety Σ of $V(\mathcal{L})$. Let Y be the normalization of Σ. Then the composition of the normalization with the bundle projection defines a finite morphism $\pi : Y \to X$ of degree 3. π will be called the *triple covering determined by* $(a, b, \mathcal{L})$. We always assume that Y is integral or, equivalently, $p(z)$ is irreducible over the function field $K(X)$ of X.

From any triple covering data (a, b), we can get a minimal one $(\tilde{a}, \tilde{b})$ as in Lemma 1.5. In fact, the triple coverings of (a, b) and $(\tilde{a}, \tilde{b})$ are isomorphic.

Proposition 3.3 ([Ta2])**.** *Let $f : Y \to X$ be a triple covering of a factorial variety. If* $\operatorname{char} k \neq 3$*, then π is determined by some minimal triple covering data $(a, b, \mathcal{L})$.*

Let $\pi : Y \to X$ be a triple covering determined by the (minimal) data $(a, b, \mathcal{L})$, and let $\mathcal{E}_\pi$ be the trace-free submodule of $\pi_*(\mathcal{O}_Y)$. Then we have

$$\pi_*(\mathcal{O}_Y) = \mathcal{O}_X \oplus \mathcal{E}_\pi. \tag{3.1}$$

We denote by $\mathcal{F}$ the syzygy sheaf of (f_1, f_2, f_3), i.e., $\mathcal{F}$ is the kernel of the following map f defined as in Corollary 1.2,

$$0 \to \mathcal{F} \to \bigoplus_{i=1}^{3} \mathcal{O}(-F_i) \xrightarrow{f} \mathcal{I}_Z \to 0, \tag{3.2}$$

where I_Z is the ideal of $\mathcal{O}_X$ generated by f_1, f_2 and f_3.

Theorem 3.4. *$\mathcal{E}_\pi \cong \mathcal{F} \otimes \mathcal{L}(T)$, where $T = \bar{\Delta} - M_0 - M_1 - M_2$.*

Proof. Tensoring (3.2) by $\mathcal{L}(T)$, we get

$$0 \to \mathcal{F} \otimes \mathcal{L}(T) \to \bigoplus_{i=1}^{3} \mathcal{L}(T - F_i) \xrightarrow{f'} \mathcal{I}_Z \otimes \mathcal{L}(T) \to 0. \tag{3.3}$$

Note first that $\mathcal{L}(T - F_1) \cong \mathcal{L}^{-1}(F_3)$ and $\mathcal{L}(T - F_2) \cong \mathcal{L}^{-2}(F_3 + M_1 + M_2)$. Hence α/f_3 and β/f_3 can be viewed respectively as bases of the invertible sheaves $\mathcal{L}(T - F_1)$ and $\mathcal{L}(T - F_2)$. So the following local map induces a morphism of sheaves:

$$\sigma : \operatorname{Ker}(f') \longrightarrow \mathcal{E}_\pi, \qquad (u, v, w) \mapsto \frac{u\alpha + v\beta}{f_3}.$$

Now by Corollary 1.2, we see that the localization of σ at any point is an isomorphism. So σ is an isomorphism of sheaves. This completes the proof. $\square$

Thus we can compute the Chern classes of $\mathcal{E}_\pi$ and $\chi(\mathcal{O}_Y)$.

Theorem 3.5. *Let $\pi : Y \to X$ be a triple covering determined by the data (a, b). Then the divisorial part of the branch locus of π is $2M_0 + M_1 + 2M_2 + M_3$. The codimension 1 locus over which π is totally ramified is $M_0 + M_2$.*

Proof. Note that π is flat over $X \setminus Z$ ($Z = F_1 \cap F_2 \cap F_3$). In the flat case, Miranda [Mir] has computed the locus locally. We use the notations $\widetilde{\cdot}$ to denote those in [Mir]. Let $x \in X \setminus Z$, and let $\{z, w\}$ be a local base of $\mathcal{E}_\pi$ as an $\mathcal{O}_{x,X}$-module. Then we have

$$z^2 = \widetilde{a}z + \widetilde{b}w + 2\widetilde{A}, \quad zw = -\widetilde{d}z - \widetilde{a}w - \widetilde{B}, \quad w^2 = \widetilde{c}z + \widetilde{d}w + 2\widetilde{C},$$

where $\widetilde{A} = \widetilde{a}^2 - \widetilde{b}\widetilde{d}$, $\widetilde{B} = \widetilde{a}\widetilde{d} - \widetilde{b}\widetilde{c}$ and $\widetilde{C} = \widetilde{d}^2 - \widetilde{c}\widetilde{a}$. Then the branch locus is defined by $\widetilde{B}^2 - 4\widetilde{A}\widetilde{C} = 0$, and the totally ramified branch locus is defined by $\widetilde{A} = \widetilde{B} = \widetilde{C} = 0$.

If $x \in X \setminus F_1$, we choose a base $z = \alpha$, $w = (-f_2\alpha + f_1\beta)/f_3$. From (2.2), the data $(\widetilde{a}, \widetilde{b}, \widetilde{c}, \widetilde{d}, \widetilde{A}, \widetilde{B}, \widetilde{C})$ can be computed as follows:

$$\left(\frac{3m_1f_2}{2\bar{a}},\ \frac{3m_1f_3}{2\bar{a}},\ \frac{m_3f_2}{18\bar{a}},\ \frac{f_3m_3}{18\bar{a}}, -\frac{1}{3}m_0m_1m_2^2\bar{a},\ 0,\ \frac{1}{81}m_0m_2^2m_3\bar{a}\right).$$

So the branch locus is defined by $\frac{1}{27}m_0^2m_1m_2^2m_3f_1^2 = 0$.

If $x \in X \setminus F_3$, we choose a base $z = \alpha$, $w = \beta$. Then $(\widetilde{a}, \widetilde{b}, \widetilde{c}, \widetilde{d}, \widetilde{A}, \widetilde{B}, \widetilde{C})$ are

$$\left(0,\ m_1m_2,\ -m_0f_2,\ \frac{1}{3}m_0m_2\bar{a}, -\frac{1}{3}m_0m_1m_2^2\bar{a},\ m_0m_1m_2f_2,\ \frac{1}{9}m_0^2m_2^2\bar{a}^2\right).$$

So the branch locus is defined by $\frac{1}{27}m_0^2m_1m_2^2m_3f_3^2 = 0$.

Hence the equation of the branch locus on $X \setminus F_1 \cap F_3$ is $m_0^2m_1m_2^2m_3 = 0$. The branch locus of π on X is determined by that on $X \setminus F_1 \cap F_3$; hence $2M_0 + M_1 + 2M_2 + M_3$ is the branch locus.

As to the totally ramified branch locus, the proof is the same. □

Corollary 3.6. *$M_1 + M_3$ is an even divisor.*

Proof. It follows from the factorization of $\delta = m_0^2m_1^3m_2^4m_3\bar{\delta}^2$. □

Finally, we are going to see when the triple covering π is flat, or $\mathcal{E}_\pi$ (equivalently, the syzygy sheaf $\mathcal{F}$) is locally free. Due to the well known Hilbert-Burch Theorem (see [Kap], p.148, Ex. 8), we have the following simple criterion.

Proposition 3.7. *We assume that X is nonsingular, and denote by $Z = Z(f)$ the subscheme defined by the ideal $\mathcal{I}_Z = \mathcal{O}_X(f_1, f_2, f_3)$. Then the following are equivalent:*

1) *π is flat.*
2) *$\mathcal{E}_\pi$ is locally free.*
3) *$Z(f)$ is locally Cohen-Macaulay and of pure codimension two.*
4) *f is locally determinantal, i.e., for any $x \in Z(f)$, there exists a 2×3 matrix M over $\mathcal{O}_x$ such that f_i is the 2×2 minor of M by leaving out the i-th column.*

On a nonsingular surface, reflexive sheaves are locally free (cf. [Ha3]), so triple coverings are always flat.

Remark 3.8. It should be very interesting to relate the triple covering determined by the data (a, b) with the elliptic curve defined by $y^2 = x^3 + ax + b$ over $K(X)$ (cf. [Lan]). We shall discuss some of the relations in a later paper.

4. Application to rank 2 vector bundles

The aim of this section is to find the relationships between rank 2 reflexive coherent sheaves and triple coverings. The following theorem can be found in [Ta2].

Theorem 4.1. *Tensoring with an invertible sheaf, any rank* 2 *reflexive sheaf on a projective factorial variety is isomorphic to the trace-free sheaf of a triple covering.*

Now from Theorem 3.4, we obtain the following well known corollary (see [TaV]).

Corollary 4.2. *Tensoring with an invertible sheaf, any rank* 2 *reflexive sheaf over a projective factorial variety is the syzygy sheaf of* 3 *coprime hypersurfaces.*

It seems to be interesting to study rank 2 vector bundles over the projective space $\mathbb{P}^n$ by considering triple coverings $\pi : Y \to \mathbb{P}^n$. For example, Lazarsfeld [Laz] proves that if $n \geq 4$ and Y is nonsingular, then $\mathcal{E}_\pi$ splits as the sum of two line bundles. In this case, sections are homogeneous polynomials in $S = k[x_0, \cdots, x_n]$.

Definition 4.3. $f = (f_1, f_2, f_3)$ is called *homogeneously determinantal* if f_1, f_2, f_3 are the 3 maximal minors of a 2×3 matrix whose elements are homogeneous polynomials in S. f is called *determinantal* if for each i, $f_1|_{x_i=1}, f_2|_{x_i=1}, f_3|_{x_i=1}$ are the 3 maximal minors of a 2×3 matrix over $S_i = k[x_0, \cdots, \widehat{x}_i, \cdots, x_n]$.

By Quillen-Suslin theorem, any rank 2 vector bundle over $U_i = \mathrm{Spec}\,(S_i) \subset \mathbb{P}^n$ is trivial (see [Qui] or [Sus]). So we have (cf. Proposition 3.7)

Corollary 4.4. *Let* $\pi : Y \to X = \mathbb{P}^n$ *be a triple covering determined by* (a, b).

1) $\mathcal{E}_\pi$ *is locally free if and only if* (f_1, f_2, f_3) *is determinantal.*

2) $\mathcal{E}_\pi$ *is locally free and split if and only if* (f_1, f_2, f_3) *is homogeneously determinantal.*

As an application, one can translate Hartshorne's conjecture on the splitness of rank two vector bundles on $\mathbb{P}^n_{\mathbb{C}}$ $(n \geq 6)$ to the following conjecture on polynomials:

Hartshorne's Conjecture. *Assume that* $f = (f_1, f_2, f_3)$ *is determinantal. If* $n \geq 6$, *then* f *is homogeneously determinantal.*

Let $\mathcal{E}$ be a rank 2 vector bundle on $\mathbb{P}^n$. The first cohomology module of $\mathcal{E}$ is a finite graded S-module defined by $H^1_*(\mathcal{E}) := \bigoplus_{k \in \mathbb{Z}} H^1(\mathcal{E}(k))$. Let $\mathcal{F}$ be the syzygy sheaf of $f = (f_1, f_2, f_3)$, let $I(f)$ be the ideal of S generated by the f_i's, and let $\overline{I(f)}$ be the saturation of $I(f)$ (cf. [Ha1], p.125, Ex. 5.10)). Then it is easy to prove that the S-module $H^1_*(\mathcal{F})$ is isomorphic to $\overline{I(f)}/I(f)$. Thus one can characterize this cohomology module as follows (cf. [Ha2], Problem 10).

Corollary 4.5. *A finite graded* S*-module* M *is the first cohomology module of a rank* 2 *vector bundle if and only if* M *is isomorphic to* $\overline{I(f)}/I(f)$ *for some determinantal* $f = (f_1, f_2, f_3)$, *where* f_1, f_2, f_3 *are pairwise coprime homogeneous polynomials.*

Because the stability of a syzygy sheaf $\mathcal{F}$ can be characterized by f (see [Ta1]), one can also give a similar characterization for the first cohomology modules of rank two stable vector bundles.

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