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COMPARISON OF 4-CLASS RANKS OF CERTAIN QUADRATIC FIELDS

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ABSTRACT. Let m be a square-free positive integer. Let $r_4(K)$ denote the 4-class rank of a quadratic field K. This paper examines how likely it is for $r_4(\mathbb{Q}(\sqrt{-m})) = r_4(\mathbb{Q}(\sqrt{m}))$ and for $r_4(\mathbb{Q}(\sqrt{-m})) = r_4(\mathbb{Q}(\sqrt{m})) + 1$.

1. INTRODUCTION

Let K be a quadratic extension of the field of rational numbers $\mathbb{Q}$. Let $C(K)$ denote the 2-class group of K in the narrow sense. It is well known that $\operatorname{rank} C(K) = t-1$, where t is the number of primes that ramify in $K/\mathbb{Q}$. Let $r_4(K)$ denote the 4-class rank of K in the narrow sense; i.e.,

$$r_4(K) = \dim_{\mathbb{F}_2}\left((C(K))^2/(C(K))^4\right) \tag{1.1}$$

where $(C(K))^i = \{c^i : c \in C(K)\}$ for positive integers i, and $\mathbb{F}_2$ is the finite field with two elements. In Equation (1.1), we are viewing the elementary abelian 2-group $(C(K))^2/(C(K))^4$ as a vector space over $\mathbb{F}_2$.

Now let m be a square-free positive integer. It is known (cf. [2], [6]) that

$$r_4(\mathbb{Q}(\sqrt{m})) \le r_4(\mathbb{Q}(\sqrt{-m})) \le r_4(\mathbb{Q}(\sqrt{m})) + 1. \tag{1.2}$$

We will consider the following question: how likely is it that $r_4(\mathbb{Q}(\sqrt{-m})) = r_4(\mathbb{Q}(\sqrt{m}))$, and how likely is it that $r_4(\mathbb{Q}(\sqrt{-m})) = r_4(\mathbb{Q}(\sqrt{m})) + 1$? A direct answer could be obtained if we could compute

$$\lim_{x\to\infty} \frac{|\{\text{square-free } m \le x : r_4(\mathbb{Q}(\sqrt{-m})) = r_4(\mathbb{Q}(\sqrt{m}))\}|}{|\{\text{square-free } m \le x\}|}$$

or

$$\lim_{x\to\infty} \frac{|\{\text{square-free } m \le x : r_4(\mathbb{Q}(\sqrt{-m})) = r_4(\mathbb{Q}(\sqrt{m}))+1\}|}{|\{\text{square-free } m \le x\}|}$$

where $|S|$ denotes the cardinality of a set S. However, computing these limits appears to be very difficult. Instead, we shall use a somewhat different approach. Although the limits we compute are not guaranteed to equal the above limits, our results do provide some insight into this question.

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First we introduce some notation. For positive integers t, nonnegative integers i, positive real numbers x, and square-free positive integers m, we define

$$\begin{aligned} A_{t;x} &= \{\mathbb{Q}(\sqrt{-m}\,) : \text{ exactly } t \text{ primes ramify in } \mathbb{Q}(\sqrt{-m}\,)/\mathbb{Q} \text{ and } m \leq x\}, \\ A_{t,i;x} &= \{\mathbb{Q}(\sqrt{-m}\,) \in A_{t;x} : r_4(\mathbb{Q}(\sqrt{-m}\,)) = i\}, \\ A^{(1)}_{t,i;x} &= \{\mathbb{Q}(\sqrt{-m}\,) \in A_{t,i;x} : r_4(\mathbb{Q}(\sqrt{-m}\,)) = r_4(\mathbb{Q}(\sqrt{m}\,))\}, \\ A^{(2)}_{t,i;x} &= \{\mathbb{Q}(\sqrt{-m}\,) \in A_{t,i;x} : r_4(\mathbb{Q}(\sqrt{-m}\,)) = r_4(\mathbb{Q}(\sqrt{m}\,)) + 1\}\ . \end{aligned}$$

We then define the following densities:

$$a_{t,i} = \lim_{x\to\infty} \frac{|A_{t,i;x}|}{|A_{t;x}|}, \tag{1.3}$$

$$a^{(1)}_{t,i} = \lim_{x\to\infty} \frac{|A^{(1)}_{t,i;x}|}{|A_{t;x}|}, \tag{1.4}$$

$$a^{(2)}_{t,i} = \lim_{x\to\infty} \frac{|A^{(2)}_{t,i;x}|}{|A_{t;x}|}. \tag{1.5}$$

Next we define the limit densities:

$$a_{\infty,i} = \lim_{t\to\infty} a_{t,i}, \tag{1.6}$$

$$a^{(1)}_{\infty,i} = \lim_{t\to\infty} a^{(1)}_{t,i}, \tag{1.7}$$

$$a^{(2)}_{\infty,i} = \lim_{t\to\infty} a^{(2)}_{t,i}. \tag{1.8}$$

It is known (cf. Equation (1.5) in [4]) that

$$a_{\infty,i} = \frac{2^{-i^2} \prod_{k=1}^{\infty} (1-2^{-k})}{\prod_{k=1}^{i} (1-2^{-k})^2} \tag{1.9}$$

for $i = 0, 1, 2, \cdots$. Furthermore, $\sum_{i=0}^{\infty} a_{\infty,i} = 1$, and, of course, $a_{\infty,i} = a^{(1)}_{\infty,i} + a^{(2)}_{\infty,i}$. To obtain the likelihood that $r_4(\mathbb{Q}(\sqrt{-m}\,)) = r_4(\mathbb{Q}(\sqrt{m}\,))$ and that $r_4(\mathbb{Q}(\sqrt{-m}\,)) = r_4(\mathbb{Q}(\sqrt{m}\,)) + 1$, we let

$$\alpha_1 = \sum_{i=0}^{\infty} a^{(1)}_{\infty,i}, \tag{1.10}$$

$$\alpha_2 = \sum_{i=0}^{\infty} a^{(2)}_{\infty,i}\ . \tag{1.11}$$

We shall prove the following theorem.

Theorem 1. *Let α_1 and α_2 be defined by Equations (1.10) and (1.11), and let $a_{\infty,i}$ be given by Equation (1.9). Then*

$$\alpha_1 = \sum_{i=0}^{\infty} 2^{-i} a_{\infty,i} \approx 0.610321 \; ;$$
$$\alpha_2 = \sum_{i=0}^{\infty} (1-2^{-i}) a_{\infty,i} \approx 0.389679 \; .$$

Remark. Theorem 1 is also valid if we use the 4-class rank in the usual sense rather than the narrow sense (cf. discussion on p. 491 of [4]).

2. Proof of Theorem 1

From the discussion on p. 491 of [4], it suffices to consider $m = p_1 \cdots p_t$ with distinct odd primes $p_1, \cdots, p_t$ and with an odd number of primes $p_i \equiv 3 \pmod 4$ when analyzing $A_{t;x}$ and its subsets in our counting arguments. For convenience we label the primes so that

$$p_i \equiv 1 \pmod 4 \text{ for } 1 \le i \le s, \tag{2.1}$$
$$p_i \equiv 3 \pmod 4 \text{ for } s+1 \le i \le t \tag{2.2}$$

where $s \ge 0$ and $t-s$ is odd. Now the 4-class rank of $K = \mathbb{Q}(\sqrt{-m}\,)$ satisfies

$$r_4(K) = t - 1 - \operatorname{rank} M'_K \tag{2.3}$$

where $M'_K = [b_{ij}]$ is the $t \times (t-1)$ matrix with entries in $\mathbb{F}_2$ defined by Legendre symbols as follows:

$$(-1)^{b_{ij}} = \begin{cases} \left(\dfrac{P_j}{p_i}\right), & \text{if } i \ne j, \\ \left(\dfrac{-m/P_j}{p_i}\right), & \text{if } i = j, \end{cases} \tag{2.4}$$

for $1 \le i \le t$ and $1 \le j \le t-1$ (cf. Equation (2.6) in [4]). Here $P_j = p_j$ if $p_j \equiv 1 \pmod 4$, and $P_j = -p_j$ if $p_j \equiv 3 \pmod 4$. As discussed on p. 492 in [4], it is also true that

$$r_4(K) = t - 1 - \operatorname{rank} M_K \tag{2.5}$$

where M_K is the $t \times t$ matrix with entries defined by Equation (2.4), except with $1 \le j \le t$ instead of $1 \le j \le t-1$. Furthermore, the sum of the entries in each row of M_K is zero, and the sum of the entries in each column of M_K is zero.

Now we let $L = \mathbb{Q}(\sqrt{m}\,)$. Since there are an odd number of primes $p_i \equiv 3 \pmod 4$ that divide m, then $m \equiv 3 \pmod 4$. So $t+1$ primes ramify in $L/\mathbb{Q}$; namely $p_1, \cdots, p_t$ and 2. The 4-class rank of L satisfies

$$r_4(L) = (t+1) - 1 - \operatorname{rank} M'_L \tag{2.6}$$

where $M'_L = [c_{ij}]$ is the $(t+1) \times t$ matrix over $\mathbb{F}_2$ whose entries satisfy

$$c_{ij} = \begin{cases} b_{ij} \text{ if } (i \ne j \text{ and } 1 \le i \le t,\ 1 \le j \le t) \text{ or if } (i = j \text{ and } 1 \le i \le s), \\ b_{ij} + 1 \text{ if } i = j \text{ and } s+1 \le i \le t, \\ 0 \text{ if } i = t+1 \text{ and 2 splits in } \mathbb{Q}(\sqrt{P_j}\,), \\ 1 \text{ if } i = t+1 \text{ and 2 remains prime in } \mathbb{Q}(\sqrt{P_j}\,)\;. \end{cases} \tag{2.7}$$

Let M_L denote the $t \times t$ matrix consisting of the first t rows of M'_L.

Lemma 1. Rank $M_L = \operatorname{rank} M_K + 1$.

Proof. Write

$$M_K = \left[\begin{array}{c:c} B_1 & B_2 \\ \hdashline B_3 & B_4 \end{array}\right]$$

where B_1 is an $s \times s$ symmetric matrix over $\mathbb{F}_2$, B_2 is an $s \times (t-s)$ matrix over $\mathbb{F}_2$, B_3 is the $(t-s) \times s$ matrix which equals B_2^T (the transpose of B_2), and B_4 is a $(t-s) \times (t-s)$ antisymmetric matrix over $\mathbb{F}_2$ (i.e., $b_{ij} = b_{ji} + 1$ for $i \neq j$). These properties follow from Equation (2.4) and quadratic reciprocity. Note that

$$\text{(2.8)} \qquad M_K^T = \left[\begin{array}{c:c} B_1^T & B_3^T \\ \hdashline B_2^T & B_4^T \end{array}\right] = \left[\begin{array}{c:c} B_1 & B_2 \\ \hdashline B_3 & B_4 + I + J \end{array}\right]$$

where I is the $(t-s) \times (t-s)$ identity matrix, and J is the $(t-s) \times (t-s)$ matrix with each entry equal to 1. Now from Equations (2.7) and (2.8) and our definition of M_L,

$$\text{(2.9)} \qquad M_L = \left[\begin{array}{c:c} B_1 & B_2 \\ \hdashline B_3 & B_4 + I \end{array}\right] = \left[\begin{array}{c:c} B_1 & B_2 \\ \hdashline B_3 & B_4 + I + 2J \end{array}\right] = M_K^T + H$$

since $2J$ is a zero matrix over $\mathbb{F}_2$, and where

$$\text{(2.10)} \qquad H = \left[\begin{array}{c:c} O & O \\ \hdashline O & J \end{array}\right] .$$

Now let $v \in \mathbb{F}_2^t$. (Think of v as a column vector.) If the last $(t-s)$ entries in v contain an even number of 1's, then

$$\text{(2.11)} \qquad M_L v = M_K^T v + Hv = M_K^T v .$$

Let

(2.12)

$$W = \{M_K^T v : v \in \mathbb{F}_2^t \text{ with an even number of 1's in the last } (t-s) \text{ entries in } v\} .$$

If the last $(t-s)$ entries in v contain an odd number of 1's, let $v_1 = v + v_2$, where $v_2 = [0, \cdots, 0, 1]^T$. Note that the last $(t-s)$ entries in v_1 contain an even number of 1's, and $v = v_1 + v_2$. Then

$$M_K^T v = M_K^T v_1 + M_K^T v_2 .$$

Clearly $M_K^T v_1 \in W$. Next, note that $M_K^T v_2 = M_K^T v_3$, where $v_3 = [1, \cdots, 1, 0]^T$, since the sum of the entries in each row of M_K^T is zero. But then $M_K^T v_3 \in W$ since v_3 has an even number of 1's in its last $(t-s)$ entries. Thus

$$M_K^T v = M_K^T v_1 + M_K^T v_3 \in W .$$

So W is the column space of M_K^T, and from Equation (2.11), we know that the column space of M_L contains W. Also, since the matrix H in Equation (2.10) has rank equal to 1, then from Equation (2.9), we know that

$$\text{(2.13)} \qquad \operatorname{rank} M_K^T \leq \operatorname{rank} M_L \leq \operatorname{rank} M_K^T + 1 .$$

Now let v_4 be the vector in $\mathbb{F}_2^t$ with each component equal to 1. Then $M_K^T v_4$ is the zero vector in $\mathbb{F}_2^t$ since the sum of the entries in each row of M_K^T is zero. Then from

Equations (2.9) and (2.10), $M_L v_4$ is the vector in $\mathbb{F}_2^t$ whose first s components are 0's and whose last $(t-s)$ components are 1's since $t-s$ is odd. Then the sum of the entries in $M_L v_4$ is 1. But then $M_L v_4$ does not belong to the column space of M_K^T since the sum of the entries in each column of M_K^T is zero. Thus from (2.13), we see that

$$\operatorname{rank} M_L = \operatorname{rank} M_K^T + 1 = \operatorname{rank} M_K + 1$$

which completes the proof of Lemma 1.

We let

$$w = \operatorname{rank} M_K = \operatorname{rank} M_L - 1 \ . \tag{2.14}$$

We now consider the $(t+1) \times t$ matrix M'_L whose first t rows form the matrix M_L. From Equation (2.7), we observe that the entries in the last row of M'_L satisfy

$$c_{(t+1)j} = \begin{cases} 0, & \text{if } p_j \equiv \pm 1 \pmod 8, \\ 1, & \text{if } p_j \equiv \pm 3 \pmod 8 \ . \end{cases}$$

Since the primes are equally distributed among the residue classes $\pm 1 \pmod 8$ and $\pm 3 \pmod 8$, it is intuitively clear that each entry in the last row of M'_L is equally likely to be a 0 or a 1. (This can be proved using character sums similar to those used to prove Propositions 2.1 and 5.1 in [4]. See [3] and [5] (or [1]) for more details on character sum calculations.) Then, of the possible 2^t matrices M'_L whose first t rows form M_L, 2^{1+w} satisfy $\operatorname{rank} M'_L = \operatorname{rank} M_L$, and $(2^t - 2^{1+w})$ satisfy $\operatorname{rank} M'_L = \operatorname{rank} M_L + 1$. From Equation (2.6) and the above discussion,

$$r_4(L) = \begin{cases} t-1-w & \text{with probability } 2^{-(t-1-w)} \\ t-2-w & \text{with probability } 1 - 2^{-(t-1-w)} \ . \end{cases} \tag{2.15}$$

Then Equations 2.5, 2.14, and 2.15 give

$$r_4(K) = \begin{cases} r_4(L) & \text{with probability } 2^{-(t-1-w)} \\ r_4(L)+1 & \text{with probability } 1 - 2^{-(t-1-w)} \ . \end{cases}$$

Then letting $i = t-1-w$ and using Equations 1.3, 1.4, and 1.5, we get

$$a_{t,i}^{(1)} = 2^{-i} a_{t,i} \quad \text{and} \quad a_{t,i}^{(2)} = (1-2^{-i}) a_{t,i} \ .$$

Taking the limit as $t \to \infty$, we get

$$a_{\infty,i}^{(1)} = 2^{-i} a_{\infty,i} \quad \text{and} \quad a_{\infty,i}^{(2)} = (1-2^{-i}) a_{\infty,i} \ .$$

Then summing over all $i \geq 0$, we get Theorem 1.

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