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ANALYTIC SETS AND THE BOUNDARY REGULARITY OF CR MAPPINGS

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Abstract. It is shown that if a continuous CR mapping between smooth real analytic hypersurfaces of finite type in $\mathbf{C}^n$ extends as an analytic set, then it extends as a holomorphic mapping.

1. Introduction

The purpose of this article is to discuss the boundary regularity of continuous CR mappings between smooth, real analytic hypersurfaces of finite type in $\mathbf{C}^n$. We are interested in obtaining a holomorphic extension under the assumption that the graph of the mapping extends as an analytic set in a sense to be made precise later. The main theorem is as follows:

Theorem 1.1. *Let $\Omega, \Omega' \subset\subset \mathbf{C}^n$ be domains and let $M \subset \Omega$, $M' \subset \Omega'$ be relatively closed, smooth, real analytic hypersurfaces of finite type. Let $f : M \to M'$ be a continuous CR mapping and suppose that $0 \in M, 0' \in M'$ and $f(0) = 0'$. If f extends as an analytic set near $(0, 0')$, then f extends holomorphically across 0.*

As an application, we recover the following theorem which was proved for $n = 2$ by Diederich-Fornaess (see [DF2]).

Theorem 1.2. *Let $\mathcal{D}, \mathcal{D}'$ be bounded, algebraic domains in $\mathbf{C}^n$ and $f : \mathcal{D} \to \mathcal{D}'$ a proper holomorphic mapping. Then f extends holomorphically across $\partial\mathcal{D}$.*

Theorem 1.1 may be considered as a version for continuous CR mappings of results contained in the works of Bedford-Bell ([BB]) and Coupet-Pinchuk ([CP]). Bedford-Bell consider the case when $f : \mathcal{D} \to \mathcal{D}'$ is known to extend C^∞ smoothly up to $\partial\mathcal{D}$ and on the other hand, Coupet-Pinchuk prove a similar result in their study of proper mappings between polynomial rigid domains. Other related results may be found in [BBR], [BJT], [H], [P], [PT] and [V].

2. Preliminary notions and terminology

We will write $z = ('z, z_n) \in \mathbf{C}^{n-1} \times \mathbf{C}$ for a point $z \in \mathbf{C}^n$. Let $\mathcal{D}, \mathcal{D}'$ be domains in $\mathbf{C}^n$. Let z, z' denote the coordinates in $\mathcal{D}, \mathcal{D}'$ respectively. A holomorphic correspondence is a complex analytic set $A \subset \mathcal{D} \times \mathcal{D}'$ of pure dimension n with

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$\overline{A} \cap (\mathcal{D} \times \partial\mathcal{D}') = \emptyset$. In this situation $\pi : A \to \mathcal{D}$, the natural projection, is a proper, finite-to-one branched covering above $\mathcal{D}$. A may also be regarded as the graph of a multivalued mapping, say $\hat{f} := \pi' \circ \pi^{-1} : \mathcal{D} \to \mathcal{D}'$, where $\pi' : A \to \mathcal{D}'$ is the natural projection. We shall henceforth make no distinction between the analytic set and the associated multivalued mapping. If $\pi' : A \to \mathcal{D}'$ is also proper, then A is said to be a proper holomorphic correspondence. The simplest example of a proper holomorphic correspondence is the graph of a proper mapping $f : \mathcal{D} \to \mathcal{D}'$. The term analytic will henceforth mean complex analytic unless stated otherwise.

After possibly shrinking Ω, Ω' in the main theorem, let $r(z, \overline{z})$, $r'(z', \overline{z}')$ be the smooth, real analytic defining functions for M, M' in Ω, Ω' respectively. We put $\Omega^+ := \{z \in \Omega : r > 0\}$, $\Omega^- := \{z' \in \Omega' : r < 0\}$ and similarly for Ω'^+ and Ω'^-. For an open set $U \subset \Omega$, $U^{\pm}$ will denote $U \cap \Omega^{\pm}$. A similar convention will be followed for the range space.

The theorem of J. M. Trepreau ([T]) shows that any continuous CR function defined on M admits at least a one sided holomorphic extension. Therefore, we may assume that after shrinking Ω, f as in the main theorem holomorphically extends to, say Ω^-. We shall still denote this extension to Ω^- by f and this leads us to consider the following general situation:

General Situation: Let Ω, Ω' and M, M' be as in the main theorem. Let $f : \Omega^- \cup M \to \Omega'$ be a mapping which is holomorphic in Ω^- and which extends continuously up to M with $f(M) \subset M'$. Suppose that $0 \in M, 0' \in M'$, $f(0) = 0'$ and that f extends as an analytic set near $(0, 0')$.

We say that f as in the general situation extends as an analytic set near $(0, 0')$ if there exist neighbourhoods U, U' of $0, 0'$ respectively and an analytic set $A \subset U \times U'$ of pure dimension n with the following property: $f(U^-) \subset U'$ and $\Gamma_f \cap (U^- \times U') \subset A$. If the projection $\pi : A \to U$ is proper for a suitable choice of neighbourhoods, then f is said to extend as a correspondence near $(0, 0')$.

If $f : \mathcal{D} \to \mathcal{D}'$ is a holomorphic mapping between domains $\mathcal{D}, \mathcal{D}'$ in $\mathbf{C}^n$, V_f will denote the analytic set in $\mathcal{D}$ described by the vanishing of the Jacobian J_f of f.

The techniques of Segre varieties will be useful and here are a few notions and properties that are referred to in the proof of the main theorem: Let $\mathcal{D}$ be a domain in $\mathbf{C}^n$ with $0 \in \partial\mathcal{D}$ and suppose that $\partial\mathcal{D}$ is smooth, real analytic near the origin. Let $r(z, \overline{z})$ be the defining function of $\partial\mathcal{D}$ in a neighbourhood of the origin, say U. If U is small enough the complexification $r(z, \overline{w})$ of r is well defined by means of a convergent power series in $U \times U$. Note that $r(z, \overline{w})$ is holomorphic in z and antiholomorphic in w. For any $w \in U$, the associated Segre variety is defined as

$$Q_w = \{z \in U : r(z, \overline{w}) = 0\}. \tag{2.1}$$

By the implicit function theorem Q_w can be written as a graph; more precisely, there exists a function $h('z, w)$ holomorphic in the $'z$ variables and antiholomorphic in w such that

$$Q_w = \{('z, z_n) \in U = {}'U \times U_n : z_n = h('z, w)\}. \tag{2.2}$$

This is a closed, complex hypersurface in U which does not depend on the choice of the defining function. For $\zeta \in Q_w$, the germ of Q_w at ζ will be denoted by ${}_\zeta Q_w$. Let $\mathcal{S} := \{Q_w : w \in U\}$ be the set of all Segre varieties and $\lambda : w \mapsto Q_w$ the so-called Segre map. We refer the reader to [DF1], [DW] for further properties of Segre varieties and for a proof of the fact that $\mathcal{S}$ admits the structure of a finite dimensional complex analytic set. Let $I_w := \{z : Q_z = Q_w\}$ be the fibre of λ over

Q_w. We say that $\mathcal{D}$ is essentially finite at 0 if I_0 contains 0 as an isolated point. In this case I_w, for all w close to 0, will be finite near 0. Furthermore λ will be a proper, finite-to-one, antiholomorphic mapping from a suitably chosen neighbourhood of the origin onto its image in $\mathcal{S}$. We shall also have occasion to use the notion of the reflected point which was introduced in [DP1]. Briefly, for a given system of coordinates near 0, the reflection of a point $w = ('w, w_n) \in \mathbf{C}^{n-1} \times \mathbf{C}$, which is close to 0, is defined to be that point which has coordinates $('w, {}^{\kappa}w_n)$ and lies on Q_w. This is a real analytic diffeomorphism and it depends on the choice of a coordinate system near 0. The reflected point will be denoted by $\kappa(w)$. For w outside $\mathcal{D}$, the component of $Q_w \cap \mathcal{D}$ which contains $\kappa(w)$ will be denoted by Q_w^c and will be referred to as the canonical component of $Q_w \cap \mathcal{D}$. Real analytic hypersurfaces of finite type are essentially finite. Moreover, essentially finite hypersurfaces do not contain germs of complex hypersurfaces.

Finally for all the notions and terminology introduced here, we simply add a prime to consider the corresponding notions in the target space.

3. Analytic sets which extend Γ_f are correspondences

The proof of Theorem 1.1 consists of two steps: first, to show that the extending analytic set A is a correspondence and then to apply the result of [DP2] which shows that all extending correspondences are holomorphic mappings.

If $A \subset U \times U'$ is the extending analytic set in the general situation, we may assume that A is irreducible without loss of generality. We begin by showing that A has points lying over U^+.

Proposition 3.1. *In the general situation, let $U_1 \subset\subset U$, $U_1' \subset\subset U'$ be arbitrarily small neighbourhoods of $0, 0'$ respectively. Then $A \cap (U_1^+ \times U_1') \neq \emptyset$.*

Proof. Let U_1, U_1' be such that $A \cap (U_1 \times U_1') \subset \overline{U_1^-} \times U_1'$. Since f is continuous, we may shrink U_1, U_1' so that $f(U_1^-) \subset U_1'$. Furthermore, there is an irreducible component of $A \cap (U_1 \times U_1')$ which contains $\Gamma_f \cap (U_1^- \times U_1')$. We still denote this irreducible component by A. Hence $A \not\subset (U_1 \cap M) \times U_1'$. Let $\mathcal{L}$ be a complex line in $\mathbf{C}^n$ which contains 0 and is transverse to M. Let $\tilde{A}$ be an irreducible component of $A \cap ((U_1^- \cap \mathcal{L}) \times U_1')$ containing $\Gamma_f \cap ((U_1^- \cap \mathcal{L}) \times U_1')$. Then $\tilde{A}$ has pure dimension 1 and it contains the point $(0, 0')$. Moreover $\tilde{A} \not\subset (U_1 \cap M) \times U_1'$. Now two cases arise: First, if $\tilde{A} \cap ((U_1 \cap M) \times U_1')$ is discrete, then we may apply the continuity principle to conclude that $(0, 0')$ is in the envelope of holomorphy of $U_1^- \times U_1'$. Second, if $\tilde{A} \cap ((U_1 \cap M) \times U_1')$ is not discrete, the finite type assumption on M shows that no open subset of $\tilde{A}$ can be contained in $\tilde{A} \cap ((U_1 \cap M) \times U_1')$. The strong disc theorem in [Vl] shows that $(0, 0')$ is again in the envelope of holomorphy of $U_1^- \times U_1'$.

Given an arbitrary $g \in \mathcal{O}(U_1^-)$, we may regard $g \in \mathcal{O}(U_1^- \times U_1')$, i.e., independent of the z' variables. Then g extends to a neighbourhood of $(0, 0')$, and the uniqueness theorem shows that the extension of g, say $\tilde{g}$, is also independent of the z' variables. Hence g is holomorphic near 0. In particular, f extends holomorphically across 0 and this finishes the proof of the proposition. ∎

One consequence of this is the following:

Proposition 3.2. *In the general situation, either f is constant or the Jacobian determinant $J_f \not\equiv 0$ in Ω^-. Moreover f holomorphically extends across an open, dense subset of M near 0.*

Proof. Let U, U' be neighbourhoods of $0, 0'$ and $A \subset U \times U'$ be the extending analytic set. Let $\pi : A \to U$ be the natural projection. Then there are points $z \in \pi(A)$ such that $\pi^{-1}(z)$ has 0 dimension. Indeed, if not, then $\pi^{-1}(z)$ is at least 1 dimensional for all $z \in \pi(A)$ and hence by [L], p. 266, it follows that

$$dim\ A \geq 1 + dim\ \pi(A) \tag{3.1}$$

and so $dim\ \pi(A) \leq n - 1$. This is a contradiction since $\pi(A)$ contains Γ_f over U^-. For $(z, z') \in A$, let $(\pi^{-1}(z))_{(z,z')}$ denote the germ of the analytic set $\pi^{-1}(z)$ at (z, z'). Let

(3.2)
$$S := \{(z, z') \in A : \pi : A \to U \text{ is not locally biholomorphic near } (z, z')\}.$$

Then $S = S_1 \cup S_2$ where

$$S_1 := \{(z, z') \in A : dim\ (\pi^{-1}(z))_{(z,z')} \geq 1\} \tag{3.3}$$

and

$$S_2 := \{(z, z') \in A : \pi : A \to U \text{ is finite-to-one near } (z, z')\}. \tag{3.4}$$

By the theorem of Cartan-Remmert (see [L], p. 271), S_1 is an analytic set in $U \times U'$ and moreover its dimension does not exceed $n - 1$. S_2 is a constructible set (see [L], p. 265) and since $\pi : A \to U$ is locally proper at each point of $A \backslash S_1$, it follows that S_2 has dimension at most $n-1$. Hence $A \backslash S$ is path connected. Fix $z \in U \cap M$ arbitrarily and let $z \in U_z \subset\subset U, f(z) \in U'_{f(z)} \subset\subset U'$ be neighbourhoods such that $f(U_z^-) \subset U'_{f(z)}$. There is an irreducible component of $A \cap (U_z \times U'_{f(z)})$, which will still be denoted by A and which contains Γ_f over U_z^-. Fix $(w_1, f(w_1)) \in (\Gamma_f \backslash S) \cap (U_z^- \times U'_{f(z)})$ and by Proposition 3.1, choose $(w_2, w_2') \in (A \backslash S) \cap (U_z^+ \times U'_{f(z)})$. Join these points by a path $\gamma \subset (A \backslash S)$. Then $\pi(\gamma) \cap M \neq \emptyset$; say $z_0 \in \pi(\gamma) \cap M$. Then f can be analytically continued along $\pi(\gamma)$ from w_1 to z_0. Hence z_0 is a point of holomorphic extendability for f. It follows from [BR] that either f is a constant or $J_f \not\equiv 0$ in Ω^-. Furthermore since z was chosen arbitrarily on M, it follows that f extends across an open, dense set of $U \cap M$. □

Remark. Proposition 3.2 also holds if M is assumed to be only essentially finite.

Henceforth Σ will denote the points of M across which f extends holomorphically. In the general situation, the continuity of f implies that we can shrink the neighbourhoods U, U' and arrange for the following two conditions to hold: $f(U^-) \subset U'$ and A, the extending analytic set is defined in a neighbourhood of $\overline{U} \times \overline{U}'$, i.e., there exists a neighbourhood of $(0, 0')$ of the form $U_0 \times U_0'$ with $0 \in U \subset\subset U_0$ and $0' \in U' \subset\subset U_0'$ such that A is a closed analytic set in $U_0 \times U_0'$. Without loss of generality the Segre map $\lambda' : U' \to \mathcal{S}'$ is proper onto its image. Define the Segre correspondence:

Definition 3.1. $A^+ = \{(w, w') \in U^+ \times U' : f(Q_w^c) \subset Q'_{w'}\}$.

To begin with, it may happen that $f(U^-)$ intersects both $U'^{\pm}$ and hence in the definition of A^+ we have to allow w' to be in all of U'. Furthermore, A^+ is not an empty set as it contains the graph of the extension of f across Σ. Let $V^+ := \pi(A^+) \subset U^+$.

Proposition 3.3. *A^+ is a closed analytic set in $U^+ \times U'$ of pure dimension n.*

Proof. First, A^+ is closed in $U^+ \times U'$. Indeed, if $(w_\nu, w'_\nu) \in A^+$ are converging to (w_0, w'_0) which is a limit point of A^+ in $U^+ \times U'$, then

$$f(Q^c_{w_\nu}) \subset Q'_{w'_\nu} \tag{3.5}$$

holds for all ν. The continuity of Segre varieties and of f shows that by passing to the limit we still have

$$f(Q^c_{w_0}) \subset Q'_{w'_0}. \tag{3.6}$$

Thus $A^+ \subset U^+ \times U'$ is closed.

Now suppose that $(w_0, w'_0) \in A^+$ is such that $Q^c_{w_0} \not\subset V_f$. By definition a point (w, w') in a neighbourhood of (w_0, w'_0) belongs to A^+ if and only if

$$f(Q^c_w) \subset Q'_{w'}. \tag{3.7}$$

This is equivalent to the statement that

$$r(z, \overline{w}) = 0 \implies r'(f(z), \overline{w}') = 0. \tag{3.8}$$

Combining this with (2.2) we get that

$$r'(f('z, h('z, \overline{w})), \overline{w}') = 0 \quad \forall\, 'z. \tag{3.9}$$

This is an infinite family of antiholomorphic functions in (w, w') describing the set A^+. By [C], section 5.7, A^+ is analytic near (w_0, w'_0).

Let $E \subset V^+$ be the set of those points w for which $Q^c_w \subset V_f$.

Claim. E is a locally finite set in U^+.

Indeed, if there is a sequence $w_\nu \to w_0 \in E$, then all the canonical components $Q^c_{w_\nu} \subset V_f$. However, V_f has only finitely many components in a fixed small ball around the reflected point $\kappa(w_0)$ and hence infinitely many w_ν should have the same Segre variety. This contradicts the finite type assumption on M. Thus so far $A^+ \cap ((U^+ \backslash E) \times U')$ is an analytic set in the domain $(U^+ \backslash E) \times U'$.

Last, consider the fibre in A^+ over an arbitrary $w_0 \in E$. Let $(w_0, w'_0) \in A^+$ be such that $w_0 \in E$. Choose a sequence $(w_\nu, w'_\nu) \in A^+ \cap ((U^+ \backslash E) \times U')$ converging to (w_0, w'_0). By the definition of A^+ and the continuity of Segre varieties we have

$$f(Q^c_{w_0}) \subset Q'_{w'_0}. \tag{3.10}$$

Fix an arbitrary $x_0 \in f(Q^c_{w_0})$. Then the above equation shows that $w'_0 \in Q'_{x_0}$ and thus the entire fibre in A^+ over w_0 is at worst the entire Segre variety Q'_{x_0}. Note that the Hausdorff dimension of Q'_{x_0} is $2n-2$. Now A^+ has dimension n and since E is locally finite, it follows by Shiffman's theorem (see [C]) that A^+ is analytic in $U^+ \times U'$. $\square$

Remark. The proof of the above proposition shows that the fibre in A^+ over points in $V^+ \backslash E$ is discrete.

Proposition 3.4. *The projection* $\pi : A^+ \cap ((V^+ \backslash E) \times U') \to V^+ \backslash E$ *is proper.*

Proof. Let $K \subset V^+ \backslash E$ be a compact set such that $\pi^{-1}(K)$ is not compact. Choose a sequence $(w_\nu, w'_\nu) \in \pi^{-1}(K)$ so that $(w_\nu, w'_\nu) \to (w_0, w'_0) \in K \times \partial U'$. Since $w_0 \in V^+$, there is a point $(w_0, \tilde{w}_0) \in A^+$. Then

$$f(Q^c_{w_\nu}) \subset Q'_{w'_\nu} \tag{3.11}$$

and passing to the limit we have

$$f(Q^c_{w_0}) \subset Q'_{w'_0}. \tag{3.12}$$

This invariance property also holds for $(w_0, \tilde{w}_0)$ and therefore

$$f(Q^c_{w_0}) \subset Q'_{\tilde{w}_0}. \tag{3.13}$$

Combining (3.12) and (3.13) and observing that $w_0 \notin E$, it follows that $Q'_{w'_0} = Q'_{\tilde{w}_0}$. This contradicts the properness of the Segre map $\lambda' : U' \to \lambda'(U') \subset \mathcal{S}'$. $\square$

Definition 3.2. $A^- = \{(w, w') \in U^- \times U' : Q'_{w'} = Q'_{f(w)}\}$.

Note that a point (w, w') belongs to A^- if and only if $w' \in {\lambda'}^{-1}(\lambda'(f(w))$. Since the Segre map $\lambda' : U' \to \mathcal{S}'$ is a proper, anti-holomorphic map onto its image in the set of all Segre varieties $\mathcal{S}'$, it follows that A^- is a closed analytic set in $U^- \times U'$ of pure dimension n. Moreover the projection

$$\pi : A^- \to U^- \tag{3.14}$$

is proper and A^- contains $\Gamma_f \cap (U^- \times U')$. Also by the definition of A^+, A^- it follows that they can be 'glued' together over points which belong to Σ. Thus $A^* := \overline{A^+ \cup A^-}$ is an analytic set in the domain $\tilde{U} \times U'$ where $\tilde{U} := V^+ \cup \Sigma \cup U^-$.

Proposition 3.5. *The projection $\pi : A^* \to \tilde{U}$ is proper.*

Proof. We first show that the projection $\pi : A^* \cap ((\tilde{U}\backslash E) \times U') \to \tilde{U}\backslash E$ is proper. Indeed, let $K \subset \tilde{U}\backslash E$ be a compact set such that $\pi^{-1}(K)$ is not compact. Choose a sequence $(w_\nu, w'_\nu) \in \pi^{-1}(K)$ converging to $(w_0, w'_0) \in K \times \partial U'$. By Proposition 3.4 and the remarks made after Definition 3.2, we may assume that $w_0 \in \Sigma$. There are two cases to be considered.

Case 1. $(w_\nu, w'_\nu) \in A^+$ after passing to a subsequence. Since $w_0 \in \Sigma$ it follows that $f(w_\nu)$ (for large ν) and $f(w_0)$ are defined and $(w_\nu, f(w_\nu)) \to (w_0, f(w_0))$. Then the invariance property

$$f(Q^c_{w_\nu}) \subset Q'_{f(w_\nu)} \tag{3.15}$$

holds for all ν. Also by the definition of A^+

$$f(Q^c_{w_\nu}) \subset Q'_{w'_\nu} \tag{3.16}$$

and combining these equations it follows from the properness of f near w_0 that $Q'_{f(w_\nu)} = Q'_{w'_\nu}$. Passing to the limit we get $Q'_{f(w_0)} = Q'_{w'_0}$ and this contradicts the properness of the Segre map $\lambda' : U' \to \lambda'(U') \subset \mathcal{S}'$.

Case 2. After passing to a subsequence $(w_\nu, w'_\nu) \in \overline{A^-}$, i.e., $w_\nu \in U^- \cup \Sigma$. In this case by the definition of A^-

$$Q'_{w'_\nu} = Q'_{f(w_\nu)}. \tag{3.17}$$

Passing to the limit we get $Q'_{f(w_0)} = Q'_{w'_0}$. Since $w_0 \in \Sigma$, $f(w_0) \in U'$ and this again contradicts the properness of the Segre map λ'.

The symmetric functions of the finitely many branches of the multivalued mapping $\pi' \circ \pi^{-1} : \tilde{U}\backslash E \to U'$ are bounded holomorphic functions in $\tilde{U}\backslash E$. Since E is locally finite they extend holomorphically across E. Thus $\pi' \circ \pi^{-1} : \tilde{U} \to U'$ is a well defined correspondence whose graph coincides with A^*. Hence the projection $\pi : A^* \to \tilde{U}$ is proper. $\square$

Using these observations we can show that:

Proposition 3.6. *A^* contains the set $A\backslash S$.*

Proof. The definition of A^* shows that it contains a component of $A \cap (\tilde{U} \times U')$. Hence we may choose $(p, p') \in A^* \cap (A \backslash S)$ and let $(q, q') \in A \backslash S$ be an arbitrary point. Now $dim S < n$ implies that $dim(\pi^{-1}(\pi(S))) < n$ and consequently $A \backslash \pi^{-1}(\pi(S))$ is path connected. Thus we may join the chosen points by a path $\gamma : [0,1] \to (A \backslash \pi^{-1}\pi(S)) \cup \{(p,p')\} \cup \{(q,q')\}$ with $\gamma(0) = (p,p')$ and $\gamma(1) = (q,q')$.

Claim. $\pi(\gamma) \in \tilde{U}$.

To prove this let

$$I \subset \{t \in [0,1] : \pi(\gamma) \in \tilde{U}\} \tag{3.18}$$

be a connected component containing 0. It is clear that I is open. Note that for $t \in I$, $\gamma(t) \in A^*$. Let us suppose that $I = [0, t_0)$ where $t_0 < 1$. Then $\pi(\gamma(t_0)) \in U \cap \partial\tilde{U}$. Since $M \backslash \Sigma \subset \pi(S)$ it follows that $\pi(\gamma)$ does not intersect $M \backslash \Sigma$. Thus the only possibility is that $\pi(\gamma(t_0)) \in \partial\tilde{U} \cap U^+$. In this situation $\gamma(t_0) \in U^+ \times \partial U'$. This is clearly a contradiction as γ lies in $U \times U'$.

Thus $I = [0,1]$ and this implies that $\pi(\gamma)$ can be lifted to a path in A^* starting at (p,p') and hence by the uniqueness theorem for analytic sets it follows that $(A \backslash S) \subset A^*$. ☐

By the construction of A^* it follows that for all $(w, w') \in (A \backslash S) \cap (U^+ \times U')$ the invariance property

$$f(Q^c_w) \subset Q'_{w'} \tag{3.19}$$

holds. Since S is nowhere dense in A it follows that

$$f(Q^c_w) \subset Q'_{w'} \tag{3.20}$$

holds for all $(w, w') \in A \cap (U^+ \times U')$.

Proposition 3.7. *In the general situation, $\pi' : A \to U'$ is proper for a suitable choice of neighbourhoods U, U'.*

Proof. It will suffice to show that the analytic set $\sigma := {\pi'}^{-1}(0')$ is 0 dimensional. Since σ is an analytic set in $U \times \{0'\}$ it can be regarded as an analytic set in U. If σ is not discrete near 0 it has an irreducible component, say σ_1, of positive dimension containing 0. First, since M is of finite type, it follows that no open subset of σ_1 is contained in M. Second, if $\sigma_1 \subset \overline{U^-} \times U'$ we may apply the strong disc theorem in [Vl] to conclude that $(0, 0')$ is in the envelope of holomorphy of $U^- \times U'$. As in Proposition 3.1, it follows that f extends holomorphically across 0. Thus we may assume that $\sigma_1 \cap U^+ \neq \emptyset$. Then by (3.20), we can write

$$f(Q^c_w) \subset Q'_{0'} \tag{3.21}$$

for all $w \in \sigma_1 \cap U^+$. Moreover the union of the canonical components of Q_w for $w \in \sigma_1 \cap U^+$ contains an open set in U^-. By Proposition 3.2 it is known that $J_f \not\equiv 0$, and hence by (3.20) and the finiteness of the Segre map $\lambda : U \to \mathcal{S}$, it follows that $\sigma_1 \cap U^+$ is a finite set. Thus σ is discrete and this shows that by suitably shrinking U, U', the projection $\pi' : A \to U'$ is proper. ☐

Since M' is of finite type, Trepreau's theorem ([T]) shows that after shrinking U', one or both of ${U'}^{\pm}$, say ${U'}^+$, has the property that all holomorphic functions defined on ${U'}^+$ extend holomorphically to U'. By the previous proposition, we may choose neighbourhoods U, U' so that $\pi : A \to U'$ is proper and at the same time

${U'}^+$ satisfies the above mentioned 'extension' property. By Proposition 3.2, choose $a \in U \cap M$ such that f extends biholomorphically near a and denote $a' := f(a)$. Choose neighbourhoods $a \in U_a$, $a' \in U'_{a'}$ so that

$$f : U_a \to U'_{a'} \tag{3.22}$$

is a biholomorphic mapping. Consider f^{-1} which is well defined in ${U'}^-_{a'}$ and may be considered as a locally holomorphic branch of $\hat{G} := \pi \circ {\pi'}^{-1} : U' \to U$. This germ of f^{-1} in ${U'}^-_{a'}$ may be analytically continued along all paths in ${U'}^-$ to give an irreducible correspondence $\hat{g} : {U'}^- \to U$. Note that the graph of $\hat{g}$ is contained in A. Furthermore, a different choice of a point of biholomorphic extendability for f in $U \cap M$ may apriori give a different correspondence defined in ${U'}^-$ by this process. Nevertheless it will suffice to work with $\hat{g}$ as above. Note that $\hat{g}$ extends continuously to $U' \cap M'$, but it is not known that the cluster set of $U' \cap M'$ under $\hat{g}$ is contained in $U \cap M$.

The aim is to show the invariance property of Segre varieties under $\hat{G}$. From (3.21) it follows that

$$f^{-1}(Q'_{f(z)} \cap U'_{a'}) = Q_z \cap U_a \tag{3.23}$$

for all $(z, f(z)) \in U_a \times U'_{a'}$. Thus the invariance property holds near (a, a') for some branch of the correspondence $\hat{g}$. Now exactly the same arguments that were used in Theorem 4.1 and Lemma 5.4 in [DP1] can be applied to show the following:

Proposition 3.8. *For all $(w, w') \in A$, the relation $\hat{G}(Q'_{w'}) \subset Q_w$ holds.*

To avoid repetition we simply provide the following reasoning given in [DP1] as to why such a statement should be true. Let $U^* := \{\omega : \overline{\omega} \in U\}$ and similarly define ${U'}^*, A^*$. For $\omega \in U^*$ let $\rho(z, \omega) := r(z, \overline{\omega})$ and similarly define $\rho'(z', \omega')$. Consider the analytic set

(3.24)

$$\mathcal{M} := \{((z, z', \omega, \omega') \in U \times U' \times U^* \times {U'}^* : (z, z') \in A, (\omega, \omega') \in A^*, \rho'(z', \omega') = 0\}.$$

Proposition 3.8 will follow if it can be shown that $\rho(z, \omega)$ vanishes on $\mathcal{M}$. Now some branch of $\hat{g}$, say g_1, extends across a' and thus for all $z' \in U'_{a'}$ we may write $r(g_1(z'), \overline{g_1(z')}) = \alpha'(z', \overline{z}')r'(z', \overline{z}')$. Here $\alpha'(z', \overline{z}')$ is some non-vanishing real analytic function. Complexification of this equation shows that

$$\rho(g_1(z'), g_1^*(\omega')) = \alpha'(z, \omega)\rho'(z', \omega') \tag{3.25}$$

where $g_1^*(\omega') := \overline{g_1(\overline{\omega}')}$. Note that $\rho(z, \omega)$ vanishes on the open set $\mathcal{M} \cap (U_a \times U'_{a'} \times U_a^* \times {U'}^*_{a'})$ and hence it would vanish everywhere on $\mathcal{M}$ if it can be shown that $\mathcal{M}$ is irreducible. This is done in Lemma 5.4 and Theorem 4.1 in [DP1].

Using this it can be shown that $\pi : A \to U$ is locally proper near $(0, 0')$. The proof is exactly the same as that of Theorem 5.1 in [DP1].

Proposition 3.9. *In the general situation, the natural projection $\pi : A \to U$ is locally proper near $(0, 0')$.*

This shows that in the general situation, both π and π' are locally proper near $(0, 0')$. Hence there are neighbourhoods $0 \in U_1 \subset\subset U_2 \subset\subset \Omega$ and $0' \in U'_1 \subset\subset U'_2 \subset\subset \Omega'$ so that $A \subset U_2 \times U'_2$ and both $\pi : A \cap (U_1 \times U'_1) \to U_1$, $\pi' : A \cap (U_2 \times U'_2) \to U'_2$ are proper. The proof of Theorem 1.1 is not complete yet; we cannot apply the result of [DP2] since it is not known that $f(U^-) \subset {U'}^-$. Let $\hat{F} := \pi' \circ \pi^{-1} : U_1 \to U'_1$

which is a finite, multivalued mapping associated with A. Note that $\hat{F}$ extends $f : U_1^- \to \Omega'$ as a correspondence. Then the proof of Theorem 4.1 in [DP1] shows that:

Proposition 3.10. *For all $(w, w') \in A \cap (U_1 \times U_1')$, the relation $\hat{F}(Q_w) \subset Q'_{w'}$ holds.*

Thus the invariance property of Segre varieties holds for both $\hat{F}$ and $\hat{G}$. This will allow us to show that:

Proposition 3.11. *With neighbourhoods of $0, 0'$ chosen as above $\hat{F}(U_1 \cap M) \subset U_1' \cap M'$ and $\hat{G}(U_2' \cap M') \subset U_2 \cap M$.*

Proof. To show that $\hat{F}(U_1 \cap M) \subset U_1' \cap M'$, choose $w_0 \in U_1 \cap M$ and $w_0' \in \hat{F}(w_0)$. Then Proposition 3.10 shows that $\hat{F}(Q_{w_0}) \subset Q'_{w_0'} = Q'_{f(w_0)}$. However, $f(w_0) \in M'$ and therefore $w_0' \in U_1' \cap M'$.

To show that $\hat{G}(U_2' \cap M') \subset U_2 \cap M$, we use Proposition 3.7 according to which $\pi' : A \to U'$ is locally proper. Therefore $f(M \cap U)$ contains an open subset of $0'$ in M'. By shrinking U_1, U_1', U_2, U_2' it follows that for all $z' \in U_2' \cap M'$ there exists $z \in \hat{G}(z')$ such that $z \in U_2 \cap M$. Now for $\tilde{z} \in \hat{G}(z'), z' \in U_2' \cap M'$, Proposition 3.8 shows that $\hat{G}(Q'_{z'}) \subset Q_{\tilde{z}}$. By the observation made before there exists $z \in \hat{G}(z') \cap M$ and hence $\hat{G}(Q'_{z'}) \subset Q_z$. Thus $Q_z = Q_{\tilde{z}}$ and this shows that $\tilde{z} \in M$. □

This shows that $f(U^-) \cap M' = \emptyset$ and by possibly interchanging the roles of ${U'}^{\pm}$ it follows that $f : U^- \to U'$ is a proper mapping which extends as a correspondence. By [DP2], it follows that f extends holomorphically across 0. This concludes the proof of Theorem 1.1.

For the proof of Theorem 1.2, we see that by [BS] f is algebraic. Furthermore, by [DF2] it is known that f also extends continuously up to $\partial\mathcal{D}$. Then by Theorem 1.1 f extends holomorphically across $\partial\mathcal{D}$.

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