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DECREASING FLOW INVARIANT SETS AND FIXED POINTS OF QUASIMONOTONE INCREASING OPERATORS

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Abstract. In this paper, we obtain some new results about the existence of multiple fixed points of a kind of quasimonotone increasing operator by the new method of decreasing flow invariant set.

1. Introduction

Let E be a real Banach space which is ordered by a cone P, i.e. $x,\ y \in E,\ x \leq y$ if and only if $y - x \in P$. P is said to be a solid cone if the interior of P(denoted as $\mathring{P}$) is nonempty. The dual cone of P, denoted as P^*, is defined as the set of all continuous linear functionals φ on E with $\varphi(x) \geq 0,\ x \geq \theta$.

Definition 1.1 ([1])**.** Let $D \subset E$, $A : D \to E$ is called quasimonotone increasing if

$$x, y \in D,\ x \leq y,\ \varphi \in P^*,\ \varphi(x) = \varphi(y) \Rightarrow \varphi(Ax) \leq \varphi(Ay).$$

In paper [2], Hu first studied the existence of fixed points of discontinuous quasimonotone increasing operators. Under the conditions that $A : R^n \to R^n$ is a quasimonotone increasing operator and that there are upper and lower solutions of A, Hu proved the existence of extremal fixed points of A. The paper [3] extended these results to the more general spaces such as $c_0,\ l^p\ (1 \leq p \leq +\infty)$, and so answered the open question in [2].

Recently, the author of paper [4] considered the existence of fixed points of continuous quasimonotone increasing operators in Banach spaces. Under the conditions that the cone is a regular solid cone, he proved a result about the existence of fixed points of continuous quasimonotone increasing operators. His method is by using the ordinary differential equations in Banach spaces.

In this paper, we will consider the multiple fixed points of continuous quasimonotone increasing operators in Hilbert space, which has been considered by few papers up to now. Our method is by using the decreasing flow invariant sets, which will be developed in the section 2 of this paper.

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2. Decreasing flow invariant sets

Let H be a Hilbert space, $f : H \to R^1$ a C^{2-0} functional, $f'(x) = x - Ax$ and $A : H \to H$ a quasimonotone increasing operator.

Consider the following IVP:

$$\begin{cases} \dfrac{dx}{dt} = Ax - x, \\ x(0) = x_0. \end{cases} \tag{2.1}$$

By the theory of ordinary differential equations in Banach spaces, (2.1) has a unique solution, denoted by $x(t,\ x_0)$, with right maximal existence interval $[0,\ \eta(x_0))$.

Definition 2.1. Let M be a subset of H. If

$$\overline{\{x(t, x_0) \mid t \in [0,\ \eta(x_0))\}} \subset M, \text{ for all } x_0 \in M,$$

then M is called a decreasing flow invariant set of f.

Obviously, we have the following Lemma 2.1 and Lemma 2.2:

Lemma 2.1. *$f(x(t,\ x_0))$ is decreasing with t on $[0,\ \eta(x_0))$.*

Lemma 2.2. *The following conclusions hold:*

(1) H is a decreasing flow invariant set of f.

(2) If $\{M_\mu | \mu \in \Lambda\}$ is a family of decreasing flow invariant sets of f, where Λ is an index set, then $\bigcup_{\mu\in\Lambda} M_\mu$ and $\bigcap_{\mu\in\Lambda} M_\mu$ are decreasing flow invariant sets of f.

(3) For any $a \in R^1$, the level sets $f_a^{\leq}$ and $f_a^{<}$ are both decreasing flow invariant sets of f.

Definition 2.2. Let M and D be decreasing flow invariant sets of f, $D \subset M$. Let

$$C_M(D) = \{x_0 | x_0 \in M, \text{ there exists } \ t' \in [0, \eta(x_0)) \text{ such that } x(t', x_0) \in D\};$$

then $C_M(D)$ is called a decreasing flow invariant set spanned by the decreasing flow invariant set D. If $D = C_M(D)$, then D is called a complete decreasing flow invariant set of f relative to M.

Remark 2.1. Clearly, we have

(1) H is a complete decreasing flow invariant set of f relative to H;

(2) If $D_1,\ D_2$ are two disjoint decreasing flow invariant sets of f, then

$$C_M(D_1) \cap C_M(D_2) = \emptyset.$$

Lemma 2.3. *Suppose that the connected set M is a decreasing flow invariant set of f, D is an open subset of M and a complete decreasing flow invariant set of f. If $D \neq M$, then $\partial_M D$, the boundary of D relative to M, is nonempty and is a complete decreasing flow invariant set of f.*

Proof. By the connectedness and the fact that $D \neq M$, we have $\partial_M D \neq \emptyset$.

We first prove that $\partial_M D$ is a decreasing flow invariant set of f. For any $x_0 \in \partial_M D$, we consider the IVP (2.1). If there exists $t' \in [0,\ \eta(x_0))$ such that $x(t', x_0) \notin \partial_M D$, then $x(t', x_0) \in D \cup (M \backslash \bar{D}^M)$, where $\bar{D}^M$ denotes the closure of D relative to M. If $x(t',\ x_0) \in D$, then $x_0 \in C_M(D) = D$, which contradicts $x_0 \in \partial_M D$ and the fact that D is an open subset of M. On the other hand, if $M \backslash \bar{D}^M$ is nonempty, $x(t', x_0) \in M \backslash \bar{D}^M$, noticing $M \backslash \bar{D}^M$ is an open set relative to M, by the continuous dependence of ordinary differential equations on initial data, we know that there exists a neighbourhood U of x_0 in M such that for any $x_0' \in U$, there exists $t'' \in [0,\ \eta(x_0'))$ such that $x(t'', x_0') \in M \backslash \bar{D}^M$. We may take $x_0' \in D \cap U$ such that $x(t'',\ x_0') \in M \backslash \bar{D}^M$, which contradicts the fact D is a decreasing flow invariant set. Thus $\{x(t, x_0) \mid t \in [0,\ \eta(x_0))\} \subset \partial_M D$. Since $\partial_M D$ is a closed set relative to M, therefore $\partial_M D$ is a decreasing flow invariant set of f.

Next, we prove that $\partial_M D$ is a complete decreasing flow invariant set of f. For any $x_0 \in M$, we suppose that there exists $t_1 \in [0,\ \eta(x_0))$ such that $x(t_1, x_0) \in \partial_M D$. Since D is a complete decreasing flow invariant set, then $x_0 \notin D$. Suppose that $x_0 \in M \backslash \bar{D}^M$. Consider the following IVP:

$$\begin{cases} \dfrac{dx}{dt} = x - Ax, \\ x(0) = x(t_1, x_0). \end{cases} \tag{2.2}$$

Let $\bar{x}(t)$ be the unique solution of (2.2). Let $\tau = t_1 - t$; then

$$\begin{aligned} \frac{dx(t_1 - t, x_0)}{dt} &= \frac{dx(\tau, x_0)}{d(t_1 - \tau)} = -\frac{dx(\tau, x_0)}{d\tau} \\ &= x(t_1 - t, x_0) - A(x(t_1 - t, x_0)). \end{aligned} \tag{2.3}$$

By the uniqueness of the solution of (2.2), we have

$$\bar{x}(t) = x(t_1 - t, x_0), \quad 0 \le t \le t_1, \tag{2.4}$$

and $\bar{x}(t_1) = x(0, x_0) = x_0$. By the continuous dependence of ordinary differential equations on initial data, there exists a neighbourhood U of $\bar{x}(0)$ in M such that for any $x_0^* \in U$, there exists $t' \in [0,\ \eta(x_0^*))$ such that $\bar{x}^*(t') \in M \backslash \bar{D}^M$, where $\bar{x}^*(t)$ denotes the unique solution of the equation $\frac{dx}{dt} = x - Ax, x(0) = x_0^*$, and $[0,\ \eta(x_0^*))$ denotes the maximal existence interval of $\bar{x}^*(t)$. Specially, we may take $x_0^* \in U \cap D$, and consider the following IVP:

$$\begin{cases} \dfrac{dx}{dt} = Ax - x, \\ x(0) = \bar{x}^*(t'). \end{cases} \tag{2.5}$$

In the same way as the proof of (2.3), (2.4), we have

$$x(t, \bar{x}^*(t')) = \bar{x}^*(t' - t), \quad 0 \le t \le t'. \tag{2.6}$$

Taking $t = t'$ in (2.6), then we have $x(t', \bar{x}^*(t')) = \bar{x}^*(0) = x_0^* \in D$. Therefore, $\bar{x}^*(t') \in C_M(D)$. Since $C_M(D) = D$, then $\bar{x}^*(t') \in D$. This contradicts $\bar{x}^*(t') \in M \backslash \bar{D}^M$. So, $x_0 \notin M \backslash \bar{D}^M$ and $x_0 \in \partial_M D$. The proof is completed. □

Lemma 2.4. *Suppose that the connected set M is a decreasing flow invariant set of f, and D is an open subset of M and a decreasing flow invariant set. Then*

(1) $C_M(D)$ is an open subset of M;

(2) if $C_M(D) \neq M$, $\inf\limits_{x \in \partial_M D} f((x) > -\infty$, then $\inf\limits_{x \in \partial_M C_M(D)} f(x) \geq \inf\limits_{x \in \partial_M D} f(x)$, where $\partial_M D$ and $\partial_M C_M(D)$ denote the boundary of D and $C_M(D)$ relative to M respectively.

Proof. By the continuous dependence of ordinary differential equations on initial data, the conclusion (1) holds. Suppose that the conclusion (2) doesn't hold; then there exists $x_0 \in \partial_M C_M(D)$ such that $f(x_0) < \inf\limits_{x \in \partial_M D} f(x)$. Since $f(x)$ is continuous, then there exists a neighbourhood U of x_0 in M such that $U \cap C_M(D) \neq \emptyset$ and for any $x_1 \in U,\ f(x_1) < \inf_{x\in\partial_M D} f(x)$. In particular, we may take $x_1 \in U \cap C_M(D)$ and consider the IVP(2.1) with initial value x_1. Then there exists $t' \in [0,\ \eta(x_1))$ such that $x(t', x_1) \in D$; therefore there exists $\bar{t} \in (0, t')$ with $x(\bar{t}, x_1) \in \partial_M D$. By Lemma 2.1, we have

$$f(x(\bar{t}, x_1)) \le f(x_1) < \inf_{x\in\partial_M D} f(x).$$

This is a contradiction. The proof is completed. □

Theorem 2.1. *Suppose that M is a closed decreasing flow invariant set of f, f satisfies the P.S. condition on M, and $\inf_{x\in M} f(x) > -\infty$. Then f has at least one critical point in M, and $c = \inf_{x\in M} f(x)$ is a critical value of f.*

Proof. The proof is standard and we only sketch it. Let $c = \inf_{x\in M} f(x)$. For any $n \in N$, there exists x_0^n such that $c \le f(x_0^n) \le c + \frac{1}{n}$. Let $x(t, x_0^n)$ be the unique solution of (2.1) and $[0,\ \eta(x_0^n))$ be the maximal existence interval. We have

$$\frac{d}{dt} f(x(t, x_0^n)) = (f'(x(t, x_0^n)), x'(t, x_0^n)) = -\|x'(t, x_0^n)\|^2 \le 0. \tag{2.7}$$

For $0 \le t_1 < t_2 < \eta(x_0^n)$, by (2.7), we have

$$\begin{aligned}\|x(t_2, x_0^n) - x(t_1, x_0^n)\| &\le \int_{t_1}^{t_2} \|x'(t, x_0^n)\| dt \le (\int_{t_1}^{t_2} \|x'(t, x_0^n)\|^2 dt)^{\frac{1}{2}} (t_2 - t_1)^{\frac{1}{2}} \\ &\le (f(x_0^n) - c)^{\frac{1}{2}} (t_2 - t_1)^{\frac{1}{2}}.\end{aligned} \tag{2.8}$$

Now, we prove that $\eta(x_0^n) = +\infty$. Otherwise, if $\eta(x_0^n) < +\infty$, by (2.8), we know that $\|x(t_2, x_0^n) - x(t_1, x_0^n)\| \to 0$ as $t_1 \to \eta(x_0^n)^-$, $t_2 \to \eta(x_0^n)^-$. Thus there exists $x^* \in M$ such that $\lim_{t\to\eta(x_0^n)^-} x(t, x_0^n) = x^*$. Consequently, the maximal interval of existence of $x(t, x_0^n)$ would be $[0,\ \eta(x_0^n) + \delta(x^*))$ for some $\delta(x^*) > 0$, which is a contradiction to the maximality of $[0, \eta(x_0^n))$. Thus, $\eta(x_0^n) = +\infty$. By (2.7), for any $t > 0$, we have

$$\int_0^t \|f'(x(t, x_0^n))\|^2 dt = f(x(0, x_0^n)) - f(x(t, x_0^n)) \le f(x_0^n) - c < +\infty.$$

Consequently, there exist $t_n \in \{x(t, x_0^n) | t \in [0,\ \eta(x_0^n))\},\ t_n \to +\infty$, such that

$$c \le f(x(t_n, x_0^n)) \le f(x_0^n) \le c + \frac{1}{n},\quad \|f'(x(t_n, x_0^n))\| \le \frac{1}{n}.$$

Then, by the *P.S.* condition, we know the conclusion holds. The proof is completed. □

By Lemma 2.3 and 2.4, we have the following Theorem 2.2.

Theorem 2.2. *Suppose that the connected set M is a decreasing flow invariant set of f, and D is an open subset of M and is also a decreasing flow invariant set. Then*

(1) $C_M(D)$ is a decreasing flow invariant set and $C_M(D)$ is an open subset of M;

(2) if $C_M(D) \neq M$, then $\partial_M C_M(D)$ is a complete decreasing flow invariant set;

(3) if $C_M(D) \neq M$ and $\inf_{x\in\partial_M D} f(x) > -\infty$, then $\inf_{x\in\partial_M C_M(D)} f(x) \ge \inf_{x\in\partial_M D} f(x)$.

For applications in the sequel, we give the following similar result.

Theorem 2.3. *Suppose that the connected set M is a decreasing flow invariant set of f, D is a closed subset of M, $\overset{\circ}{D} \neq \emptyset$ ($\overset{\circ}{D}$ denotes the interior of D relative to M), and D is also a decreasing flow invariant set. Also suppose that f satisfies the P.S. condition on $\partial_M D$ and has no critical point on $\partial_M D$ and $\inf\limits_{x\in\partial_M D} f(x) > -\infty$. Then the conclusions (1)–(3) in Theorem 2.2 hold.*

Proof. We only need to prove that $C_M(D)$ is an open subset of M. For any $x_0 \in C_M(D)$, it is easy to prove that there exist $t' \in [0,\ \eta(x_0))$ such that $x(t', x_0) \in \overset{\circ}{D}$. Then by the continuous dependence of ordinary differential equations on initial data, we know that $C_M(D)$ is an open subset of M. The proof is completed. □

Remark 2.2. The main results in this section come from [7].

3. Fixed points of quasimonotone increasing operators

In this section, we will discuss the multiple fixed points of quasimonotone increasing operator.

Lemma 3.1 ([5])**.** *Let P be a cone in E, and also a distance set. Assume that*

(1) $u, v \in C^1[[t_0, t_0+a), E]$, $f \in C[[t_0, t_0+a)\times E, E]$ and $f(t, x)$ is quasimonotone increasing in x for each $t \in [t_0, t_0 + a)$;

(2) $u'(t) - f(t, u(t)) \le v'(t) - f(t, v(t)),\quad t \in [t_0, t_0 + a)$;

(3) $\|f(t, x) - f(t, y)\| \le g(t, \|x - y\|)$, $x \in E\backslash P$, $y \in \partial P$, where g is a uniqueness function.

Then $u(t_0) \le v(t_0)$ implies that $u(t) \le v(t),\ t \in [t_0, t_0 + a)$.

Now, let us formulate some conditions.

(H_1) Let H be a Hilbert space, $P \subset H$ a solid cone, and $f : P \to R^1$ a C^{2-0} functional. Suppose that $f'(x) = x - Ax$, $A : P \to P$ is quasimonotone increasing, and f satisfies the $P.S.$ condition on P.

(H_2) There exist $x_0 \in \overset{\circ}{P},\ y_0 \in \overset{\circ}{P}$ such that $Ax_0 \le x_0,\ Ay_0 \ge y_0$.

Remark 3.1. Under the condition (H_1), by Lemma 3 in [6], we know that P is a decreasing flow invariant set.

Theorem 3.1. *Suppose that (H_1), (H_2) hold and that*

(1) $y_0 \not\le x_0$;

(2) $\inf\limits_{x\in D_1} f(x) > -\infty$, f has no critical points on $\partial_P D_1$, where $D_1 = [\theta, x_0]$.

Then A has at least two fixed points on P.

Proof. Let $D_2 = \{x \in P : x \ge y_0\}$. Since H is a Hilbert space and P is a closed convex set, we know that P is a distance set. It follows from Lemma 3.1 that D_1 and D_2 are two decreasing flow invariant sets of f. Since $y_0 \not\le x_0$, then $D_1 \cap D_2 = \emptyset$. By the definition of $C_P(D_1)$, we know that $C_P(D_1) \cap D_2 = \emptyset$, $C_P(D_1) \neq P$. It follows from Theorem 2.3 that $\partial_P C_P(D_1)$ is a decreasing flow invariant set and that

$$c_1 = \inf_{x\in\partial_P C_P(D_1)} f(x) \ge \inf_{x\in\partial_P D_1} f(x) > -\infty.$$

Let $c_2 = \inf\limits_{x\in D_1} f(x)$. By Theorem 2.1, c_1, c_2 are critical values and there exist x_1,x_2

such that

$$x_1 \in \partial_P C_P(D_1), \quad x_2 \in D_1; \quad f(x_i) = c_i, \quad i = 1, 2.$$

Clearly, x_1, x_2 are two fixed points of A on P. The proof is completed. □

Theorem 3.2. *Suppose that* (H_1), (H_2) *hold and that*
(1) $y_0 \leq x_0$;
(2) there exists a $z_0 \in D_2$ *with* $z_0 \not\leq x_0$ *such that*

$$\max\{f(\theta), f(z_0)\} < \inf_{x \in \partial_P(D_1 \cap D_2)} f(x), \tag{3.1}$$

where $D_1 = [\theta, x_0]$, $D_2 = \{x \in P | x \geq y_0\}$;
(3) f *has no critical points on* $\partial_P(D_1 \cap D_2)$; $\inf_{x \in D_1 \cup D_2} f(x) > -\infty$.
Then A *has at least five fixed points.*

Proof. Let $D_3 = D_1 \cap D_2$. It follows from Lemma 3.1 that D_1 and D_2 are two decreasing flow invariant sets of f. Thus D_3 is also a decreasing flow invariant set. Then, by (3.1), we easily prove that $\theta \notin C_{D_1}(D_3)$, $z_0 \notin C_{D_2}(D_3)$. It follows from Theorem 2.3 that $\partial_{D_1} C_{D_1}(D_3)$, $\partial_{D_2} C_{D_2}(D_3)$ are two decreasing flow invariant sets of f.

Let $F_1 = D_1 \backslash C_{D_1}(D_3)$, $F_2 = D_2 \backslash C_{D_2}(D_3)$. It is easy to prove that F_i $(i = 1, 2)$ are decreasing flow invariant sets of f.

Let $c_1 = \inf_{x \in \partial_{D_1} C_{D_1}(D_3)} f(x)$, $c_2 = \inf_{x \in \partial_{D_2} C_{D_2}(D_3)} f(x)$, $c_3 = \inf_{x \in F_1} f(x)$, $c_4 = \inf_{x \in F_2} f(x)$, $c_5 = \inf_{x \in D_3} f(x)$. By Theorem 2.1, we know that c_i $(i = 1, 2, 3, 4, 5)$ are critical values, and there exist five points x_i $(i = 1, 2, 3, 4, 5)$ such that $f(x_i) = c_i$. Clearly, x_i $(i = 1, 2, 3, 4, 5)$ are five fixed points of A. The proof is completed. □

Remark 3.2. In Theorems 3.1 and 3.2, we give the results on the multiple fixed points of a quasimonotone increasing operator under the condition that the lower solutions may not be less than the upper solutions, which is different from all the known results on the fixed points of a quasimonotone increasing operator. Our method is also different from all the known results.

Remark 3.3. Clearly, our results can be applied to the case when A is increasing.

Example. Let $H = R^2$, $P = R^+ \times R^+$. Consider the mapping $A : R^2 \to R^2$,

$$A((x_1, x_2)) = (\frac{1}{8}(1 + \sin x_1) + \frac{1}{4}x_2, \frac{1}{4}x_1 + 5\pi(1 + \cos x_2)), \quad (x_1, x_2) \in R^2.$$

It is easy to see that $A : P \to P$ is a quasimonotone increasing operator. Let $f(x_1, x_2) = \frac{1}{2}(x_1^2 + x_2^2) - g(x_1, x_2)$, $g(x_1, x_2) = \frac{1}{8}x_1 - \frac{1}{8}\cos x_1 + \frac{1}{4}x_1 x_2 + 5\pi x_2 + 5\pi \sin x_2$. Then we have

$$\begin{aligned}(\frac{\partial g(x_1, x_2)}{\partial x_1}, \frac{\partial g(x_1, x_2)}{\partial x_2}) &= (\frac{1}{8}(1 + \sin x_1) + \frac{1}{4}x_2, \frac{1}{4}x_1 + 5\pi(1 + \cos x_2)) \\ &= A((x_1, x_2)),\end{aligned}$$

$$\left(\frac{\partial f(x_1,x_2)}{\partial x_1},\frac{\partial f(x_1,x_2)}{\partial x_2}\right)=(x_1,x_2)-\left(\frac{\partial g(x_1,x_2)}{\partial x_1},\frac{\partial g(x_1,x_2)}{\partial x_2}\right)$$
$$=x-Ax$$

where $x=(x_1,x_2)$. Let $x_0=(\frac{\pi}{2},\pi)$, $y_0=(\pi,8\pi)$. Then we have

$$\begin{aligned}Ax_0&=(\frac{1}{8}(1+\sin\frac{\pi}{2})+\frac{1}{4}\pi,\frac{1}{8}\pi+5\pi(1+\cos\pi))\\&=(\frac{1}{4}+\frac{\pi}{4},\frac{1}{8}\pi)<(\frac{\pi}{2},\pi)=x_0,\\Ay_0&=(\frac{1}{8}(1+\sin\pi)+2\pi,\frac{1}{4}\pi+5\pi(1+\cos 8\pi))\\&=(\frac{1}{8}+2\pi,10.25\pi)>(\pi,8\pi)=y_0.\end{aligned}$$

It is easy to see that $y_0>x_0$, and so $y_0\not\leq x_0$. We also easily know that $f(x_1,x_2)>-\frac{9}{8},(x_1,x_2)\in[0,\ \frac{\pi}{2}]\times[0,\ \pi]$. Therefore $\inf\limits_{x\in[\theta,\ x_0]}f(x)>-\frac{9}{8}$. It is easy to check that

$$A(\frac{\pi}{2},y)\neq(\frac{\pi}{2},y),\quad A(x,\pi)\neq(x,\pi),\ 0\leq x\leq\frac{\pi}{2},\ 0\leq y\leq\pi.$$

So, f has no critical point on the boundary of $[0,\ \frac{\pi}{2}]\times[0,\ \pi]$. By Theorem 3.1, we know that A has at least two fixed points on P.

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