PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 129, Number 9, Pages 2789–2797
S 0002-9939(01)06068-3

DETERMINACY OF SMOOTH GERMS WITH REAL ISOLATED LINE SINGULARITIES

BOHAO SUN AND LESLIE C. WILSON

(Communicated by Jozef Dodziuk)

Abstract. The germ of a smooth real-valued function on Euclidean space is called a real isolated line singularity if its singular set is a nonsingular curve, its Jacobian ideal is Łojasiewicz at the singular set, and its Hessian determinant restricted to the singular set is Łojasiewicz at 0. Consider the set of all germs whose singular set contains a fixed nonsingular curve L. We prove that such a germ f is infinitely determined among all such germs with respect to composition by diffeomorphisms preserving L if, and only if, the Jacobian ideal of f contains all germs which vanish on L and are infinitely flat at 0 if, and only if, f is a real isolated line singularity.

§1. Introduction

If f is a complex holomorphic germ in n variables with an isolated singularity at 0, then $V(f) := f^{-1}(0)$ is a hypersurface germ with an isolated singularity at 0. These of course have been widely studied. One of the properties of such an f is that it is finitely determined: there is a k such that $j^k g(0) = j^k f(0)$ implies that there is a germ of a biholomorphic h such that $g \circ h = f$ (i.e., g is "right-equivalent" to f). More recently there has been much interest in studying varieties with non-isolated singularities. The simplest such varieties are the zero sets of an f whose singular set $\Sigma(f)$ is a nonsingular curve with transversal cross-section away from 0 a Morse singularity; these are called "isolated line singularities". Siersma, in [Si], proved the finite determinacy of these inside the space of functions whose singular sets contain $\Sigma(f)$. In this paper we prove the analogue of this latter result for real analytic or smooth functions.

Let $\mathbf{E}_k$ be the ring of all germs at 0 of smooth functions on $\mathbf{R}^k$, $\mathbf{m}_k$ be the maximal ideal of $\mathbf{E}_k$ and $J_k^l = \mathbf{E}_k/\mathbf{m}_k^{l+1}$ be the jet space. Denote by $\mathcal{R}$ the group of all smooth local diffeomorphisms ϕ at 0 on $\mathbf{R}^k$ which fix the origin, i.e., $\phi(0) = 0$. $\mathcal{R}$ acts on $\mathbf{E}_k$ by composition; $f, g \in \mathbf{E}_k$ are *$\mathcal{R}$-equivalent* if they are in the same $\mathcal{R}$ orbit. One says that f is *finitely* (respectively, *infinitely*) *$\mathcal{R}$-determined* if there is an $l \in \mathbf{N}$ (respectively, $l = \infty$) such that $j^l g(0) = j^l f(0)$ implies that g and f are $\mathcal{R}$-equivalent. If f is a function of $x_1, \dots, x_k$, let $J(f)$ denote the Jacobian ideal $(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_k})\mathbf{E}_k$. If f is real analytic, $f_{\mathbf{C}} : \mathbf{C}^k \to \mathbf{C}$ denotes the complexification of f.

Received by the editors January 5, 2000.

1991 *Mathematics Subject Classification.* Primary 58K40; Secondary 32S05.

Key words and phrases. Non-isolated singularities, line singularities, infinite determinacy, Lojasiewicz inequality.

The following results are classical (a good reference is [Wa]).

Theorem 1. *Suppose $f \in \mathbf{E}_n$. The following are equivalent:*

(1) *f is finitely $\mathcal{R}$-determined*;
(2) *$J(f) \supset \mathbf{m}_n^l$ for some $l \in \mathbf{N}$*;
(3a) (*if f is analytic*) *$f_{\mathbf{C}}$ has an isolated singularity at* 0.

Theorem 2. *Suppose $f \in \mathbf{E}_n$. The following are equivalent*:

(1) *f is infinitely $\mathcal{R}$-determined*;
(2) *$J(f) \supset \mathbf{m}_n^\infty$*;
(3a) (*if f is analytic*) *f has an isolated singularity at* 0.

The assumption in (3a) of Theorem 2 that f be analytic can be dropped if we replace the condition of isolated singularity by a Łojasiewicz inequality. One says that a continuous germ g at 0 is *Łojasiewicz on A at S* if for one (and hence for every) representative G of g, there is a neighborhood U of 0 and constants $C, r > 0$ such that $|G(x)| \geq Cd(x,S)^r$ for all $x \in U \cap A$, where $d(x,S) = \inf_{y\in S} |x-y|$ (necessarily S contains $g^{-1}(0)$). We omit saying "on A" if A contains a neighborhood of 0. A finitely generated ideal I is Łojasiewicz at S if it contains an element which is Łojasiewicz at S. As pointed out in section V.4 of [To], it is equivalent to check for any set of generators of the ideal whether the sum of the squares of the generators or the sum of the absolute values of the generators is Łojasiewicz at S. Then Theorem 2 is true if we replace condition (3a) by

$$J(f) \text{ is Łojasiewicz at } 0. \tag{3}$$

It is a well-known result of Łojasiewicz that every analytic germ at 0 is Łojasiewicz at its zero set. Hence condition (3) reduces to condition (3a) when f is analytic.

Before stating the analogous theorems for isolated line singularities, we need some more notation and some preliminary results.

Let $z = (x,y) = (x, y_1, \ldots, y_n) = (z_0, z_1, \ldots, z_n)$ be the coordinate system of $\mathbf{R}^{n+1}$, where $x \in \mathbf{R}$ and $y \in \mathbf{R}^n$, and let $|y| = \sum_{i=1}^n |y_i|, |z| = \sum_{i=0}^n |z_i|$. Let $f : (\mathbf{R}^{n+1}, 0) \to (\mathbf{R}, 0)$ be a germ of a smooth function with a smooth 1-dimensional critical set $\Sigma(f)$. Necessarily $f = 0$ on $\Sigma(f)$. After a change of coordinates we can assume $\Sigma(f) = L = \{(x,y) : y = 0\}$, so $f(z) \in (y)^2\mathbf{E}_{n+1}$, where $(y)^2\mathbf{E}_{n+1}$ denotes the ideal in $\mathbf{E}_{n+1}$ generated by all y_iy_j, $1 \leq i,j \leq n$. For any ring R, let $\mathbf{M}(R,n)$ denote the space of all $n \times n$ matrices with entries in R, and let $\mathbf{S}(R,n)$ denote the subspace of $\mathbf{M}(R,n)$ consisting of all symmetric matrices.

Let $\mathcal{R}_L$ denote the subgroup of all local diffeomorphisms ϕ at 0 on $\mathbf{R}^{n+1}$ which fix the origin and leave the x-axis invariant, i.e., $\phi(0) = 0$ and $\phi(L) = L$. The group $\mathcal{R}_L$ acts on $(y)^2\mathbf{E}_{n+1}$ by composing on the right. Let $\mathcal{R}_L \cdot f$ denote the orbit of $f \in (y)^2\mathbf{E}_{n+1}$.

Definition. f and g in $(y)^2\mathbf{E}_{n+1}$ are said to be *$\mathcal{R}_L$-equivalent* if there is some $\phi \in \mathcal{R}_L$ such that $f = g \circ \phi$, i.e., $\mathcal{R}_L \cdot f = \mathcal{R}_L \cdot g$.

Definition. $f \in (y)^2\mathbf{E}_{n+1}$ is *k-determined in $(y)^2\mathbf{E}_{n+1}$* if $f + (y)^2\mathbf{m}_{n+1}^{k-1} \subset \mathcal{R}_L \cdot f$, f is *finitely determined in $(y)^2\mathbf{E}_{n+1}$* if f is k-determined in $(y)^2\mathbf{E}_{n+1}$ for some k, and f is *∞-determined in $(y)^2\mathbf{E}_{n+1}$* if $f + (y)^2\mathbf{m}_{n+1}^\infty \subset \mathcal{R}_L \cdot f$.

It is shown in [Si] that

$$\tau(f) = \{a\frac{\partial f}{\partial x} + \sum_l b_l \frac{\partial f}{\partial y_l} : a \in \mathbf{m}_{n+1}, b_l \in (y)\mathbf{E}_{n+1}, 1 \leq l \leq n\}$$

is the tangent space at f to the orbit $\mathcal{R}_L \cdot f$.

Let $f = \sum_{i,j} y_i y_j f_{ij}$, with $f_{ij} = f_{ji}$. Let $D(x,y) = \det(f_{ij})$ and, by abuse of notation, let $D(x) = D(x,0)$; $D(x)$ is the determinant of the Hessian matrix of f with respect to y on L, i.e., the matrix of second partials of f with respect to the y variables, evaluated on L. Thus $D(x)$ is independent of the choice of the f_{ij}'s.

Definition. $f \in \mathbf{E}_{n+1}$ has a *real isolated line singularity* if:

(1) $\Sigma(f)$ is a nonsingular curve through 0;
(2) $J(f)$ is Łojasiewicz at $\Sigma(f)$; and
(3) $D|\Sigma(f)$ is Łojasiewicz at 0.

Remark. The second condition in the above definition does not imply the third by the following example. Obviously the third one can't imply the second.

Examples. Let $f(x,y) = x(y_1^4 + y_2^4)$ and let $g(x,y) = y_1^3 + y_2^2$. Both have singular set the x-axis, and their Jacobian ideals are Łojasiewicz at the x-axis since they are analytic. In both cases, $D(x) = 0$ for all x, so condition (3) of the definition fails.

Here are some functions which are isolated line singularities: y^2, xy^2, x^2y^2, $x^2y^2 + y^r$, $r \geq 3$. The following function is a real isolated line singularity, but its complexification is not an isolated line singularity: $(y_1^2 + y_2^2)(x^2 + y_1^2 + y_2^2)$.

Lemma 1.1. *Assuming that* $\Sigma(f) = \{y = 0\}$, *condition* (2) *in the definition of real isolated line singularity can be replaced by*: *for every* $r > 0$, $J(f)$ *is Łojasiewicz at* 0 *on* $\{z = (x,y) : |y| \geq |x|^r\}$.

Proof. It is routine to see that (2) implies the condition of this Lemma. We shall show that (3) implies:

(∗) there is some $r > 0$ such that $J(f)$ is Łojasiewicz at L on $H_r = \{|y| \leq |x|^r\}$.

Hence (3) and the condition of this Lemma together imply (2).

Suppose that (3) holds and that (∗) fails. Then, for each $p \in \mathbf{N}$, $J(f)$ fails to be Łojasiewicz at L on H_p. Let $d_y f = (\frac{\partial f}{\partial y_1}, \ldots, \frac{\partial f}{\partial y_n})$ and let $d_y^2 f$ denote the Hessian matrix of f with respect to y. Then there exists a sequence $z_p = (x_p, y_p) \to 0$ with $|y_p| \leq |x_p|^p$ such that $|d_y f(z_p)| \leq |y_p|^p$. Let S_p denote the line segment joining $w_p = (x_p, 0)$ to z_p, let r be the distance from L, and let $\frac{\partial g}{\partial r}$ denote the derivative of any function g with respect to r. Since $d_y f(w_p) = 0$, the Mean Value Theorem applied to $\frac{\partial f}{\partial y_i}|S_p$ implies that there exist $z_{p,i} = (x_p, y_{p,i}) \in S_p$ such that

$$\begin{aligned} |\frac{\partial^2 f}{\partial r \partial y_i}(z_{p,i})| &= |\frac{\partial f}{\partial y_i}(z_p)|/|y_p| \\ &\leq |y_p|^{p-1} \leq |x_p|^{p(p-1)}. \end{aligned}$$

Note that $\frac{\partial}{\partial r} d_y f = u_r \cdot d_y^2 f$, where u_r is the unit radial vector in $L^\perp$. It is not hard to see that $D(x) = \det d_y^2 f(x,0)$ Łojasiewicz at 0 implies that $u_r \cdot d_y^2 f(x,0)$ is Łojasiewicz at 0; hence, there is an $R > 0$ such that $|\frac{\partial}{\partial r} d_y f(w_p)| \geq |x_p|^R \geq$

$2n|x_p|^{p(p-1)}$ for p sufficiently large. There exist $z'_{p,i} \in S_p$ such that

$$\sum_i \left|\frac{\partial^3 f}{\partial r^2 \partial y_i}(z'_{p,i})\right| = \sum_i \frac{\left|\frac{\partial^2 f}{\partial r \partial y_i}(z_{p,i}) - \frac{\partial^2 f}{\partial r \partial y_i}(w_p)\right|}{|y_{p,i}|} \geq \frac{|x_p|^R}{2|x_p|^p} \to \infty$$

as $p \to \infty$. This is impossible since f is C^3. ☐

Theorem 3 ([Si]). *For $f \in (y)^2\mathbf{E}_{n+1}$, the following conditions are equivalent*:

(1) *f is finitely determined in $(y)^2\mathbf{E}_{n+1}$.*
(2) *$\tau(f) \supset (y)^2\mathbf{m}^k_{n+1}$ for some k.*
(3) *$J(f) \supset (y)\mathbf{m}^k_{n+1}$ for some k.*
(4a) *(f analytic) $f_{\mathbf{C}}$ has an isolated line singularity.*

Actually Siersma only considered complex analytic functions, but it is easy to check that (1), (2) and (3) are still equivalent in the smooth case.

Siersma's result has also been generalized to the case in which $\Sigma(f)$ is a curve with a singularity at 0 (see [Pe] and [IzMats]).

The principal result of this paper extends Theorem 3 to real isolated line singularities and infinite determinacy:

Theorem 4. *For $f \in (y)^2\mathbf{E}_{n+1}$, the following conditions are equivalent*:

(1) *f is infinitely determined in $(y)^2\mathbf{E}_{n+1}$.*
(2) *$\tau(f) \supset (y)^2\mathbf{m}^\infty_{n+1}$.*
(3) *$J(f) \supset (y)\mathbf{m}^\infty_{n+1}$.*
(4) *f has a real isolated line singularity.*

(4) $\Rightarrow$ (3) is proved in §2, (3) $\Rightarrow$ (2) is trivial, (2) $\Rightarrow$ (1) is proved in §3 and (1) $\Rightarrow$ (4) is proved in §4.

§2. THE JACOBIAN OF f

Let $\mathcal{M}$ be a finitely generated $\mathbf{E}_k$ submodule of $\mathbf{E}^p_k$. A set of generators $f_1, \dots, f_r$ of $\mathcal{M}$ gives rise to a $p \times r$ matrix M whose columns are the f_j's; then $M(\mathbf{E}^r_k) = \mathcal{M}$. Let $F_0(M)$ denote the ideal in $\mathbf{E}_k$ generated by the $p \times p$ minors of M. It is easy to see that this ideal is independent of the choice of generators (use the Cauchy-Binet formula for the determinant of the product of two matrices)—it is called the 0th Fitting ideal of the module $\mathcal{M}$.

Lemma 2.1. *$\mathcal{M} \supset (\mathbf{m}^\infty_k)^p$ iff $F_0(\mathcal{M}) \supset \mathbf{m}^\infty_k$.*

Proof. **"if":** Let M be a matrix whose columns generate $\mathcal{M}$. For any $p \times p$ matrix A, $A(\operatorname{adj} A) = (\det A)I$; hence $A(\mathbf{E}^p_k) = \det A \cdot \mathbf{E}^p_k$. Applying this to the $p \times p$ submatrices of M, we conclude $\mathcal{M} \supset F_0(\mathcal{M})\mathbf{E}^p_k$.

"only if": Pick $u \in \mathbf{m}^\infty_k$. There exist $u_1, \dots, u_p \in \mathbf{m}^\infty_k$ such that $u = u_1 \cdot \dots \cdot u_p$ (see for example [To]). Let A be the diagonal matrix with $u_1, \dots, u_p$ on the diagonal. By assumption, there is an $r \times p$ matrix N such that $A = MN$. By the Cauchy-Binet formula, $u = \det A$ is in $F_0(\mathcal{M})$. ☐

Let ψ: $\mathbf{E}^n_{n+1} \to (y)\mathbf{E}_{n+1}$ be the map defined by $(f_1, f_2, \cdots, f_n) \mapsto \sum_{i=1}^n y_i f_i$. Then $\ker\psi$ is the *module of relations* among the functions $y_1, \dots, y_n \in \mathbf{E}_{n+1}$. Let $e_i \in \mathbf{E}^n_{n+1}$ have a 1 in the i-th component and 0's elsewhere, for $1 \leq i \leq n$.

Lemma 2.2. $\ker\psi$ *is generated by the trivial relations* $\{y_i e_j - y_j e_i : 1 \leq j < i \leq n\}$.

Proof. Let $\mathbf{O}_k$ denote the ring of analytic germs at 0 in $\mathbf{R}^k$ and let α: $\mathbf{O}_{n+1}^n \to (y)\mathbf{O}_{n+1}$ be the map defined by $(f_1, f_2, \cdots, f_n) \mapsto \sum_{i=1}^n y_i f_i$.

Because $y_1, \ldots, y_n$ is a regular sequence in $\mathbf{O}_{n+1}$, the analytic relations $\ker\alpha$ are generated by the trivial relations (see I.5.1 of [To]). But $\mathbf{E}_{n+1}$ is a flat $\mathbf{O}_{n+1}$-module (Corollary VI.1.3 of [To]). It follows from Proposition I.4.2 of [To] that $(\ker\alpha) \otimes_{\mathbf{O}_{n+1}} \mathbf{E}_{n+1} = \ker(\alpha \otimes_{\mathbf{O}_{n+1}} \mathbf{E}_{n+1}) = \ker\psi$, i.e., the smooth relations $\ker\psi$ are also generated by the trivial relations. □

Let $R(f)$ denote the matrix whose columns are these generators of $\ker\psi$.

Next we wish to determine $\psi^{-1}J(f)$ for $f \in (y)^2\mathbf{E}_{n+1}$. We write $f = \sum_{i=1}^n y_i f_i$, and $f_i = \sum_{j=1}^n y_j f_{ij}$ with $f_{ij} = f_{ji}$. Then $\frac{\partial f}{\partial x} = \sum_{i=1}^n y_i \frac{\partial f_i}{\partial x}$ and $\frac{\partial f}{\partial y_j} = f_j + \sum_{i=1}^n y_i \frac{\partial f_i}{\partial y_j} = \sum_{i=1}^n y_i(\frac{\partial f_i}{\partial y_j} + f_{ij})$. Let $\mathcal{M}(f)$ be the module generated by the columns

$$\begin{pmatrix} \dfrac{\partial f_1}{\partial x} \\ \dfrac{\partial f_2}{\partial x} \\ \cdot \\ \cdot \\ \cdot \\ \dfrac{\partial f_n}{\partial x} \end{pmatrix}, \quad \begin{pmatrix} \dfrac{\partial f_1}{\partial y_1} + f_{11} \\ \dfrac{\partial f_2}{\partial y_1} + f_{21} \\ \cdot \\ \cdot \\ \cdot \\ \dfrac{\partial f_n}{\partial y_1} + f_{n1} \end{pmatrix}, \quad \cdots, \quad \begin{pmatrix} \dfrac{\partial f_1}{\partial y_n} + f_{1n} \\ \dfrac{\partial f_2}{\partial y_n} + f_{2n} \\ \cdot \\ \cdot \\ \cdot \\ \dfrac{\partial f_n}{\partial y_n} + f_{nn} \end{pmatrix},$$

and let $M(f)$ be the corresponding matrix. Let $M(f)|R(f)$ be the matrix formed by adjoining these two matrices (so this matrix generates the submodule $\mathcal{M}(f)+\ker\psi$).

Thus

$$(y_1, \ldots, y_n) \cdot M(f) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y_1}, \ldots, \frac{\partial f}{\partial y_n}\right)$$

generates $J(f)$. Therefore

$$\psi^{-1}J(f) = \mathcal{M}(f) + \ker\psi.$$

Let $I(f) = F_0(\mathcal{M}(f) + \ker\psi)$.

Proposition 2.3. *$J(f) \supset (y)\mathbf{m}_{n+1}^\infty$ if, and only if, $I(f)$ is Łojasiewicz at* 0.

Proof. $J(f) \supset (y)\mathbf{m}_{n+1}^\infty$ iff $\mathcal{M}(f)+\ker\psi \supset \psi^{-1}((y)\mathbf{m}_{n+1}^\infty) = \mathbf{m}_{n+1}^\infty(\mathbf{E}_{n+1}^n)+\ker\psi$ iff $\mathcal{M}(f)+\ker\psi \supset \mathbf{m}_{n+1}^\infty(\mathbf{E}_{n+1}^n)$ iff $I(f) \supset \mathbf{m}_{n+1}^\infty$ (by Lemma 2.1) iff $I(f)$ is Łojasiewicz at 0 (by Proposition V.4.3 of [To]). □

Proof of (4) ⇒ (3). By assumption, $|\frac{\partial f}{\partial x}| + \sum |\frac{\partial f}{\partial y_j}|$ is Łojasiewicz at L, and $D(x) = D(x,0) = \det(f_{i,j}(x,0))$ is Łojasiewicz at 0. By Proposition 2.3, it suffices to prove that $I(f)$ is Łojasiewicz at 0.

For each i and j, $1 \le i \le n$ and $1 \le j \le n+1$, let $M_{i,j}$ denote the $n \times n$ submatrix of $M(f)|R(f)$ whose first column is the j-th column of $M(f)$ and whose other columns are the relations $y_i e_k - y_k e_i$, $1 \le k < i$, and $y_k e_i - y_i e_k$, $i < k \le n$. Then $\det M_{i,1} = \pm y_i^{n-2}\frac{\partial f}{\partial x}$ and $\det M_{i,j+1} = \pm y_i^{n-2}\frac{\partial f}{\partial y_j}$ for $1 \le j \le n$. Thus

$\Delta = \sum |M_{i,j}| = (\sum |y_i|^{n-2})(|\frac{\partial f}{\partial x}| + \sum |\frac{\partial f}{\partial y_j}|)$ is a sum of absolute values of elements of $I(f)$ which is Łojasiewicz at L. It follows that Δ is Łojasiewicz at 0 on the set $\{(x,y) : |y| \geq C|x|^r\}$, for any positive constants C and r. We need to find an element of $I(f)$ which is Łojasiewicz on the complementary set

$$H_{r,C} = \{(x,y) : |y| \leq C|x|^r\}.$$

Let $A = \left(\frac{\partial f_i}{\partial y_j} + f_{ij}\right)$. Since $\frac{\partial f_i}{\partial y_j} = f_{ij} + \sum_k y_k \frac{\partial f_{ik}}{\partial y_j}$, $A = (2f_{ij} + \sum_k y_k \frac{\partial f_{ik}}{\partial y_j})$; hence $\det A = 2^n(D(x,y) + b(x,y))$ for some $b \in (y)\mathbf{E}_{n+1}$. Let $b = \sum_{i=1}^n y_i b_i$.

Also $D(x,y) - D(x) \in (y)\mathbf{E}_{n+1}$, so it can be written as $\sum_{i=1}^n y_i a_i$. Thus

$$D(x) = 2^{-n} \det A - \sum_{i=1}^{n} y_i(a_i + b_i).$$

By assumption, there exist $C, r > 0$ such that

$$C|x|^r \leq |D(x)| \leq 2^{-n}|\det A| + \sum |y_i||a_i + b_i|$$

near 0. There is a constant $L > 0$ such that $|a_i + b_i| \leq L$ near 0. Consequently, on the set $H_{r,C/(2L)}$,

$$\sum |y_i||a_i + b_i| \leq L \sum |y_i| \leq \frac{C}{2}|x|^r,$$

which implies that $2^{-n}|\det A| \geq \frac{C}{2}|x|^r$.

We conclude that $\Delta + |\det A|$ is a sum of absolute values of elements of $I(f)$ which is Łojasiewicz at 0, which implies that $I(f)$ is Łojasiewicz at 0, as desired. □

§3. THE TANGENT SPACE $\tau(f)$ TO THE ORBIT OF f IN $(y)^2\mathbf{E}_{n+1}$

Suppose that $f, g \in (y)^2\mathbf{E}_{n+1}$ and $j^\infty f(0) = j^\infty g(0)$. Let $u = g - f$. It follows from Proposition V.2.3 of [To] that there exist $u_{ij} \in \mathbf{m}_{n+1}^\infty$ such that $u = \sum_{i,j} y_i y_j u_{ij}$ with $u_{ij} = u_{ji}$. Let $F = f + tu, 0 \leq t \leq 1$. For any $t_0 \in [0,1]$, let

$$\tau^*(F)_{(0,t_0)} = \{a\frac{\partial F}{\partial x} + \sum_{l=1}^{n} b_l \frac{\partial F}{\partial y_l} : a \in \mathbf{m}_{n+1}\mathbf{E}_{n+2}(0,t_0), b_l \in (y)\mathbf{E}_{n+2}(0,t_0)\},$$

where $\mathbf{E}_{n+2}(0,t_0)$ denotes the smooth germs at $(0,t_0) \in \mathbf{R}^{n+2}$.

We are trying to show that g is $\mathcal{R}_L$-equivalent to f. It suffices to show that there exists for each $t \in [0,1]$ an $h_t \in \mathcal{R}_L$ such that $F \circ (h_t, t) = f$. A standard argument (see [Math] or [Mart]) shows that we can find h_t if, for each $t_0 \in [0,1]$, $u = \partial F/\partial t \in \tau^*(F)_{(0,t_0)}$. Thus, it suffices to show that $\tau^*(F)_{(0,t_0)} \supset (y)^2\mathbf{m}_{n+1}^\infty\mathbf{E}_{n+2}(0,t_0)$.

By assumption, $\tau(f) \supset (y)^2\mathbf{m}_{n+1}^\infty$. Therefore,

$$\tau(f)\mathbf{E}_{n+2}(0,t_0) \supset (y)^2\mathbf{m}_{n+1}^\infty\mathbf{E}_{n+2}(0,t_0).$$

If $\eta \in \tau^*(F)_{(0,t_0)}$, then there exist $a \in \mathbf{m}_{n+1}\mathbf{E}_{n+2}(0,t_0)$ and $b_l \in (y)\mathbf{E}_{n+2}(0,t_0)$, $1 \leq l \leq n$, such that

$$\eta = a\frac{\partial F}{\partial x} + \sum_l b_l \frac{\partial F}{\partial y_l}.$$

Let

$$\eta_1 = a\frac{\partial f}{\partial x} + \sum_l b_l \frac{\partial f}{\partial y_l} \in \tau(f)\mathbf{E}_{n+2}(0, t_0).$$

Then

$$\eta - \eta_1 = at\frac{\partial u}{\partial x} + \sum_l b_l t \frac{\partial u}{\partial y_l} \in (y)^2 \mathbf{m}_{n+1}^{\infty}\mathbf{E}_{n+2}(0, t_0),$$

as required.

§4. THE PROOF OF (1) ⇒ (4)

In this section we assume that f is infinitely determined in $(y)^2\mathbf{E}_{n+1}$.

Let $\phi = (\phi_0, \phi_1, \ldots, \phi_n) \in \mathcal{R}_L$; then there exist $a, a_{ij}, b_j \in \mathbf{E}_{n+1}$ such that $\phi_i = \sum_{j=1}^n a_{ij} y_j$, $1 \le i \le n$, and $\phi_0 = ax + \sum_{j=1}^n y_j b_j$. Write $\psi = (\phi_1, \ldots, \phi_n)^T$ (where T denotes "transpose"), $\beta = (b_1, \ldots, b_n)$ and $A = (a_{ij})$. Then $\phi = \left(\begin{smallmatrix}\phi_0 \\ \psi\end{smallmatrix}\right) = \left(\begin{smallmatrix} a & \beta \\ 0 & A\end{smallmatrix}\right)\left(\begin{smallmatrix} x \\ y\end{smallmatrix}\right)$. Since ϕ is a diffeomorphism, the matrix $\left(\begin{smallmatrix} a & \beta \\ 0 & A\end{smallmatrix}\right)$ is invertible, and its inverse is

$$\begin{pmatrix} a^{-1} & -a^{-1}\beta A^{-1} \\ 0 & A^{-1} \end{pmatrix}.$$

Lemma 4.1. *Let ϕ, A and a_{ij} be as above. If $f = \sum_{i,j} f_{ij} y_i y_j$, $g = f \circ \phi$ and $(g_{ij}) = A^T (f_{ij} \circ \phi) A$, then $g = \sum_{i,j} g_{ij} y_i y_j$.*

Proof.

$$\begin{aligned} g &= \sum_{i,j} \phi_i \phi_j (f_{ij} \circ \phi) \\ &= \sum_{ij} \Big(\sum_k a_{i,k} y_k\Big)\Big(\sum_l a_{jl} y_l\Big)(f_{ij} \circ \phi) \\ &= \sum_{k,l} \Big(\sum_{i,j} a_{ik} (f_{ij} \circ \phi) a_{jl}\Big) y_k y_l \\ &= \sum_{k,l} g_{kl} y_k y_l. \end{aligned}$$

□

Corollary. *If f and g are $\mathcal{R}_L$-equivalent, then*

$$\det(g_{ij}(x, 0)) \in \mathbf{m}_1^{\infty} \quad \textit{if, and only if,} \quad D(x) \in \mathbf{m}_1^{\infty}.$$

Lemma 4.2. *If the sequence $\{x_p\}$ satisfies (1) $0 < x_p < \frac{1}{2}$ and (2) $x_{p+1} < \frac{1}{3} x_p$, and the sequence $\{A_p = (a_{ij}^p)\}$ of symmetric matrices satisfies $|a_{ij}^p| < x_p^p$ for all i, j, then there is some $u \in (y)^2 \mathbf{m}_{n+1}^{\infty}$ such that $\frac{\partial^2 u}{\partial y_i \partial y_j}(x_p, 0) = a_{ij}^p$ for all i, j.*

Proof. Let B_p be the interval $(\frac{2}{3} x_p, \frac{4}{3} x_p)$; then $B_p \cap B_q = \emptyset$ for $p \ne q$. Let μ_p be a smooth function on $\mathbf{R}$ with the following properties: (1) $\mu_p(x_p) = 1, \mu_p = 0$ outside B_p and (2) there exist positive constants C_k depending only on k (not on x_p) such that $|\mu_p^{(k)}(x)| \le \frac{C_k}{x_p^k}$ for every $k \in \mathbf{N}$ (see Lemma IV.3.3 of [To]).

Now let $u(z) = \sum_{p=1}^{\infty}\sum_{i,j=1}^{n} y_i y_j \mu_p(x)\frac{a_{ij}^p}{2}$, for $z = (x, y)$. Then u is smooth outside the hyperplane $x = 0$. Let k denote the multi-index (k_0, k'), with $k' = (k_1, \ldots, k_n)$. Let e_i denote the n-tuple with 1 in the i-th place, 0 elsewhere. Then

$$(*) \qquad \frac{\partial^{|k|}u}{\partial z^k}(z) = \begin{cases} 0, & x \notin \cup_p B_p \quad \text{or} \quad |k'| > 2, \\ \sum_p \sum_{i,j} y_i y_j \mu_p^{(k_0)}(x)\dfrac{a_{ij}^p}{2}, & x \in B_p \quad \text{and} \quad k' = 0, \\ \sum_p \sum_{j=1}^{n} y_j \mu_p^{(k_0)}(x) a_{ij}^p, & x \in B_p \quad \text{and} \quad k' = e_i, \\ \sum_p \sum_{j=1}^{n} \mu_p^{(k_0)}(x) a_{ij}^p, & x \in B_p \quad \text{and} \quad k' = e_i + e_j. \end{cases}$$

In particular, $\frac{\partial^2 u}{\partial y_i \partial y_j}(x_p, 0) = a_{ij}^p$ and $u(x, 0) = \frac{\partial u}{\partial y_i}(x, 0) = 0$ for $x \neq 0$. Since

$$|\mu_p^{(k)}(x) a_{ij}^p| \le \frac{C_k}{x_p^k} x_p^p = C_k x_p^{p-k},$$

from $(*)$ it follows that $\lim_{z\to 0} \frac{\partial^{|k|}u}{\partial z^k}(z) = 0$. By the Lemma on page 87 of [Mart], $u \in C^\infty(\mathbf{R}^{n+1})$. So $u \in (y)^2\mathbf{m}_{n+1}^\infty$. □

Lemma 4.3. *There is a sequence* $x_p \neq 0$, $x_p \to 0$, *such that* $D(x_p) \neq 0$.

Proof. Assume $D(x) = 0$ for all x. Choose a sequence $x_p > 0$ as in Lemma 4.2. Let $\Sigma = \{A \in \mathbf{M}(\mathbf{R}, n) : \det A = 0\}$. Choose $A_p = (a_{ij}^p) \in \mathbf{S}(\mathbf{R}, n) \setminus \Sigma$ with $|a_{ij}^p - f_{ij}(x_p, 0)| < |x_p|^p$ for all i, j. By Lemma 4.2, there exists $u \in (y)^2\mathbf{m}_{n+1}^\infty$ with $\frac{\partial^2 u}{\partial y_i \partial y_j}(x_p, 0) = a_{ij}^p - f_{ij}(x_p, 0)$. Then $\det((f + u)_{ij}(x_p, 0)) = \det A_p \neq 0$. By Lemma 4.1, $D(\phi(x_p, 0)) \neq 0$, which is a contradiction. □

Note that $D(x)$ is not Łojasiewicz if, and only if, $D(x) \in \mathbf{m}_1^\infty$.

Lemma 4.4. *For each* k, *there is at least one* $k \times k$ *minor of the matrix* $(f_{ij}(x, 0))$ *which is not flat at* 0. *In particular* $D(x) \notin \mathbf{m}_1^\infty$.

Proof. Since f is ∞-determined in $(y)^2\mathbf{E}_{n+1}$, $g(z) = f(z) - \sum' y_i y_j f_{ij}(x, 0)$ is $\mathcal{R}_L$-equivalent to f, where the summation is taken over the set $\{(i, j) : f_{ij}(x, 0) \in \mathbf{m}_1^\infty\}$. By the Corollary of Lemma 4.1 we may assume that $f_{ij}(x, 0)$ is either 0 or not in $\mathbf{m}_1^\infty$. By Lemma 4.3, $D(x)$ is not identically equal to 0, so at least one of the entries is not 0.

We will prove the result by induction. The result clearly holds for $k = 1$. Assume it holds for k. Assume $B = (f_{i_s j_t}(x, 0))$ is a $k \times k$ submatrix of $(f_{ij}(x, 0))$, whose determinant is not in $\mathbf{m}_1^\infty, 1 \le k < n$. Take any $l \notin \{j_1, \ldots, j_k\}$. Look at the matrix

$$A = \begin{pmatrix} f_{1j_1}(x,0) & \ldots & f_{1j_k}(x,0) & f_{1l}(x,0) \\ \ldots & \ldots & \ldots & \ldots \\ f_{nj_1}(x,0) & \ldots & f_{nj_k}(x,0) & f_{nl}(x,0) \end{pmatrix}.$$

For any $m \notin \{i_1, \ldots, i_k\}$, let

$$A_m = \begin{pmatrix} f_{i_1 j_1}(x,0) & \ldots & f_{i_1 j_k}(x,0) & f_{i_1 l}(x,0) \\ \ldots & \ldots & \ldots & \ldots \\ f_{i_k j_1}(x,0) & \ldots & f_{i_k j_k}(x,0) & f_{i_k l}(x,0) \\ f_{m j_1}(x,0) & \ldots & f_{m j_k}(x,0) & f_{ml}(x,0) \end{pmatrix}.$$

If $\det A_m \in \mathbf{m}_1^\infty$ for all $m \notin \{i_1, \ldots, i_k\}$, then $g = f - \sum_{m \notin \{i_1,\ldots,i_k\}} y_m y_l \frac{\det A_m}{\det B}$ is $\mathcal{R}_L$-equivalent to f, since $\frac{\det A_m}{\det B} \in \mathbf{m}_1^\infty \subset \mathbf{m}_{n+1}^\infty$. Denote the corresponding submatrices of $(g_{ij}(x,0))$ by A'_m and A'; then $\det A'_m = \det A_m - \frac{\det A_m}{\det B} \det B = 0$.

Since $\det B(x) \notin \mathbf{m}_1^\infty$, $\det B(x) \neq 0$ for small $x \neq 0$. So for these x's, any row of A' is a linear combination of the first k rows of A_m; hence any maximal minor of $A'(x)$ is 0 for all small $x \neq 0$. By continuity it is also 0 at 0. Therefore $\det(g_{ij}(x,0)) = 0$ for all small x. By Lemma 4.1 this will imply that $D(x) = 0$, which contradicts Lemma 4.3. □

Proof of (1) ⇒ (4). By Lemma 4.4, $D(x)$ must be Łojasiewicz at 0.

By Lemma 1.1, it suffices to show, for each $r > 0$, that $J(f)$ is Łojasiewicz at 0 on $A_r = \{z : |y| \geq |x|^r\}$.

Suppose this fails. Then there exists $r > 0$ and a sequence $z_p \in A_r$ such that $|df(z_p)| \leq |z_p|^p$. Without loss of generality, we can assume $|y_{p+1}| < |y_p|/3$ for all p. Let B_p be the open ball centered at z_p of radius $|y_p|/3$. The B_p's are disjoint, and 0 is the only limit point in the x-axis of the union of these balls. By Lemma IV.3.3 of [To], there exists for each p a smooth function μ_p which is 1 on a neighbourhood of z_p, zero outside B_p, and satisfying: for each multi-index I there exists a constant C_I independent of z_p such that $|D^I \mu_p(x)| \leq C_I / |y_p|^{|I|}$.

By Sard's Theorem there exist regular values w_p of f such that $|w_p - f(z_p)| \leq |z_p|^p$. Let g_p be the linear function such that $g_p(z_p) = w_p - f(z_p)$ and $dg_p(z_p) = -df(z_p)$. Let $u = \sum_p \mu_p g_p$. Then u is a smooth function, $u \in \mathbf{m}_{n+1}^\infty$ and, since u is identically 0 on a neighborhood of $\{y = 0, x \neq 0\}$, $u \in \mathbf{m}_L^\infty \subset (y)^2 \mathbf{E}_{n+1}$. But $f + u$ can't be $\mathcal{R}_L$-equivalent to f, since $(f+u)(z_p) = w_p$ is a critical value of $f + u$ but is not a critical value of f. □

References

[IzMats] S. Izumiya and S. Matsuoka, *Notes on smooth function germs on varieties*, Proc. Amer. Math. Soc. **97** (1986), 146–150. MR **87g:**58013

[Mart] J. Martinet, *Singularities of smooth functions and maps*, Cambridge University Press, New York, 1982. MR **83i:**58018

[Math] J. N. Mather, *Stability of C^∞ mappings, III: finitely-determined map germs*, Inst. Hautes Études Sci. Publ. Math. **35** (1968), 279–308. MR **43:**1215a

[Pe] R. Pellikaan, *Finite determinacy of functions with non-isolated singularities*, Proc. London Math. Soc. **57** (1988), 357–382. MR **89i:**32014

[Si] Dirk Siersma, *Isolated line singularities*, Proc. Sympos. Pure Math. **40** (1983), 485–496. MR **85d:**32017

[To] J. C. Tougeron, *Idéaux de fonctions différentiables*, Springer-Verlag, New York, 1972. MR **55:**13472

[Wa] C. T. C. Wall, *Finite determinacy of smooth map-germs*, Bull. London Math. Soc. **13** (1981), 481–539. MR **83i:**58020

[Wi] L. C. Wilson, *Infinitely determined map germs*, Canad. J. Math. **XXXIII, No.3** (1981), 671–684. MR **82k:**58023

Department of Mathematics, University of Hawaii, Honolulu, Hawaii 96822

Department of Mathematics, University of Hawaii, Honolulu, Hawaii 96822
E-mail address: `les@math.hawaii.edu`