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UNIFORMLY MORE POWERFUL TESTS FOR HYPOTHESES ABOUT LINEAR INEQUALITIES WHEN THE VARIANCE IS UNKNOWN

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Abstract. Let $\mathbf{X}$ be a p-dimensional normal random vector with unknown mean μ and covariance matrix $\Sigma = \sigma^2\Sigma_0$, where Σ_0 is a known matrix and σ^2 an unknown parameter. This paper gives a test for the null hypothesis that μ lies either on the boundary or in the exterior of a closed, convex polyhedral cone versus the alternative hypothesis that μ lies in the interior of the cone. Our test is uniformly more powerful than the likelihood ratio test.

1. Introduction

Let $\mathbf{X} = (X_1, \ldots, X_p)'$, $p \geq 2$, be a p-dimensional normal random vector with unknown mean $\mu = (\mu_1, \ldots, \mu_p)'$ and covariance matrix $\Sigma = \sigma^2\Sigma_0$, where Σ_0 is a known matrix. Let $b_1, \ldots, b_m$ be m ($m \geq 2$) specified p-dimensional vectors and consider the problem of testing

$$H_0^* : \min_{1\leq i\leq m} b_i'\mu \leq 0 \text{ versus } H_1^* : \min_{1\leq i\leq m} b_i'\mu > 0, \tag{1.1}$$

assuming that $\{b_1, \ldots, b_m\}$ is without positive relations and has no redundant vectors (Sasabuchi [7]). This is equivalent to testing the null hypothesis that μ lies either on the boundary or in the exterior of a closed, convex polyhedral cone versus the alternative hypothesis that μ lies in the interior of the cone. Common applications of the testing problem (1.1) arise in the setting of equivalence testing (Berger and Hsu [2]) and clinical trials of combination therapies (Laska and Meisner [3]).

Sasabuchi ([7], [8]) derived the likelihood ratio test (LRT) for a problem very similar to (1.1), and Berger [1] showed that this test was also the LRT for (1.1). The LRT is biased, and the level α is attained only in the limit if one $b_i'\mu = 0$ and $b_j'\mu \to \infty$ for all $j \neq i$ (Sasabuchi [7], [8]). Berger [1], Liu and Berger [4], and McDermott and Wang [6] constructed classes of size-α tests that are uniformly more powerful than the LRT for the case of σ^2 known. Berger [1] also demonstrated that his approach does not yield a size-α test for the case of σ^2 unknown.

In view of Liu and Berger [4], it is clear that when $m = 2$, without loss of generality, the problem can be transformed to that of observing $(U, V)' \sim$

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$N_2((\mu_1, \mu_2)', \sigma^2 I)$ and testing

$$H_0 : \min\{\mu_1 + r\mu_2,\, \mu_2\} \leq 0 \text{ versus } H_1 : \min\{\mu_1 + r\mu_2,\, \mu_2\} > 0, \tag{1.2}$$

with $-\infty < r < \infty$. In this paper, an approach based on conditioning arguments is used to construct tests for (1.2) that are uniformly more powerful than the LRT for the case of σ^2 unknown. This approach to test construction can be extended to the general problem (1.1), that is, $m \geq 2$ and $\Sigma_0 \neq I$, by linear transformation and by applying the intersection-union method of Liu and Berger [4].

Throughout the paper, $P_{(\mu_1,\mu_2)}(\cdot)$ will be used to denote the probability measure with the indicated means and variance-covariance matrix I.

2. Derivation of tests uniformly more powerful than the LRT

In this section, an approach based on conditioning is used to derive tests that are uniformly more powerful than the LRT assuming $\Sigma = \sigma^2 I$, with σ^2 unknown. Let $\mathcal{C} = \{(U, V) : U + rV \geq 0,\, V \geq 0\}$. The following theorem due to Berger [1] is important in our derivation.

Theorem 2.1. *Under H_0, for any fixed $\sigma^2 > 0$ and $A \subset \mathcal{C}$,*

$$\sup_{\mu_1,\mu_2} P_{(\mu_1,\mu_2)}(A) = \max\left\{\sup_{\mu_1\geq 0} P_{(\mu_1,0)}(A), \sup_{\mu_2\geq 0} P_{(-r\mu_2,\mu_2)}(A)\right\}.$$

Let S^2 be a random variable independent of (U, V) such that S^2/σ^2 has a χ^2 distribution with ν degrees of freedom, and let $W^2 = S^2 + U^2 + V^2$. Also, let T be a random variable such that $\sqrt{\nu}T$ has a t-distribution with ν degrees of freedom. For $\alpha < 0.5$, Sasabuchi ([7], [8]) showed that the critical region of the LRT is

$$A_0 \equiv \left\{(U, V, S) : (U + rV)/\left(S\sqrt{1 + r^2}\right) \geq t,\, V/S \geq t\right\}, \tag{2.1}$$

where t is the upper 100α percentile of the distribution of T. Note that $t > 0$, since $\alpha < 0.5$. In view of Theorem 2.1, any test with critical region $R \subset \mathcal{C}$ has size α if and only if the following is satisfied:

(2.2)

$$\alpha = \max\left\{\sup_{\mu_1\geq 0,\, \sigma^2>0} P_{(\mu_1,0;\sigma^2)}\left[(U, V) \in R\right], \sup_{\mu_2\geq 0,\, \sigma^2>0} P_{(-r\mu_2,\mu_2;\sigma^2)}\left[(U, V) \in R\right]\right\},$$

where the probability measure now depends on the additional parameter σ^2 in the obvious way. Let

$$X = \frac{U + rV}{\sqrt{1 + r^2}}, \quad Y = \frac{-rU + V}{\sqrt{1 + r^2}}, \quad \nu_1 = \frac{\mu_1 + r\mu_2}{\sqrt{1 + r^2}}, \quad \nu_2 = \frac{-r\mu_1 + \mu_2}{\sqrt{1 + r^2}}. \tag{2.3}$$

Then $X \sim N(\nu_1, \sigma^2)$, $Y \sim N(\nu_2, \sigma^2)$, and X, Y, and S^2 are mutually independent. Thus

$$\sup_{\mu_2\geq 0,\, \sigma^2>0} P_{(-r\mu_2,\mu_2;\sigma^2)}\left[(U, V) \in R\right] = \sup_{\nu_2\geq 0,\, \sigma^2>0} P_{(0,\nu_2;\sigma^2)}\left[(X, Y) \in R'\right],$$

where R' is the appropriately transformed critical region.

Note that A_0, the critical region for the LRT, satisfies (2.2) because the following inequalities hold for every u, y, and w, the observed values of U, Y, and W, respectively:

$$
\begin{aligned}
&P_{(\mu_1,0)}\left((U+rV)/\left(S\sqrt{1+r^2}\right) \geq t,\ V/S \geq t \,|\, U=u, W=w\right) \leq \alpha, \\
&P_{(0,\nu_2)}\left((rX+Y)/\left(S\sqrt{1+r^2}\right) \geq t,\ X/S \geq t \,|\, Y=y, W=w\right) \leq \alpha.
\end{aligned} \tag{2.4}
$$

Because the above inequalities are strict for some values of u, y, and w, the rejection region of the LRT may be expanded while preserving the size of the test to be less than or equal to α. The general approach that follows is similar to that of McDermott and Wang [6], but with further conditioning on W to deal with the nuisance parameter σ^2.

2.1 Derivation for $r \leq 0$. Denote $\gamma = |r|$. The critical region of the LRT is

$$
A_0 = \left\{(U,V,S) : (U-\gamma V)/(S\sqrt{1+\gamma^2}) > t\ ,\ V/S > t\right\}. \tag{2.5}
$$

Let (u^*, v^*) be the solution to the system

$$
\begin{cases} u^2 + v^2 = w^2, \\ u - \gamma v = 0. \end{cases}
$$

Then $v^* = w/\sqrt{1+\gamma^2}$ and $u^* = \gamma v^*$ (see Figures 1 and 2).

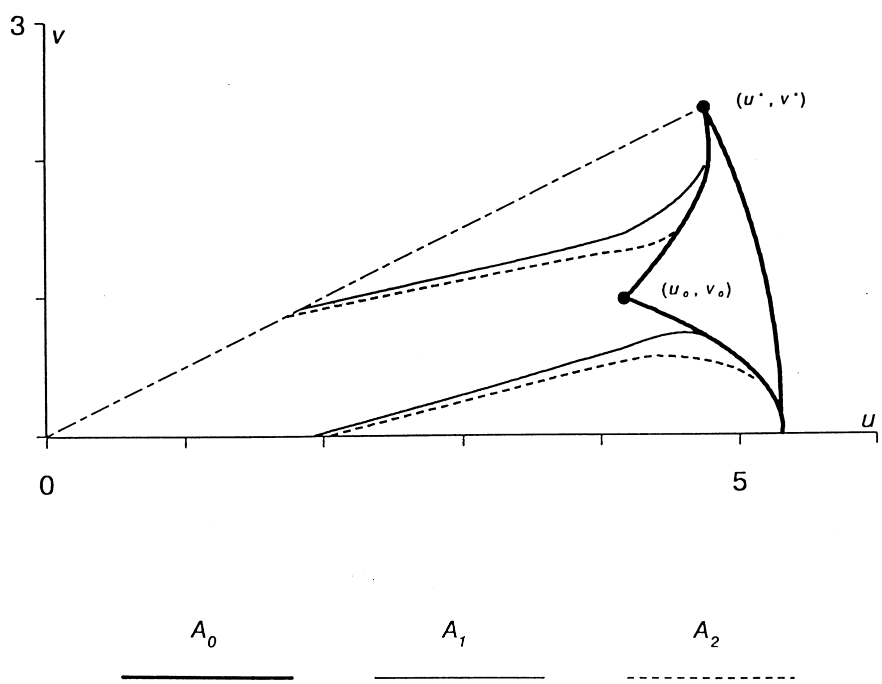

FIGURE 1. Illustration of $A_0 \cup (A_1 \cap A_2)$, the critical region of the uniformly more powerful test for $r < 0$ and $u_0 < u^*$, conditional on $W = w$. The values used for computation were $\alpha = 0.20$, $\nu = 8$, $r = -2$, and $w^2 = 28.2$.

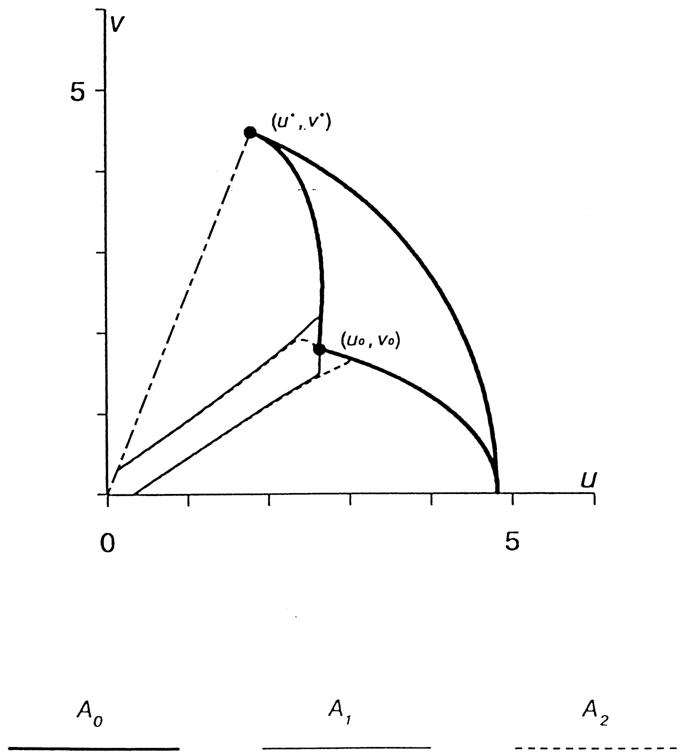

FIGURE 2. Illustration of $A_0 \cup (A_1 \cap A_2)$, the critical region of the uniformly more powerful test for $r < 0$ and $u_0 > u^*$, conditional on $W = w$. The values used for computation were $\alpha = 0.10$, $\nu = 8$, $r = -0.4$, and $w^2 = 23.2$.

Let (u_0, v_0) be the intersection point of the curves $(u - \gamma v)/(\sqrt{1+\gamma^2}s) = t$ and $v/s = t$, which are the boundary curves of A_0 (see Figures 1 and 2). That is, (u_0, v_0) is the solution to the system

$$\begin{cases} (u-\gamma v)/\sqrt{(1+\gamma^2)(w^2-u^2-v^2)} = t, \\ v/\sqrt{w^2-u^2-v^2} = t, \end{cases}$$

subject to $v > 0$ and $v\gamma < u$. The solution is

$$v_0 = tw/\sqrt{1+t^2+t^2(\gamma+\sqrt{1+\gamma^2})^2}, \quad u_0 = (\gamma+\sqrt{1+\gamma^2})v_0.$$

2.1.1 The case of $u_0 \le u^*$. It can be easily verified that A_0 is contained in the positive orthant. Note also that

(2.6)
$$\begin{aligned} &\left\{(U,V,S) : V > 0,\, U \le u^*,\, U - \gamma V > 0,\, (U-\gamma V)/S > \sqrt{1+\gamma^2}t\right\} \\ &= \left\{(U,V,S) : V > 0,\, \gamma V < U \le u^*,\, (U-\gamma V)^2 > (1+\gamma^2)t^2(W^2-U^2-V^2)\right\} \\ &= \{(U,V,S) : V > 0,\, \gamma V < U \le u^*, \\ &\qquad V < \left[\gamma U - \sqrt{d(1+\gamma^2)t^2(W^2-U^2)+(\gamma^2-d)U^2}\right]/d\right\}, \end{aligned}$$

where $d = t^2+\gamma^2+\gamma^2t^2$. The last equation in (2.6) holds because the critical region is under the curve $v = \gamma u/d - \sqrt{d(1+\gamma^2)t^2(w^2-u^2) - du^2 + \gamma^2u^2}/d$ (see Figure 1). Now define, for $u \le u^*$,

$$g(u,w) = \gamma u/d - \sqrt{d(1+\gamma^2)t^2(w^2-u^2) - du^2 + \gamma^2u^2}/d.$$

From (2.5) and (2.6), it follows that

$$A_0 \cap \{(U,V,S) : U \le u^*\} = \{(U,V,S) : U \le u^*,\, tS < V < g(U,W)\}.$$

Consequently, for $u \le u^*$,

$$\begin{aligned} &P_{(\mu_1,0)}(A_0\,|U=u,W=w) \\ &= \begin{cases} 0 & \text{if } u < u_0, \\ P_{(\mu_1,0)}[tS < V < g(U,W)\,|\,U=u,W=w] & \text{if } u_0 \le u \le u^*. \end{cases} \end{aligned} \tag{2.7}$$

Note that when $u \le u^*$,

$$\begin{aligned} &\{V : 0 < V < g(U,W)\,|\,U=u,W=w\} \\ &= \big\{V : V > 0,\, V^2/(W^2-U^2-V^2) \\ &\qquad < g^2(U,W)/[W^2-U^2-g^2(U,W)]\,|\,U=u,W=w\big\}. \end{aligned}$$

By Lukacs' theorem [5], $V^2/(W^2-U^2-V^2)$ is independent of (U,W). Therefore, when $u \in [u_0,u^*]$,

$$\begin{aligned} &P_{(\mu_1,0)}(A_0\,|\,U=u,W=w) \\ &=P_{(\mu_1,0)}\left(t < V/S < g(U,W)/\sqrt{W^2-U^2-g^2(U,W)}\,|\,U=u,W=w\right) \\ &=G\left(g(u,w)/\sqrt{w^2-u^2-g^2(u,w)}\right) - G(t) \\ &=G\left(g(u,w)/\sqrt{w^2-u^2-g^2(u,w)}\right) - (1-\alpha), \end{aligned} \tag{2.8}$$

where $G(\cdot)$ is the distribution function of T. Define the type I error probability of the LRT, conditioning on U and W, as $\alpha_1(u,w) = P_{(\mu_1,0)}(A_0\,|\,U=u,W=w)$. From (2.7) and (2.8), it follows that

$$\alpha_1(u,w) = \begin{cases} 0 & \text{if } 0 < u < u_0, \\ G\left(g(u,w)/\sqrt{w^2-u^2-g^2(u,w)}\right) - (1-\alpha) & \text{if } u_0 \le u < u^*, \\ \xi(u,w) & \text{if } u \ge u^*, \end{cases}$$

where $\xi(u,w) = P_{(\mu_1,0)}(A_0\,|\,U = u > u^*, W = w) \le P_{(\mu_1,0)}(V/S > t) = \alpha$. Note that $\alpha_1(u,w) < \alpha$ for every (u,w) such that $u \in (0,u^*)$, and $\alpha_1(u,w) \le \alpha$ for $u > u^*$. To compensate for this shortcoming, we want to find two functions $f_1(u,w)$ and $f_2(u,w)$ satisfying the following conditions:

(i) If $u \in [u_0,u^*)$, then $f_1(u,w) < t$ and $f_2(u,w) > g(u,w)/\sqrt{w^2-u^2-g^2(u,w)}$;
(ii) $P_{(\mu_1,0)}[f_1(U,W) < V/S < f_2(U,W)\,|\,U=u,W=w] = \alpha$ for $u \le u^*$.

Clearly, if f_1 and f_2 satisfy the above conditions, then $A_0 \subset \{(U,V,S) : f_1(U,W) < V/S \le f_2(U,W)\}$ and $P_{(\mu_1,0)}[f_1(U,W) < V/S \le f_2(U,W)\,|\,U=u,W=w] = \alpha$ for $u \le u^*$.

We first consider the case when $u \in [u_0,u^*)$. It can be easily verified that the functions f_1 and f_2 satisfying the following equations also satisfy the above conditions (i) and (ii):

$$P_{(\mu_1,0)}(f_1 < V/S < t \,|\, U = u, W = w) = k[\alpha - \alpha_1(u,w)], \tag{2.9}$$

$$\begin{aligned} &P_{(\mu_1,0)}[g(U,W)/\sqrt{W^2 - U^2 - g^2(U,W)} < V/S < f_2 \,|\, U = u, W = w] \\ &\quad = (1-k)[\alpha - \alpha_1(u,w)], \end{aligned} \tag{2.10}$$

where $0 < k < 1$. From (2.9) and Lukacs' theorem [5], it follows that

$$G(t) - G(f_1) = k[\alpha - \alpha_1(u,w)].$$

This equation may be solved for f_1 to obtain

$$f_1(u,w) = G^{-1}\left[1 - \alpha - k\alpha + k\alpha_1(u,w)\right].$$

Similarly, (2.10) can be solved to obtain

$$f_2(u,w) = G^{-1}\left\{G\left[g(u,w)/\sqrt{w^2 - u^2 - g^2(u,w)}\right] + (1-k)\left[\alpha - \alpha_1(u,w)\right]\right\}.$$

Next, we discuss the case when $u \in (0, u_0)$. Following Liu and Berger [4], it is desired to have the line segment joining the origin and the point (u_0, v_0) included in the critical region. Note that the equation of this line is $v = (v_0/u_0)u$. Therefore, two functions $h_1(u,w)$ and $h_2(u,w)$ need to be defined that satisfy the following equations:

$$P_{(\mu_1,0)}(h_1 < V < v_0 U/u_0 \,|\, U = u, W = w) = k[\alpha - \alpha_1(u,w)] = k\alpha, \tag{2.11}$$

$$P_{(\mu_1,0)}(v_0 U/u_0 < V < h_2 \,|\, U = u, W = w) = (1-k)[\alpha - \alpha_1(u,w)] = (1-k)\alpha. \tag{2.12}$$

To solve (2.11), note that

$$\begin{aligned} &\{V : h_1 < V < v_0 u/u_0 \,|\, U = u, W = w\} \\ &= \left\{V : h_1/\sqrt{w^2 - u^2 - h_1^2} < V/S < (v_0 u/u_0)/\sqrt{w^2 - u^2 - (v_0 u/u_0)^2}\right\}. \end{aligned} \tag{2.13}$$

Define $f_1 = h_1/\sqrt{w^2 - u^2 - h_1^2}$. In view of (2.13), it is clear that (2.11) can be rewritten as

$$P_{(\mu_1,0)}\left[f_1 < V/S \le (v_0 U/u_0)/\sqrt{W^2 - U^2 - (v_0 U/u_0)^2} \,|\, U = u, W = w\right] = k\alpha. \tag{2.14}$$

The above equation may be solved to obtain

$$f_1(u,w) = G^{-1}\left\{G\left[(v_0 u/u_0)/\sqrt{w^2 - u^2 - (v_0 u/u_0)^2}\right] - k\alpha\right\}.$$

Similarly, we can define $f_2 = h_2/\sqrt{w^2 - u^2 - h_2^2}$ and rewrite (2.12) as

$$P_{(\mu_1,0)}\left[(v_0 U/u_0)/\sqrt{W^2 - U^2 - (v_0 U/u_0)^2} < V/S \le f_2 \,|\, U = u, W = w\right]. \tag{2.15}$$

Equation (2.12) may then be solved to obtain

$$f_2(u,w) = G^{-1}\left\{(1-k)\alpha + G\left[(v_0 u/u_0)/\sqrt{w^2 - u^2 - (v_0 u/u_0)^2}\right]\right\}.$$

Recall that when $u \geq u^*$, $P_{(\mu_1,0)}(V/S > t) = \alpha$. Thus, in summary,

(2.16)
$$f_1(u,w) = \begin{cases} G^{-1}\left\{G\left[(v_0u/u_0)/\sqrt{w^2-u^2-(v_0u/u_0)^2}\right] - k\alpha\right\} & \text{if } 0 < u < u_0, \\ G^{-1}[1-\alpha-k\alpha+k\alpha_1(u,w)] & \text{if } u_0 \leq u < u^*, \\ t & \text{if } u \geq u^*, \end{cases}$$

$$f_2(u,w) = \begin{cases} G^{-1}\left\{(1-k)\alpha + G\left[(v_0u/u_0)/\sqrt{w^2-u^2-(v_0u/u_0)^2}\right]\right\} & \text{if } 0 < u < u_0, \\ G^{-1}\left\{G\left[\dfrac{g(u,w)}{\sqrt{w^2-u^2-g^2(u,w)}}\right] + (1-k)[\alpha-\alpha_1(u,w)]\right\} & \text{if } u_0 \leq u < u^*, \\ \infty & \text{if } u \geq u^*. \end{cases}$$

Next, recall the definitions of X, Y, ν_1, and ν_2 in (2.3). The critical region of the LRT can be expressed as $A_0 = \{(U,V,S) : X/S > t,\ (\gamma X+Y)/(S\sqrt{1+\gamma^2}) > t\}$. Note that X, Y, and W are functions of (U,V,S). Define

$$A_1 = \{(U,V,S) : f_1(U,W) < V/S \leq f_2(U,W)\} \cap \mathcal{C},$$
$$A_2 = \{(U,V,S) : f_1(Y,W) < X/S \leq f_2(Y,W)\} \cap \mathcal{C}.$$

From the above derivation (see (2.9), (2.10), (2.14) and (2.15)), it follows that $P_{(\mu_1,0)}(A_1 \,|\, U=u, W=w) \leq \alpha$ for all real numbers u and $w > 0$. In fact, when $u > 0$, $P_{(\mu_1,0)}(A_1 \,|\, U=u, W=w) = \alpha$. Similarly, $P_{(\gamma\mu_2,\mu_2)}(A_2 \,|\, Y=y, W=w) \leq \alpha$ for all real numbers y and $w > 0$. Therefore,

$$P_{(\mu_1,0)}(A_1 \cap A_2) \leq E[P_{(\mu_1,0)}(A_1 \,|\, U,W)] \leq \alpha,$$
$$P_{(\gamma\mu_2,\mu_2)}(A_1 \cap A_2) \leq E[P_{(\gamma\mu_2,\mu_2)}(A_2 \,|\, Y,W)] \leq \alpha.$$

Note also that $A_0 \subset A_1 \cap A_2$ (see conditions (i) and (ii) for f_1 and f_2 given above). Thus, by Theorem 2.1, the test with critical region $A_1 \cap A_2$ is a size-α test that is uniformly more powerful than the LRT.

2.1.2 The case of $u_0 > u^*$.

In this case we have, similar to (2.6),

(2.17)
$$\begin{aligned} &\left\{(U,V,S) : V > 0,\ U < u_0,\ (U-\gamma V)/S > t\sqrt{1+\gamma^2}\right\} \\ &= \{(U,V,S) : V > 0,\ \gamma V < U < u_0,\ (U-\gamma V)^2/(W^2-U^2-V^2) > t^2(1+\gamma^2)\} \\ &= \left\{(U,V,S) : U > u_0,\ V > \gamma U/d + \sqrt{d(1+\gamma^2)t^2(W^2-U^2) - dU^2 + \gamma^2U^2}/d\right\}. \end{aligned}$$

Note that the conditional critical region is above the curve

$$v = \gamma u/d + \sqrt{d(1+\gamma^2)t^2(w^2-u^2) - du^2 + \gamma^2u^2}/d$$

(see Figure 2). Let $g(u,w) = \gamma u/d + \sqrt{d(1+\gamma^2)t^2(w^2-u^2) - du^2 + \gamma^2u^2}/d$ be a function defined for $u^* < u \leq u_0$. From (2.1) and (2.17), it follows that

$$A_0 \cap \{(U,V,S) : U \leq u_0\} = \{(U,V,S) : U \leq u_0,\ V > g(U,W)\}.$$

Consequently, for $u \le u_0$,

$$P_{(\mu_1,0)}(A_0 \,|U = u, W = w) = \begin{cases} 0 & \text{if } u \le u^*, \\ P_{(\mu_1,0)}[V > g(U,W)\,|\,U = u, W = w] & \text{if } u^* < u \le u_0. \end{cases} \tag{2.18}$$

Note that when $u^* < u \le u_0$,

$$\begin{aligned} &\{V : V > g(U,W)\,|\,U = u, W = w\} \\ &= \left\{V : V^2/(W^2 - U^2 - V^2) > g^2(U,W)/[W^2 - U^2 - g^2(U,W)]\,|\,U = u, W = w\right\}. \end{aligned}$$

Therefore, again using Lukacs' theorem [5], when $u \in (u^*, u_0]$, the type I error probability of the LRT when $\mu_2 = 0$, conditioning on U and W, is

$$\alpha_1(u,w) = \begin{cases} 0 & \text{if } \ 0 < u \le u^*, \\ 1 - G\left(g(u,w)/\sqrt{w^2 - u^2 - g^2(u,w)}\right) & \text{if } \ u^* < u \le u_0\ , \\ \xi(u,w) & \text{if } \ u > u_0, \end{cases} \tag{2.19}$$

where $\xi(u,w) = P_{(\mu_1,0)}(A_0\,|\,U = u > u_0, W = w) \le P_{(\mu_1,0)}(V/S > t) = \alpha$. Clearly, $\alpha_1(u,w) < \alpha$ for all (u,w) such that $u \le u_0$, and $\alpha_1(u,w) \le \alpha$ for $u > u_0$. To compensate for this shortcoming, we want to find two functions $f_1(u,w)$ and $f_2(u,w)$ such that

$$P_{(\mu_1,0)}(f_1 < V/S \le f_2\,|\,U = u, W = w) = \alpha - \alpha_1(u,w),$$

with the line segment joining the origin and the point (u_0, v_0) being contained in the set $\{(U,V,S) : f_1 < V/S < f_2\}$.

To accomplish this, we first define two functions h_1 and h_2 by the following two equations:

$$\begin{aligned} &P_{(\mu_1,0)}\{h_1 < V \le v_0U/u_0\,|\,U = u, W = w\} = k[\alpha - \alpha_1(u,w)], \\ &P_{(\mu_1,0)}\{v_0U/u_0 < V \le h_2\,|\,U = u, W = w) = (1-k)[\alpha - \alpha_1(u,w)]. \end{aligned}$$

Let $f_i(u,w) = h_i(u,w)/\sqrt{w^2 - u^2 - h_i^2(u,w)}$ for $i = 1, 2$. The above equations are equivalent to

$$\begin{aligned} &P_{(\mu_1,0)}\Big[f_1(u,w) < V/S < (v_0u/u_0)/\sqrt{w^2 - u^2 - (v_0u/u_0)^2}\Big] = k[\alpha - \alpha_1(u,w)], \\ &P_{(\mu_1,0)}\Big[(v_0u/u_0)/\sqrt{w^2 - u^2 - (v_0u/u_0)^2} < V/S \le f_2(u,w)\Big] = (1-k)[\alpha - \alpha_1(u,w)], \end{aligned}$$

since V/S is independent of (U,W).

The above equations may be solved to obtain, for $0 < u \le u_0$,

$$\begin{aligned} f_1(u,w) &= G^{-1}\left\{G\left[(v_0u/u_0)/\sqrt{w^2 - u^2 - (v_0u/u_0)^2}\right] - k[\alpha - \alpha_1(u,w)]\right\}, \\ f_2(u,w) &= G^{-1}\left\{G\left[(v_0u/u_0)/\sqrt{w^2 - u^2 - (v_0u/u_0)^2}\right] + (1-k)[\alpha - \alpha_1(u,w)]\right\}, \end{aligned}$$

where $\alpha_1(u,w)$ is the piecewise-defined function given in (2.19).

Finally, as in Subsection 2.1.1, recall the definitions of X and Y in (2.3) and define

$$\begin{aligned} A_1 &= \{(U,V,S) : f_1(U,W) < V/S \le f_2(U,W)\} \cap \mathcal{C}, \\ A_2 &= \{(U,V,S) : f_1(Y,W) < X/S \le f_2(Y,W)\} \cap \mathcal{C}. \end{aligned}$$

It follows from the same argument given in Subsection 2.1.1 that the test with critical region $A_0 \cup (A_1 \cap A_2)$ is a size-α test that is uniformly more powerful than the LRT.

2.2 Tests for $r > 0$. In this section, a test is given that is uniformly more powerful than the LRT when $r > 0$. Details of the derivation are omitted here, but they are very similar to those given in Section 2.1. Let $(u_0, v_0) \in \mathcal{C}$ be the solution to the following system of equations:

$$\begin{cases} (u + rv)^2 = t^2(1 + r^2)(w^2 - u^2 - v^2), \\ v^2 = t^2(w^2 - u^2 - v^2). \end{cases}$$

This system can be solved to obtain $u_0 = ctw/\sqrt{1 + t^2c^2 + t^2}$ and $v_0 = u_0/c$, where $c = \sqrt{1 + r^2} - r$. Thus, the equation of the line joining $(0, 0)$ and (u_0, v_0) is $v = u/c$. Let $d = t^2 + r^2 + r^2t^2$, $k \in (0, 1)$, and

$$\begin{aligned} g(u, w) &= \left\{-ru + \sqrt{d(1 + r^2)t^2(w^2 - u^2) - du^2 + r^2u^2}\right\}/d, \\ \alpha_1(u, w) &= 1 - G\left(g(u, w)/\sqrt{w^2 - u^2 - g^2(u, w)}\right), \\ f_1(u, w; k) &= G^{-1}\left\{G\left(u/\sqrt{c^2w^2 - u^2c^2 - u^2}\right) - (1 - k)[\alpha - \alpha_1(u, w)]\right\}, \\ f_2(u, w; k) &= G^{-1}\left\{G\left(u/\sqrt{c^2w^2 - u^2c^2 - u^2}\right) + k[\alpha - \alpha_1(u, w)]\right\}. \end{aligned}$$

Note that $u_0 \equiv u_0(w)$ depends on w. Define X and Y as in (2.3), and let

$$\begin{aligned} A_1 &= \{0 < U < u_0(W),\ f_1(U, W; k) < V/S \le f_2(U, W; k)\}, \\ A_2 &= \{0 < Y < u_0(W),\ f_1(Y, W; 1 - k) < X/S \le f_2(Y, W; 1 - k)\}. \end{aligned}$$

Then the test with critical region $A_0 \cup (A_1 \cap A_2 \cap \mathcal{C})$ is uniformly more powerful than the LRT.

Again, extensions to the general problem (1.1) with $m > 2$ and $\Sigma_0 \ne I$ can be achieved by linear transformation and by applying the intersection-union method described in Section 4 of Liu and Berger [4].

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