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PERIODIC POINTS AND NORMAL FAMILIES

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ABSTRACT. Let $\mathcal{F}$ be the family of all functions which are holomorphic in some domain and do not have periodic points of some period greater than one there. It is shown that $\mathcal{F}$ is quasinormal, and the sequences in $\mathcal{F}$ which do not have convergent subsequences are characterized. The method also yields a new proof of the result that transcendental entire functions have infinitely many periodic points of all periods greater than one.

1. Introduction and main results

Let X, Y be sets and let $f : X \to Y$ be a function. The iterates $f^n : X_n \to Y$ are defined by $X_1 := X, f^1 := f$ and $X_n := f^{-1}(X_{n-1} \cap Y), f^n := f^{n-1} \circ f$ for $n \in \mathbb{N}$, $n \geq 2$. Note that $X_2 = f^{-1}(X_1 \cap Y) \subset X = X_1$ and thus $X_{n+1} \subset X_n \subset X$ for all $n \in \mathbb{N}$.

A point $x \in X$ is called a *periodic point of period* n of f if $x \in X_n$ and $f^n(x) = x$, but $f^m(x) \neq x$ for $1 \leq m \leq n-1$. A periodic point of period 1 is called a *fixed point*. The periodic points of period n are thus the fixed points of f^n which are not fixed points of f^m for any m less than n.

The following result is due to P. C. Rosenbloom [16].

Theorem A. *Let f be a transcendental entire function and let $n \in \mathbb{N}$, $n \geq 2$. Then f^n has infinitely many fixed points.*

This result generalized a result of P. Fatou [11, p. 345] dealing with the case $n = 2$. Clearly, polynomials which are not of the form $f(z) = z + c$, $c \in \mathbb{C}$, and thus their iterates, also have fixed points.

M. Essén and S. Wu [9] proved a corresponding normality criterion, thereby answering a question of L. Yang [18, Problem 8].

Theorem B. *Let $D \subset \mathbb{C}$ be a domain and let $\mathcal{G}$ be the family of all holomorphic functions $f : D \to \mathbb{C}$ for which there exists $n = n(f) > 1$ such that f^n has no fixed point. Then $\mathcal{G}$ is normal.*

The polynomial $p(z) = -z + z^2$ has no periodic point of period 2. The following result of I. N. Baker [2] shows that it is essentially the only polynomial of degree greater than one where periodic points of some period are missing.

Theorem C. *Let f be a polynomial of degree $d \geq 2$ and let $n \in \mathbb{N}$, $n \geq 2$. Suppose that f has no periodic point of period n. Then $d = n = 2$. Moreover, there exists a linear transformation L such that $f(z) = L^{-1}(p(L(z)))$, with $p(z) = -z + z^2$.*

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The following generalization of Theorem A was conjectured in [13, Problem 2.20] and proved in [3, Theorem 1] and [4, §1.6, Satz 2].

Theorem D. *Let f be a transcendental entire function and let $n \in \mathbb{N}$, $n \geq 2$. Then f has infinitely many periodic points of period n.*

Here we shall be concerned with normal family analogues of these results. The method used will also lead to a new proof of Theorem D.

Let $D \subset \mathbb{C}$ be a domain, $m \in \mathbb{N}$, $m \geq 2$. By $\mathcal{F}_m = \mathcal{F}_m(D)$ we denote the family of all holomorphic functions $f : D \to \mathbb{C}$ for which there exists $n = n(f) \geq m$ such that f has no periodic point of period n. Clearly $\mathcal{F}_{m+1} \subset \mathcal{F}_m$ for $m \geq 2$. In particular, $\mathcal{F}_3 \subset \mathcal{F}_2$.

Since all linear transformations are contained in $\mathcal{F}_m$, for all $m \geq 2$, none of the families $\mathcal{F}_m$ is normal. Also, if L is a linear transformation and if $p(z) = -z + z^2$, then $L^{-1} \circ p \circ L \in \mathcal{F}_2$. We shall show that in a certain sense all obstructions to normality arise from these examples.

Recall that a family $\mathcal{F}$ of functions holomorphic in a domain D is called *quasinormal* (cf. [7, 14, 17]) if for each sequence (f_k) in $\mathcal{F}$ there exists a subsequence (f_{k_j}) and a finite set $E \subset D$ such that (f_{k_j}) converges locally uniformly in $D \backslash E$. If the cardinality of the exceptional set E can be bounded independently of the sequence (f_k), and if q is the smallest such bound, then we say that $\mathcal{F}$ is quasinormal of *order* q.

Note that the maximum principle implies that if a sequence (f_k) of functions holomorphic in D converges locally uniformly in $D\backslash\{z_1, \ldots, z_q\}$, but not in D, then $f_k \to \infty$ in $D\backslash\{z_1, \ldots, z_q\}$.

Theorem 1. *$\mathcal{F}_2(D)$ is quasinormal of order* 1*.*

To compare this result with Theorem B of Essén and Wu [9] we note that our hypotheses are weaker than theirs; that is, $\mathcal{G} \subset \mathcal{F}_2(D)$. As a consequence, we obtain only quasinormality instead of normality.

The following result characterizes the sequences in $\mathcal{F}_2(D)$ without convergent subsequences.

Theorem 2. *Let (f_k) be a sequence in $\mathcal{F}_2(D)$ which does not have a convergent subsequence. Then there exists a compact set $K \subset D$ such that $f_k \to \infty$ locally uniformly in $D\backslash K$. Moreover, for sufficiently large k there exists a simply-connected domain $\Omega_k \subset K$ satisfying $\min_{z \in K\backslash\Omega_k} |f_k(z)| \to \infty$ and there exists a quasiconformal homeomorphism $\phi_k : \mathbb{C} \to \mathbb{C}$ such that if p_k is defined by $p_k(z) = 2z$ for $f_k \in \mathcal{F}_3$ and $p_k(z) = -z + z^2$ for $f_k \in \mathcal{F}_2\backslash\mathcal{F}_3$, then $f_k(z) = \phi_k^{-1}(p_k(\phi_k(z)))$ for $z \in \Omega_k$.*

Thus Theorem 2 says that if a sequence (f_k) does not have a convergent subsequence, then $f_k \to \infty$ outside certain small domains, and in these domains f_k is quasiconformally conjugate to one of two specific polynomials.

2. Preliminaries for the proofs

As in [3, 9], one of the central tools comes from the Ahlfors theory of covering surfaces. (An account of the Ahlfors theory can be found in [1], [12, Chapter 5] or [15, Chapter XIII].) To state the result we need, let $D \subset \mathbb{C}$ be a domain and let $f : D \to \mathbb{C}$ be holomorphic. Given a Jordan domain $V \subset \mathbb{C}$, we say that f has an *island* over V if $f^{-1}(V)$ has a component whose closure is contained in D. Note that if U is such a component, then $f|_U : U \to V$ is a proper map.

Lemma 1. *Let $D \subset \mathbb{C}$ be a domain and let $D_1, D_2 \subset \mathbb{C}$ be Jordan domains with $\overline{D_1} \cap \overline{D_2} = \emptyset$. Let $\mathcal{F}$ be a family of functions holomorphic in D which is not normal. Then there exists a function $f \in \mathcal{F}$ which has an island over D_1 or D_2.*

Lemma 1 follows from [12, Theorem 5.5] (applied with a domain D_3 containing ∞) and [12, Theorem 6.6]. For a different proof of Lemma 1 see [5, §5.1].

Another important concept used in [3, 9], and again used here, is that of a polynomial-like map. By definition, if $U, V \subset \mathbb{C}$ are bounded, simply-connected domains with $\overline{U} \subset V$, and if $f : U \to V$ is a proper map (of degree d), then the triple (f, U, V) is called a *polynomial-like map* (of *degree* d). The fundamental result about polynomial-like maps is the following (see [6, Theorem VI.1.1] or [8, Theorem 1]).

Lemma 2. *Let (f, U, V) be a polynomial-like map of degree d. Then there exists a polynomial p of degree d and a quasiconformal map $\phi : \mathbb{C} \to \mathbb{C}$ such that $f(z) = \phi^{-1}(p(\phi(z)))$ for all $z \in U$. Moreover, $\phi(U)$ contains the filled Julia set of p and thus, in particular, all periodic points of p.*

If $d = 1$, then p has the form $p(z) = az + b$ with $a, b \in \mathbb{C}$. It is not difficult to see that $|a| > 1$. With $L(z) = z + b/(a-1)$ we have $p(z) = L^{-1}(aL(z))$. With $\psi(z) = z|z|^{\alpha} e^{i\beta \log|z|}$ where $\alpha = \log 2/\log|a| - 1$ and $\beta = -\arg a/\log|a|$ we have $az = \psi^{-1}(2\psi(z))$. Replacing ϕ by $\psi \circ L \circ \phi$ we obtain the following:

Remark. If $d = 1$ in Lemma 2, then we can choose $p(z) = 2z$ there.

A simple consequence of Lemma 2 (which, however, can also be proved directly) is the following result.

Lemma 3. *Every polynomial-like map has a fixed point.*

Let $f : D \to \mathbb{C}$ be holomorphic and let $U, V \subset \mathbb{C}$ be Jordan domains. It will be convenient to use the notation $U \overset{f}{\rightsquigarrow} V$ if $f|_{D \cap U}$ has an island over V. Note that if $U \overset{f}{\rightsquigarrow} V$ and $V \overset{g}{\rightsquigarrow} W$, then $U \overset{g \circ f}{\rightsquigarrow} W$. Also, if $V \overset{f}{\rightsquigarrow} V$, then there exists $U \subset V$ such that (f, U, V) is a polynomial-like map. Lemma 3 thus yields the following result.

Lemma 4. *If $V \overset{f}{\rightsquigarrow} V$, then f has a fixed point in V.*

We shall use some elementary concepts from graph theory. We mention that graph theoretic arguments like the ones below are also implicit in [9], even though they are not formalized this way.

For a set V and a set $E \subset V \times V$ we call the pair $G = (V, E)$ a digraph. The elements of V are called vertices and those of E are called edges. (Note that in contrast to the usual terminology we allow loops; that is, edges e of the form $e = (v, v)$ with $v \in V$.)

Let $n \in \mathbb{N}$ and $w = (v_0, v_1, \ldots, v_n) \in V^{n+1}$. Then w is called a *closed walk of length* n if $(v_{k-1}, v_k) \in E$ for $k \in \{1, \ldots, n\}$ and $v_0 = v_n$. Note that we have not excluded the case that $v_j = v_k$ for $j, k \in \{1, \ldots, n\}$, $j \neq k$. We call a closed walk $w = (v_0, v_1, \ldots, v_n)$ *primitive* if there do not exist $p \in \mathbb{N}$, $1 \leq p < n$, such that $p|n$ and $v_j = v_k$ for all $j, k \in \{1, \ldots, n\}$ satisfying $p|(j-k)$. A primitive closed walk is thus a closed walk which is not obtained by running through a closed walk of smaller length several times. Finally recall that the outdegree of a vertex v is defined to be the cardinality of $\{u \in V : (v, u) \in E\}$.

Lemma 5. *Let $n \in \mathbb{N}$ and let $G = (V, E)$ be a digraph. Suppose that there exist $u, v \in V$ such that $u \neq v$ and $\{(v, v), (u, v), (v, u)\} \subset E$. Then G contains a primitive closed walk of length n.*

Proof. Since (v, v) is closed walk of length 1 we may assume that $n \geq 2$. Let $v_0 := v_n := u$ and $v_j := v$ for $j \in \{1, \ldots, n-1\}$. Then $(v_0, \ldots, v_n)$ is a primitive closed walk of length n in G. □

Remark. The hypotheses of Lemma 5 are satisfied if the cardinality of V is at least 2 and G is the complete digraph; that is, if $E = V \times V$.

Lemma 6. *Let $q, n \in \mathbb{N}$, $q \geq 4$, $n \geq 2$. Let $G = (V, E)$ be a digraph with q vertices such that the outdegree of each vertex is at least $q-1$. Then G contains a primitive closed walk of length n.*

Proof. Suppose first that there exists $v \in V$ such that $(v, v) \in E$. Since the outdegree of v is at least 3 there exist $v_1, v_2 \in V \setminus \{v\}$ such that $v_1 \neq v_2$ and $(v, v_j) \in E$ for $j \in \{1, 2\}$. If $(v_1, v) \in E$ or $(v_2, v) \in E$, then we can apply Lemma 5 and the conclusion follows. If $(v_1, v) \notin E$ and $(v_2, v) \notin E$, then $\{(v_1, v_1), (v_1, v_2), (v_2, v_1)\} \subset E$, and again the conclusion follows from Lemma 5.

Suppose now that $(v, v) \notin E$ for all $v \in V$. Then $E = (V \times V) \setminus \{(v, v) : v \in V\}$. Let $v \in V$ and $v_0 := v_n := v$. For $j \in \{1, \ldots, n-1\}$ we choose $v_j \in V \setminus \{v\}$ inductively such that $v_j \neq v_{j-1}$ for $j \in \{1, \ldots, n-1\}$. Then $(v_0, \ldots, v_n)$ is a primitive closed walk of length n in G. □

Given pairwise disjoint Jordan domains $D_1, \ldots, D_q \subset \mathbb{C}$ and a holomorphic function f we shall apply Lemmas 5 and 6 to the digraph $G(f, \{D_j\}_{j=1}^q)$ whose vertex set V and edge set E are given by $V = \{D_j : j = 1, \ldots, q\}$ and $E = \{(D_j, D_k) \in V \times V : D_j \overset{f}{\rightsquigarrow} D_k\}$.

Lemma 7. *If $G(f, \{D_j\}_{j=1}^q)$ contains a primitive closed walk of length n, then f has a periodic point of period n in each D_j belonging to the walk.*

Proof. Let $(D_{j_0}, D_{j_1}, \ldots, D_{j_n})$ be a primitive closed walk of length n. By Lemma 4 and the remarks made before Lemma 4 there exists a fixed point $z_0 \in D_{j_0}$ of f^n such that $f^k(z_0) \in D_{j_k}$ for $k \in \{1, \ldots, n-1\}$. It follows from the definition of primitivity that z_0 has period n. □

3. Proof of Theorems 1 and 2

Proof of Theorem 1. We may assume that D is bounded. Let $q \in \mathbb{N}$, let $z_1, \ldots, z_q \in D$ be distinct and let (f_k) be a sequence in $\mathcal{F}_2$ such that no subsequence of (f_k) is normal in a neighborhood of some z_j. We will show that $q = 1$. This implies that $\mathcal{F}_2(D)$ is quasinormal, and in fact quasinormal of order at most 1.

For $j \in \{1, \ldots, q\}$ let D_j be a disk around z_j such that $\overline{D_j} \subset D$ and $\overline{D_i} \cap \overline{D_j} = \emptyset$ for $i \neq j$. We consider the digraphs $G_k = G(f_k, \{D_j\}_{j=1}^q)$.

First we show that $q < 4$. We assume that this is not the case. Lemma 1 implies that the outdegree of each vertex of G_k is at least $q-1$ if k is sufficiently large. Lemma 6 implies that for these k there is a primitive closed walk of length n for each $n \geq 2$. From Lemma 7 we deduce that f_k has a periodic point of period n for each $n \geq 2$, provided k is large enough. This is a contradiction. Hence $q < 4$.

We may now assume that $q < 4$ has been chosen maximal so that (f_k) is normal in $D\backslash\{z_1, \ldots, z_q\}$. It follows that $f_k \to \infty$ locally uniformly in $D\backslash\{z_1, \ldots, z_q\}$. This implies that $f_k(\partial D_i) \cap \overline{D_j} = \emptyset$ for $i, j \in \{1, \ldots, q\}$ and large k. Since no subsequence of (f_k) is normal, we also have $f_k(D_i) \cap D_j \neq \emptyset$ for $i, j \in \{1, \ldots, q\}$ and large k. It follows that $D_i \overset{f_k}{\rightsquigarrow} D_j$ for $i, j \in \{1, \ldots, q\}$ and large k. Hence G_k is the complete digraph if k is sufficiently large. On the other hand, since $f_k \in \mathcal{F}_2$ for all $k \in \mathbb{N}$, there exists $n_k \in \mathbb{N}$, $n_k \geq 2$, such that f has no periodic point of period n_k. Thus G_k does not contain a primitive closed walk of length n_k by Lemma 7. Lemma 5 and the remark following it now show that $q = 1$. □

Proof of Theorem 2. Again we may assume that D is bounded. Since no subsequence of (f_k) converges to ∞ there exist $M, \delta > 0$ and a sequence (z_k) in D such that $\operatorname{dist}(z_k, \partial D) \geq \delta$ and $|f_k(z_k)| \leq M$ for all k. We define $H_k := \{z \in \mathbb{C} : |z - z_k| < \delta/2\}$ and $m_k := \min_{z\in\partial H_k} |f_k(z)|$. Choose $\zeta_k \in \partial H_k$ with $m_k = |f_k(\zeta_k)|$.

Suppose that (m_k) has a bounded subsequence $\left(m_{k_j}\right)$. Passing to further subsequences if necessary, we may assume that $z_{k_j} \to z_0 \in D$ and $\zeta_{k_j} \to \zeta_0 \in D$. Since $\left(f_{k_j}\right)$ is quasinormal of order 1 by Theorem 1, we may also assume that $\left(f_{k_j}\right)$ converges in $D\backslash\{z_1\}$ for some $z_1 \in D$. As noted before Theorem 1, we actually have $f_{k_j} \to \infty$ in $D\backslash\{z_1\}$. Since $\left|f_{k_j}\left(z_{k_j}\right)\right| < M$ this implies that $z_1 = z_0$ and hence that $m_{k_j} = \left|f_{k_j}\left(\zeta_{k_j}\right)\right| \to \infty$. This is a contradiction. Hence (m_k) does not have a bounded subsequence; that is, $m_k \to \infty$.

Let $\Delta_k := \{z \in \mathbb{C} : |z| < m_k/2\}$. Then $f_k(\partial H_k) \cap \overline{\Delta_k} = \emptyset$ for all k. For large k we have $m_k > 2M$ and thus $f_k(z_k) \in \Delta_k$ so that $f_k(H_k) \cap \Delta_k \neq \emptyset$. Hence $H_k \overset{f_k}{\rightsquigarrow} \Delta_k$ for large k. Since D is bounded and $m_k \to \infty$ we also have $\overline{H_k} \subset \Delta_k$ for large k. For these k we put $\Omega_k := f_k^{-1}(\Delta_k) \cap H_k$. Let $D_1, \ldots, D_q$ be components of Ω_k. Then $G_k = G(f_k, \{D_j\}_{j=1}^q)$ is the complete graph, and thus $q = 1$ by Lemmas 5 and 7. Hence Ω_k is connected and, by the maximum principle, even simply-connected.

Thus (f, Ω_k, Δ_k) is a polynomial-like map. By Lemma 2 there exists a quasiconformal map $\phi_k : \mathbb{C} \to \mathbb{C}$ and a polynomial p_k such that $f_k(z) = \phi_k^{-1}(p_k(\phi_k(z)))$ for $z \in \Omega_k$. Now $f_k \in \mathcal{F}_m(D)$ implies that $p_k \in \mathcal{F}_m(\mathbb{C})$. From Theorem C we thus deduce that if $f_k \in \mathcal{F}_3(D)$, then p_k has degree 1. By the remark following Lemma 2 we may thus assume that $p_k(z) = 2z$ in this case. From Theorem C we can also deduce that if $f_k \in \mathcal{F}_2(D)\backslash\mathcal{F}_3(D)$, then p_k has the form $p_k(z) = L_k^{-1}(p(L_k(z))$, with $p(z) = -z + z^2$ and a linear transformation L_k. Replacing ϕ_k by $L_k \circ \phi_k$ we obtain the conclusion. □

4. A proof of Theorem D

Let (a_k) be a sequence tending to ∞ such that $(f(a_k))$ is bounded. Define $f_k : \mathbb{C} \to \mathbb{C}$ by $f_k(z) = f(a_k z)/a_k$. Then $f_k(0) \to 0$ and $f_k(1) \to 0$.

Suppose now that f has only finitely many periodic points of period n. Theorem 1 implies that (f_k) is quasinormal of order 1 in $\mathbb{C}\backslash\{0\}$. It is not difficult to show that no subsequence of (f_k) is normal at 0, and this implies that $f_k \to \infty$ in $\mathbb{C}\backslash\{0, 1\}$.

For sufficiently large k we have $|f_k(0)| < 2$, $|f_k(1)| < 2$, $\min_{|z|=1/2} |f_k(z)| > 2$ and $\min_{|z-1|=1/2} |f_k(z)| > 2$. This implies that there are components D_0 and D_1 of $f_k^{-1}(\{z \in \mathbb{C} : |z| < 2\})$ satisfying $0 \in D_0 \subset \{z \in \mathbb{C} : |z| < \frac{1}{2}\}$ and

$1 \in D_1 \subset \{z \in \mathbb{C} : |z-1| < \frac{1}{2}\}$. It follows that $G(f_k, \{D_j\}_{j=0}^1)$ is the complete graph. From Lemmas 5 and 7 we deduce that f_k has a periodic point of period n in D_1, and thus f has a periodic point of period n in $\{z \in \mathbb{C} : |z-a_k| < \frac{1}{2}|a_k|\}$ for large k. This is a contradiction. □

5. Repelling periodic points

Let $z_0 \in \mathbb{C}$ be a periodic point of period n of the holomorphic function f. Then z_0 is called *repelling* if $|(f^n)'(z_0)| > 1$.

M. Essén and S. Wu [10, Theorem 1] have generalized their Theorem B as follows.

Theorem E. *Let $D \subset \mathbb{C}$ be a domain and let $\mathcal{G}$ be the family of all holomorphic functions $f : D \to \mathbb{C}$ for which there exists $n = n(f) > 1$ such that f^n has no repelling fixed point. Then $\mathcal{G}$ is normal.*

Theorems C and D also have generalizations dealing with repelling periodic points.

Theorem F. *Let f be a polynomial of degree $d \geq 2$ and let $n \in \mathbb{N}$. Suppose that f has no repelling periodic point of period n. Then one of the following cases holds:*

(i) $n = 1$, $d \geq 2$,
(ii) $n = 2$, $d = 2$,
(iii) $n = 2$, $d = 3$,
(iv) $n = 2$, $d = 4$,
(v) $n = 3$, $d = 2$.

Theorem G. *Let f be a transcendental entire function and let $n \in \mathbb{N}$, $n \geq 2$. Then f has infinitely many repelling periodic points of period n.*

Theorem F is proved in [4, §1.4, Satz 1] while Theorem G can be found in [3, Theorem 1] and [4, §1.6, Satz 2]. Examples in [4, §1.4] show that each of the five exceptional cases listed in Theorem F does occur.

Here we give the following result.

Theorem 3. *Let $D \subset \mathbb{C}$ be a domain and let $\mathcal{F}$ be the family of all holomorphic functions $f : D \to \mathbb{C}$ for which there exists $n = n(f) > 1$ such that f has no repelling periodic point of period n. Then $\mathcal{F}$ is quasinormal.*

It seems possible to describe the non-normal sequences in $\mathcal{F}$ similarly as in Theorem 2, but we omit this here.

We only sketch the proof of Theorem 3. Given a holomorphic function $f : D \to \mathbb{C}$ and a Jordan domain $V \subset \mathbb{C}$, we say that f has a *simple island* over V if $f^{-1}(V)$ has a component U with $\overline{U} \subset D$ such that $f|_U : U \to V$ is bijective. If, in addition, $\overline{U} \subset V$, then (f, U, V) is a polynomial-like map of degree 1 and, as already mentioned after Lemma 2 (and also easily proved directly), f has a repelling fixed point in U in this case.

Instead of Lemma 1 we now use the following result from the Ahlfors theory.

Lemma 8. *Let $D \subset \mathbb{C}$ be a domain and let D_1, D_2, D_3 be Jordan domains with pairwise disjoint closures. Let $\mathcal{F}$ be a family of functions holomorphic in D which is not normal. Then there exists a function $f \in \mathcal{F}$ which has a simple island over D_1, D_2 or D_3.*

We also replace Lemma 6 by the following one.

Lemma 9. *Let $q, n \in \mathbb{N}$, $q \geq 6$, $n \geq 2$. Let G be a digraph with q vertices such that the outdegree of each vertex is at least $q-2$. Then G contains a primitive closed walk of length n.*

With these modifications, the proof of Theorem 3 proceeds similar to that of Theorem 1. In fact one obtains that $\mathcal{F}$ is quasinormal of order less than 6. We omit the details.

Essén and Wu [9, §4, Remark 1] have raised the question whether the conclusion of Theorem B holds if one replaces "holomorphic" by "meromorphic" in the definition of $\mathcal{G}$. They show [9, §4, Remark 2] that the resulting family $\mathcal{G}$ is still quasinormal of order less than 17.

Similarly we may replace "holomorphic" by "meromorphic" in Theorem 3. To see this we note that in the meromorphic case an analogue of Lemma 8 holds if we take five domains D_j instead of three domains – this is the celebrated "Ahlfors five islands theorem". And if we assume in Lemma 9 only that the outdegree of each vertex is at least $q-4$, then the conclusion still holds if q is sufficiently large.

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References

[1] L. V. Ahlfors, Zur Theorie der Überlagerungsflächen, *Acta Math.* 65 (1935), 157-194, and *Collected Papers*, Birkhäuser, Boston, Basel, Stuttgart, 1982, Vol. I, pp. 214-251. MR **84k:**01066a

[2] I. N. Baker, Fixpoints of polynomials and rational functions, *J. London Math. Soc.* 39 (1964), 615-622. MR **30:**230

[3] W. Bergweiler, Periodic points of entire functions: proof of a conjecture of Baker, *Complex Variables Theory Appl.* 17 (1991), 57-72. MR **92m:**30048

[4] ———, Periodische Punkte bei der Iteration ganzer Funktionen. Habilitationsschrift, Rheinisch-Westfälische Techn. Hochsch., Aachen 1991.

[5] ———, A new proof of the Ahlfors five islands theorem, *J. Analyse Math.* 76 (1998), 337-347. MR **99m:**30032

[6] L. Carleson and T. W. Gamelin, *Complex Dynamics*, Springer, New York, Berlin, Heidelberg, 1993. MR **94h:**30033

[7] C.-T. Chuang, *Normal Families of Meromorphic Functions*, World Scientific, Singapore, 1993.

[8] A. Douady and J. H. Hubbard, On the dynamics of polynomial-like mappings, *Ann. Sci. École Norm. Sup.* (4) 18 (1985), 287-343. MR **87f:**58083

[9] M. Essén and S. Wu, Fix-points and a normal family of analytic functions, *Complex Variables Theory Appl.* 37 (1998), 171-178. MR **99m:**30069

[10] ———, Repulsive fixpoints of analytic functions with applications to complex dynamics, *J. London Math. Soc.* (2) 62 (2000), 139–148.

[11] P. Fatou, Sur l'itération des fonctions transcendantes entières, *Acta Math.* 47 (1926), 337-360.

[12] W. K. Hayman, *Meromorphic Functions*, Clarendon Press, Oxford, 1964. MR **29:**1337

[13] ———, *Research Problems in Function Theory*, Athlone Press, London, 1967. MR **36:**359

[14] P. Montel, *Leçons sur les familles normales des fonctions analytiques et leurs applications*, Gauthier-Villars, Paris, 1927.

[15] R. Nevanlinna, *Eindeutige analytische Funktionen*, Springer, Berlin, Göttingen, Heidelberg, 1953. MR **15:**208c

[16] P. C. Rosenbloom, L'itération des fonctions entières, *C. R. Acad. Sci. Paris* 227 (1948), 382-383. MR **10:**187a

[17] J. L. Schiff, *Normal Families*, Springer, New York, Berlin, Heidelberg, 1993. MR **94f:**30046
[18] L. Yang, Some recent results and problems in the theory of value-distribution, in *Proceedings of the Symposium on Value Distribution Theory in Several Complex Variables*, (W. Stoll, ed.), Univ. of Notre Dame Press, Notre Dame Math. Lect. 12 (1992), 157-171. MR **94i:**30029

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