

POWER LINEAR KELLER MAPS OF DIMENSION THREE

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ABSTRACT. In this paper it is proved that a power linear Keller map of dimension three over a field of characteristic zero is linearly triangularizable.

Let K be a field. A *polynomial map* F in dimension n over K is an n -tuple (F_1, F_2, \dots, F_n) of polynomials in $K[X_1, X_2, \dots, X_n]$. If G is another polynomial map of the same dimension, then the composition of F and G is defined by

$$F \circ G = (F_1(G_1, G_2, \dots, G_n), \dots, F_n(G_1, G_2, \dots, G_n)).$$

The polynomial map F is *invertible* if there exists a polynomial map G such that $F \circ G$ and $G \circ F$ are both identities. It is *Keller* if the determinant of its Jacobian is a nonzero element in K , i.e., $\det JF \in K^*$. By the chain rule for Jacobians, invertible polynomial maps are Keller maps. The famous Jacobian conjecture states that if $\text{char } K = 0$, then any Keller map is invertible (see, e.g., [1] or [4]).

A polynomial map F is *power linear* if it is of the form $(X_1 + A_1^{d_1}, X_2 + A_2^{d_2}, \dots, X_n + A_n^{d_n})$ where A_i is a linear form in X_1, X_2, \dots, X_n and $d_i \geq 2$ for all i . It is *cubic linear* if it is power linear where $d_i = 3$ for all i . Druzkowski [2] showed that in the case $\text{char } K = 0$ if cubic linear Keller maps are invertible, then the Jacobian conjecture would be true. A polynomial map is *triangular* if it is of the form $(X_1 + p_1, X_2 + p_2, \dots, X_n + p_n)$ where p_i is a polynomial in $K[X_{i+1}, X_{i+2}, \dots, X_n]$. It is *linearly triangularizable* if there exists a linear invertible polynomial map ϕ such that $\phi \circ F \circ \phi^{-1}$ is triangular. A polynomial map is *elementary* if it is of the form $(X_1, \dots, X_{i-1}, X_i + p, X_{i+1}, \dots, X_n)$ where $p \in K[X_1, \dots, X_{i-1}, \hat{X}_i, X_{i+1}, \dots, X_n]$. It is *tame* if it can be written as a composition of invertible linear maps and elementary maps. It is not hard to see that linearly triangularizable maps are tame and tame maps are invertible. The tame generators conjecture asserts that an invertible polynomial map is tame. This is proved in dimension two by Jung [5] and van der Kulk [6]. For dimensions beyond two, it is an open problem.

In this note we prove that power linear Keller maps in dimension three over a field of characteristic zero are linearly triangularizable (hence tame and invertible) proving both the Jacobian and the tame generators conjecture in this case. (It is worth noting that if the degrees of the components of these maps are all equal to three, then the result is a special case of that of Wright [7] which states that all cubic homogeneous polynomial maps are linearly triangularizable.)

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Theorem. *Let K be a field of characteristic zero and let $F = (X + A^{d_1}, Y + B^{d_2}, Z + C^{d_3})$ be a polynomial map over K with $d_i \geq 2$ for all i where A, B, C are linear forms in X, Y, Z . If $\det JF \in K^*$, then F is linearly triangularizable.*

Proof. Let $A = a_1X + a_2Y + a_3Z$, $B = b_1X + b_2Y + b_3Z$, $C = c_1X + c_2Y + c_3Z$. For convenience let

$$[xy]_{pq} = \begin{bmatrix} x_p & x_q \\ y_p & y_q \end{bmatrix}$$

and

$$[xyz]_{pqr} = \begin{bmatrix} x_p & x_q & x_r \\ y_p & y_q & y_r \\ z_p & z_q & z_r \end{bmatrix}.$$

We also let $(xy)_{pq} = \det [xy]_{pq}$ and $(xyz)_{pqr} = \det [xyz]_{pqr}$. By expanding $\det JF$ and collecting homogeneous components we see that $\det JF \in K^*$ (or, equivalently, $\det JF = 1$) is equivalent to

$$(J) \quad \begin{aligned} & d_1 a_1 A^{d_1-1} + d_2 b_2 B^{d_2-1} + d_3 c_3 C^{d_3-1} + d_1 d_2 (ab)_{12} A^{d_1-1} B^{d_2-1} \\ & + d_1 d_3 (ac)_{13} A^{d_1-1} C^{d_3-1} + d_2 d_3 (bc)_{23} B^{d_2-1} C^{d_3-1} \\ & + d_1 d_2 d_3 (abc)_{123} A^{d_1-1} B^{d_2-1} C^{d_3-1} = 0. \end{aligned}$$

(Case 1) Suppose two of A, B, C are zero. After linear conjugation by an appropriate permutation map, we may assume that $B = C = 0$. (For example if $A = B = 0$, then, after conjugating F with (Z, Y, X) , we get $F_1 = (Z, Y, X) \circ F \circ (Z, Y, X) = (X + \bar{C}^{d_3}, Y, Z)$ where $\bar{C}(X, Y, Z) = C(Z, Y, X)$.) By (J), we have $d_1 a_1 A^{d_1-1} = 0$. Since $d_1 \geq 2$, either $a_1 = 0$ or $A = 0$. In both cases, F is triangular.

(Case 2) Suppose only one of A, B, C is zero. As before, after conjugating with a permutation map, we may assume that $C = 0$. Suppose A, B are similar, i.e., $B = rA$ for some $r \in K^*$, and $d_1 = d_2$. Then $F_1 = \phi \circ F \circ \phi^{-1} = (X + \bar{A}^{d_1}, Y, Z)$ where $\phi = (X, Y - r^{d_1} X, Z)$ and the result follows from (Case 1). Otherwise either A, B are not similar or $d_1 \neq d_2$. Now (J) implies the following:

$$(1) \quad d_1 a_1 A^{d_1-1} + d_2 b_2 B^{d_2-1} = 0,$$

$$(2) \quad d_1 d_2 (ab)_{12} A^{d_1-1} B^{d_2-1} = 0.$$

Since $d_i > 0$, (1) implies $a_1 = b_2 = 0$ and (2) implies $(ab)_{12} = 0$. Consequently, the matrix

$$\begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix}$$

has a row of zeros. After conjugating F with (Y, X, Z) , we may assume that $a_1 = b_1 = b_2 = 0$, i.e., F is triangular.

(Case 3) Suppose none of A, B, C are zero. After conjugating with a permutation map we may assume that $d_1 \geq d_2 \geq d_3$. There are two cases.

(i) $(d_1 - 1) \neq (d_2 - 1) + (d_3 - 1)$. Then (J) is equivalent to

$$(J1) \quad d_1 a_1 A^{d_1-1} + d_2 b_2 B^{d_2-1} + d_3 c_3 C^{d_3-1} = 0,$$

$$(J2) \quad d_1 d_2 (ab)_{12} A^{d_1-1} B^{d_2-1} + d_1 d_3 (ac)_{13} A^{d_1-1} C^{d_3-1} \\ + d_2 d_3 (bc)_{23} B^{d_2-1} C^{d_3-1} = 0,$$

$$(J3) \quad d_1 d_2 d_3 (abc)_{123} A^{d_1-1} B^{d_2-1} C^{d_3-1} = 0.$$

First note that at most two of A, B, C are similar. For if A, B, C are similar, then we may assume d_1, d_2, d_3 are distinct or else it is reduced to (Case 2) by a linear conjugation as before. Henceforth we may assume $d_1 > d_2 > d_3$. Now (J1) implies that $a_1 = b_2 = c_3 = 0$ and so $A = B = C = 0$, contradicting the assumption.

Suppose first that only two of A, B, C are similar. Then, after linear conjugation with an appropriate permutation map, we may assume that $C = rB$ for some $r \in K^*$ and that A, B are not similar. If $d_2 = d_3$, then, as before, a linear conjugation reduces the problem to (Case 2). Otherwise after a linear conjugation, we may assume that $d_3 > d_2$. After substituting $C = rB$ and dividing through by B^{d_2-1} in (J2) we have

$$(d_1 d_2 (ab)_{12} + d_1 d_3 (ac)_{13} r^{d_3-1} B^{d_3-d_2}) A^{d_1-1} = -d_2 d_3 (bc)_{23} r^{d_3-1} B^{d_3-1}.$$

Since $d_3 \geq 2$, B divides the right side, hence the left side. Since A is not similar to B , we have $d_1 d_2 (ab)_{12} + d_1 d_3 (ac)_{13} r^{d_3-1} B^{d_3-d_2} = d_2 d_3 (bc)_{23} r^{d_3-1} = 0$ and so $(ab)_{12} = (ac)_{13} = (bc)_{23} = 0$. Substituting $C = rB$ into (J1), we have

$$d_1 a_1 A^{d_1-1} = -d_2 b_2 B^{d_2-1} - d_3 c_3 r^{d_3-1} B^{d_3-1}.$$

Since $d_i \geq 2$ for all i , B divides the left side. But A, B are not similar, so both sides are zero. Consequently, $a_1 = 0$ and, as $d_2 \neq d_3$, $b_2 = c_3 = 0$. Since $C = rB$, this implies that $b_3 = c_2 = 0$. Note that $b_1 \neq 0$; otherwise $B = 0$. So $(ab)_{12} = 0$ implies $a_2 = 0$ and, similarly, $(ac)_{13} = 0$ implies $a_3 = 0$. Therefore $A = 0$, a contradiction.

It remains to treat the case where no two of A, B, C are similar. After linear conjugation with an appropriate permutation map, we may assume $d_1 \geq d_2 \geq d_3$. From (J2), we have

$$d_1 d_2 (ab)_{12} A^{d_1-1} B^{d_2-1} + d_1 d_3 (ac)_{13} A^{d_1-1} C^{d_3-1} = -d_2 d_3 (bc)_{23} B^{d_2-1} C^{d_3-1}.$$

Since $d_1 > 1$, A divides the right side. Since A does not divide either B or C , $(bc)_{23} = 0$ and, since $A \neq 0$, $d_1 d_2 (ab)_{12} B^{d_2-1} + d_1 d_3 (ac)_{13} C^{d_3-1} = 0$. But since $d_2 > 1$ and B, C are not similar, we have $(ab)_{12} = (ac)_{13} = 0$. By (J3), $(abc)_{123} = 0$ so $\text{rank}[abc]_{123} < 3$. Since A, B are not similar, $C = p_1 A + p_2 B$. Therefore, $(ac)_{13} = p_2 (ab)_{13}$, $(bc)_{23} = -p_1 (ab)_{23}$. Since, by assumption, $p_1 \neq 0$ and $p_2 \neq 0$, we have $(ab)_{12} = (ab)_{13} = (ab)_{23} = 0$. Hence the first two rows of $[abc]_{123}$ are linearly dependent, i.e., A is similar to B , a contradiction.

(ii) $(d_1 - 1) = (d_2 - 1) + (d_3 - 1)$. Then (J) is equivalent to (J3) and the following:

$$(K1) \quad d_2 b_2 B^{d_2-1} + d_3 c_3 C^{d_3-1} = 0,$$

$$(K2) \quad d_1 a_1 A^{d_1-1} + d_2 d_3 (bc)_{23} B^{d_2-1} C^{d_3-1} = 0,$$

$$(K3) \quad d_1 d_2 (ab)_{12} A^{d_1-1} B^{d_2-1} + d_1 d_3 (ac)_{13} A^{d_1-1} C^{d_3-1} = 0.$$

As before if A, B, C are similar, then we may assume that $d_1 > d_2 > d_3$. Therefore (K1) implies $b_2 = c_3 = 0$; hence $b_3 = c_2 = 0$ and consequently $(bc)_{23} = 0$. Using this in (K2) we have $a_1 = 0$ and so $A = B = C = 0$, a contradiction.

Suppose only two of A, B, C are similar. We may also assume as before that the powers of these similar forms are not equal. We may further assume B, C are similar. Then (K1) implies $b_2 = c_3 = 0$, (K2) implies $a_1 = (bc)_{23} = 0$, (K3) implies $(ab)_{12} = (ac)_{13} = 0$ and (J3) implies $(abc)_{123} = 0$.

It is easy to arrive at the same conclusion if no two of A, B, C are similar. By [3, Lemma 1.2], there exists a permutation matrix P such that $P[abc]_{123}P^{-1}$ is upper triangular with 0's on the main diagonal. Hence F is linearly triangularizable. \square

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