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DERIVED SUBGROUPS AND CENTERS OF CAPABLE GROUPS

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ABSTRACT. A group G is said to be capable if it is isomorphic to the central factor group $H/\mathbf{Z}(H)$ for some group H. It is shown in this paper that if G is finite and capable, then the index of the center $\mathbf{Z}(G)$ in G is bounded above by some function of the order of the derived subgroup G'. If G' is cyclic and its elements of order 4 are central, then, in fact, $|G : \mathbf{Z}(G)| \leq |G'|^2$.

1. Introduction

Recall that a group G is said to be **capable** if there exists some group H such that $H/\mathbf{Z}(H)$ is isomorphic to G. Of course, there are groups that are not capable (nontrivial cyclic groups, for example), and so the condition that a group is capable imposes certain restrictions on its structure. It is known, for example, that if G is a finite capable p-group with $|G'| = p$, then $|G : \mathbf{Z}(G)| = p^2$. (I would like to thank A. Mann for informing me of this result, a special case of which appears in [1].)

As extraspecial p-groups show, there is no general upper bound on the index of the center of a finite group in terms of the order of its derived subgroup. In this paper, we prove that there is such a bound for all capable groups.

Theorem A. *There exists a function $B(n)$ defined on the natural numbers such that if G is finite and capable, then $|G : \mathbf{Z}(G)| \leq B(|G'|)$.*

As an immediate consequence, we have the following general result. (Recall that the **second center** of a group G is the preimage in G of $\mathbf{Z}(G/\mathbf{Z}(G))$.)

Corollary B. *Let G be an arbitrary finite group. Then the index of the second center of G is bounded above by some function of $|G'|$.*

Proof. The group $G/\mathbf{Z}(G)$ is capable, and $|(G/\mathbf{Z}(G))'| \leq |G'|$. The result follows by applying Theorem A to $G/\mathbf{Z}(G)$. □

We shall not attempt to find the optimal function $B(n)$ in Theorem A, but in the special case where G' is cyclic and all elements of order 4 in G' are central, we obtain the best possible bound.

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Theorem C. *Let G be finite and capable, and suppose that G' is cyclic and that all elements of order 4 in G' are central in G. Then $|G : \mathbf{Z}(G)| \leq |G'|^2$, and equality holds if G is nilpotent.*

Unfortunately, we have been unable to decide whether or not the assumption about elements of order 4 in Theorem C is really necessary.

We mention that we are aware of two other related papers in the literature. In [3], H. Heineken considers capable groups G for which G' is central and elementary abelian of order p^n, where p is a prime number. In Proposition 3, he shows that if $n = 2$, then $|G : \mathbf{Z}(G)| \leq p^5$. Heineken also asserts that for arbitrary n, if $p > 2$, then there exist examples where $|G : \mathbf{Z}(G)| = p^m$, where $m = 2n + \binom{n}{2}$. In [4], Heineken and D. Nikolova show that under certain very restrictive additional conditions, the index $|G : \mathbf{Z}(G)|$ cannot exceed p^m, where $m = 2n + \binom{n}{2}$. (In order to obtain this bound, the authors assume that G has exponent p and that $G' = \mathbf{Z}(G)$.)

I would like to thank A. Moreto for informing me of the existence of the papers [2] and [3], and for a number of helpful conversations on the subject of this paper. It was the referee who told me about [4], and I thank him too.

2. The general case

In this section, we work toward a proof of Theorem A. We begin with a couple of preliminary lemmas, which must surely be known. We thank D. S. Passman for helping us to find a proof of the first of these results.

(2.1) Lemma. *Let G be a finite capable group. Then there is a finite group H such that $H/\mathbf{Z}(H) \cong G$.*

Proof. Since G is capable, there is by definition a possibly infinite group H such that $H/Z \cong G$, where $Z = \mathbf{Z}(H)$. By choosing one element in each of the finitely many cosets of Z in H, we can produce a finitely generated subgroup K of H such that $H = ZK$. Then $\mathbf{Z}(K) \subseteq \mathbf{Z}(H) = Z$, and so $\mathbf{Z}(K) = K \cap Z$. It follows that $K/\mathbf{Z}(K) = K/(K \cap Z) \cong ZK/Z = H/Z \cong G$, and we can therefore replace H by K and assume that H is finitely generated.

Now $|H : Z| < \infty$, and hence by Schreier's theorem, the abelian group Z is finitely generated, and thus we can write $Z = T \times F$, where T is finite and F is torsion free. Certainly $Z/F \subseteq \mathbf{Z}(H/F)$, and we claim that equality holds here. To see this, let $h \in H$ be central modulo F, so that $[H, h] \subseteq F$, and in particular $[H, h]$ is central in H. It follows that the map $x \mapsto [x, h]$ defines a homomorphism from H into F. Since Z is in the kernel of this homomorphism and H/Z is finite, we see that $[H, h]$ is a finite subgroup of F. But F is torsion free, and thus $[H, h] = 1$ and $h \in Z$.

Now H/F is a finite group, and we have $(H/F)/\mathbf{Z}(H/F) = (H/F)/(Z/F) \cong H/Z \cong G$. This completes the proof. $\square$

(2.2) Lemma. *Let $A \subseteq G$, where A is abelian, and suppose that $|G : A| = m < \infty$ and that $|G'| = n < \infty$. Then*

$$|G : \mathbf{Z}(G)| \leq m^{1+\log(n)},$$

where the logarithm is to the base 2.

Proof. First, we argue that we can choose a subset $X \subseteq G$ such that $G = \langle X, A \rangle$ and $|X| \leq \log(m)$. To prove this, write $A_0 = A$ and recursively construct subgroups A_i such that $A_i = \langle A_{i-1}, x_i \rangle$, where x_i is chosen arbitrarily in $G - A_{i-1}$ as long as $A_{i-1} < G$. We thus have $A = A_0 < A_1 < \cdots < A_r = G$ for some integer $r \leq \log(|G : A|) = \log(m)$. The set $X = \{x_i \mid 1 \leq i \leq r\}$ has the desired properties.

Each conjugacy class of G is contained in some coset of G' in G, and thus each of the classes of G has cardinality no larger than $|G'|$. It follows that $|G : \mathbf{C}_G(x)| \leq |G'|$ for each element $x \in G$, and thus $|G : \mathbf{C}_G(X)| \leq |G'|^{|X|}$. Since A is abelian and $G = \langle A, X \rangle$, we see that $A \cap \mathbf{C}_G(X) \subseteq \mathbf{Z}(G)$, and thus

$$\begin{aligned} |G : \mathbf{Z}(G)| &\leq |G : A||A : A \cap \mathbf{C}_G(X)| \\ &\leq |G : A||G : \mathbf{C}_G(X)| \\ &\leq |G : A||G'|^{|X|} \\ &\leq mn^{\log(m)} = mm^{\log(n)}, \end{aligned}$$

and the result follows. □

Of course, if we know the smallest prime divisor p of $m = |G : A|$ in the situation of Lemma 2.2, we can work with logarithms to the base p, and thereby obtain a better bound.

Every group with trivial center is clearly capable, and our next result establishes Theorem A for such groups.

(2.3) Theorem. *There is a function $F(n)$ defined on the natural numbers such that if $\mathbf{Z}(G) = 1$ and $|G'| = n < \infty$, then $|G| \leq F(n)$.*

Proof. Let $C = \mathbf{C}_G(G')$, and write $m = |G : C|$, so that m is bounded above by some function of n. (For example, since G/C is isomorphically embedded in $\mathrm{Aut}(G')$, it follows that $m \leq n!$.) We have $[G, C] \subseteq G'$, and thus $[G, C, C] = 1$. We conclude by the three subgroups lemma that C' centralizes G, and since we are assuming that $\mathbf{Z}(G) = 1$, we see that $C' = 1$ and C is abelian.

We can now apply Lemma 2.2 to the abelian subgroup $C \subseteq G$ of index m, and we conclude that $|G| = |G : \mathbf{Z}(G)| \leq m^{1+\log(n)}$. The result follows since m is bounded in terms of n. □

We mention that a bound significantly better than the inequality $m \leq n!$ is available in the situation of Theorem 2.3. If we argue as in the proof of Lemma 2.2, we can find a generating set X for G' with $|X| \leq \log(n)$, where $n = |G'|$. Then $C = \mathbf{C}_G(G') = \mathbf{C}_G(X)$, and it follows that $m = |G : C| \leq |G'|^{|X|} \leq n^{\log(n)}$. (A version of this easy argument goes back at least as far as 1939, where it appears in the proof of statement (35) of the paper [6], by H. Wielandt.)

Before we begin the proof of Theorem A, we recursively define the relevant function $B(n)$. We start by setting $B(1) = 1$, and we assume that we have fixed some particular bounding function $F(n)$ as in Theorem 2.3. (We assume, as we may, that $F(1) = 1$ and that F is monotonically increasing.) If $n > 1$, we let M be the maximum of the quantities $B(n/q)$, where q runs over prime divisors of n, and we set

$$B(n) = \max\{F(n),\ (nM)^{1+\log(n)}\}.$$

It is easy to see that with this definition, we have $B(m) \leq B(n)$ whenever m divides n, and of course, $F(n) \leq B(n)$ for all n.

Proof of Theorem A. We are given a finite capable group $G = H/Z$, where $Z = \mathbf{Z}(H)$. Write $U = H'Z$ and let $n = |G'| = |U/Z|$. By Lemma 2.1, we can assume that H is finite, and we show that $|G : \mathbf{Z}(G)| \leq B(n)$ by induction on $|H|$. (Note that if $H = 1$, then $G = 1$ and the inequality is trivially true.)

If $Z = 1$, then $H = G$, and by Lemma 2.3, we have $|G : \mathbf{Z}(G)| = |G| \leq F(n) \leq B(n)$. There is nothing further to prove in this case, and so we assume that $Z > 1$, and we choose a subgroup $T \subseteq Z$ of prime order p. Let $Y/T = \mathbf{Z}(H/T)$, and note that $Y \supseteq Z$ and $H/Y \cong (H/T)/\mathbf{Z}(H/T)$ is capable.

Suppose first that $Y \cap U = Z$. Let $A/Y = \mathbf{Z}(H/Y)$ and note that $UY/Y = (H/Y)'$, and this subgroup has order n. Since $|H/T| < |H|$, we can apply the inductive hypothesis to deduce that $|H : A| \leq B(n)$. But $[H, A] \subseteq Y \cap H' \subseteq Y \cap U = Z$, and thus $A/Z \subseteq \mathbf{Z}(H/Z)$. This shows that $|G : \mathbf{Z}(G)| \leq |H : A| \leq B(n)$, as desired.

We can now assume that $Y \cap U > Z$. Let y be an element of $Y \cap U$ that does not lie in Z and set $C = \mathbf{C}_H(y) < H$. We have $[H, Y] \subseteq T \subseteq Z$, and thus the map $h \mapsto [h, y]$ defines a homomorphism from H into T with kernel C, and we see that $C \triangleleft H$ and $|H : C|$ divides $|T| = p$. It follows that $|H : C| = p$ and also, since H/C is abelian, we have $U \subseteq C$. Because $[H, Y] \subseteq Z$, we also have $1 = [h, y]^p = [h, y^p]$ for all elements $h \in H$. We deduce that $y^p \in Z$, and thus y has order p modulo Z.

Let $X = \mathbf{Z}(C)$ and observe that $y \in X \cap U$, and hence $|X \cap U : Z|$ is divisible by p and $|U : X \cap U|$ is a divisor of $|U : Z|/p = n/p$. Now $(H/X)' = UX/X \cong U/(X \cap U)$, and thus $|(H/X)'|$ divides n/p. It follows that C/X is a capable group whose derived subgroup has order dividing $|(H/X)'|$, which in turn divides n/p.

Write $V/X = \mathbf{Z}(C/X)$. Since $C < H$, the inductive hypothesis applies, and we conclude that $|C : V| \leq B(|(C/X)'|) \leq B(n/p)$. Now let $h \in H - C$ and write $S/X = \mathbf{C}_{V/X}(h)$. Since H/C has prime order, we see that h generates H modulo C, and thus $S/X \subseteq \mathbf{Z}(H/X)$. But $|(H/X)'| \leq n/p$, and thus $|(H/X) : \mathbf{C}_{H/X}(h)| \leq n/p$, and we have $|V : S| \leq n/p$. Now $|H : C| = p$ and $|C : V| \leq B(n/p)$, and so we see that $|H : S| \leq nB(n/p)$. In particular, $|H : S| \leq nM$, where M is the maximum value of $B(n/q)$, as q runs over all prime divisors of n.

Since $S/X \subseteq \mathbf{Z}(H/X)$, we have $[H, S] \subseteq X$. But $S \subseteq C$ and $X = \mathbf{Z}(C)$, and so $[H, S, S] \subseteq [X, C] = 1$, and thus we see by the three subgroups lemma that S' centralizes H. We conclude that $S' \subseteq Z$, and thus S/Z is abelian. Since $|H : S| \leq nM$ and $|(H/Z)'| = n$, Lemma 2.2 yields that $|G : \mathbf{Z}(G)| = |(H/Z) : \mathbf{Z}(H/Z)| \leq (nM)^{1+\log(n)}$. In particular, $|G : \mathbf{Z}(G)| \leq B(n)$, as required. □

3. Cyclic derived subgroups

We begin work toward a proof of Theorem C by studying groups that are not necessarily capable, but which have a cyclic derived subgroup.

(3.1) Lemma. *Let G be finite and assume that G' is a cyclic p-group for some prime p. If $G' \cap \mathbf{Z}(G)$ is nontrivial, then G has a normal p-complement.*

Proof. Let $P \in \mathrm{Syl}_p(G)$. Then $G' \subseteq P$, and so $P \triangleleft G$ and G has a p-complement H. By Fitting's lemma, we can write $G' = \mathbf{C}_{G'}(H) \times [G', H]$, and we observe that the first factor is nontrivial since we are assuming that $G' \cap \mathbf{Z}(G) > 1$. Since the cyclic p-group G' is indecomposable, we conclude that $[G', H] = 1$, and thus $[P, H] = [P, H, H] = 1$, where the first equality follows because $(|H|, |P|) = 1$. We conclude that P normalizes H, and thus $H \triangleleft G$, as required. □

(3.2) Theorem. *Let G be finite and assume that G' is cyclic. Let π be the set of prime divisors of $|G' \cap \mathbf{Z}(G)|$ and let b be the π'-part of $|G'|$. Then:*

(a) *G has a normal π-complement M and G/M is nilpotent.*
(b) *$|M : M \cap \mathbf{Z}(G)|$ divides $b\varphi(b)$, where φ is Euler's function.*
(c) *$|G : \mathbf{Z}(G)|$ divides $b\varphi(b)|G : V|$, where $V/M = \mathbf{Z}(G/M)$.*

Proof. Let $p \in \pi$ and note that since G' is cyclic, the derived subgroup of $G/\mathbf{O}_{p'}(G)$ is a cyclic p-group. Also, since $G' \cap \mathbf{Z}(G) \not\subseteq \mathbf{O}_{p'}(G)$ by the definition of the set π, we conclude that the group $G/\mathbf{O}_{p'}(G)$ satisfies the hypotheses of Lemma 3.1, and hence it has a normal p-complement. It follows that G has a normal p-complement, and since this is true for every prime $p \in \pi$, we see that G has a normal π-complement M and that G/M is nilpotent. This proves (a).

Now let $B = M \cap G'$ and write $C = \mathbf{C}_M(B)$. Since B is cyclic of order b and $B \triangleleft G$, we see that $|M : C|$ divides $\varphi(b)$, and thus to prove (b), it suffices to show that $|C : C \cap \mathbf{Z}(G)|$ divides b.

Since $C \triangleleft G$, we have $[G, C] \subseteq C \cap G' \subseteq M \cap G' = B$, and thus $[G, C, C] \subseteq [B, C] = 1$. By the three subgroups lemma, it follows that $[C', G] = 1$, and thus $C' \subseteq G' \cap \mathbf{Z}(G)$. But C is a π'-group, and by the definition of π it follows that no prime divisor of $|C|$ divides the order of $G' \cap \mathbf{Z}(G)$. We conclude that $C' = 1$ and C is abelian.

Now let $Q \in \mathrm{Syl}_q(C)$, where q is an arbitrary prime in π', and note that $Q \triangleleft G$ since C is abelian and $C \triangleleft G$. Since $q \in \pi'$ is arbitrary, we see that to establish that $|C : C \cap \mathbf{Z}(G)|$ divides b, as claimed, it suffices to show that the q-part of this index divides b, or equivalently, that $|Q : Q \cap \mathbf{Z}(G)|$ divides b.

Now, $[G, M] \subseteq B \subseteq C$, and thus $M/C \subseteq \mathbf{Z}(G/C)$. Since G/M is nilpotent, it follows that G/C is nilpotent, and we let K/C be the normal q-complement of G/C. Now K acts on Q, and we prove next that $\mathbf{C}_Q(K) \subseteq \mathbf{Z}(G)$. To this end, we let $L = [\mathbf{C}_Q(K), G]$ and we note that $L \subseteq G'$ and also $L \subseteq \mathbf{C}_Q(K)$ since $\mathbf{C}_Q(K) \triangleleft G$. Since K centralizes L and $L \triangleleft G$, we see that the q-group G/K acts on the q-group L, and thus if $L > 1$, we have $1 < \mathbf{C}_L(G) \subseteq G' \cap \mathbf{Z}(G)$. This is a contradiction since $q \notin \pi$, and it follows that $L = 1$, and thus $\mathbf{C}_Q(K) \subseteq \mathbf{Z}(G)$, as claimed, and thus, in fact $\mathbf{C}_Q(K) = Q \cap \mathbf{Z}(G)$.

Since $Q \subseteq C$ and C is abelian, the q'-group K/C acts on Q, and hence we have $Q = [Q, K] \times \mathbf{C}_Q(K)$ by Fitting's lemma. We have established that $\mathbf{C}_Q(K) = Q \cap \mathbf{Z}(G)$, and therefore $|Q : Q \cap \mathbf{Z}(G)| = |[Q, K]|$, and this divides b since $[Q, K] \subseteq B$. This shows that $|C : C \cap \mathbf{Z}(G)|$ divides b, as claimed, and the proof of (b) is complete.

Finally, to prove (c), we let $V/M = \mathbf{Z}(G/M)$ and $W = \mathbf{C}_V(B)$. (Note that $\mathbf{Z}(G) \subseteq W$ and that $W \cap M = C$.) Since B is cyclic of order b, we see that $|V : W|$ divides $\varphi(b)$, and thus $|G : W|$ divides $|G : V|\varphi(b)$. It suffices, therefore, to show that $|W : \mathbf{Z}(G)|$ divides b.

First, we prove that a Hall π-subgroup H of W is central in G by showing separately that H centralizes the Hall π'-subgroup M of G and that it centralizes a Hall π-subgroup S of G, where S is chosen to contain H. We have $[M, H] \subseteq M \cap G' = B$, and thus $[M, H] = [M, H, H] \subseteq [B, H] = 1$, as desired, where the last equality follows since $H \subseteq W = \mathbf{C}_V(B)$. Also, $[H, G] \subseteq M$ since $H \subseteq V$, and thus since $H \subseteq S$, we have $[H, S] \subseteq M \cap S = 1$. This shows that $H \subseteq \mathbf{Z}(G)$, as claimed, and thus $|W : \mathbf{Z}(G)|$ is a π'-number. We now have $W = (W \cap M)\mathbf{Z}(G) = C\mathbf{Z}(G)$, and thus $|W : \mathbf{Z}(G)| = |C : C \cap \mathbf{Z}(G)|$, which, as we have seen, is a divisor of b. This completes the proof. □

Next, we quote a known result.

(3.3) Lemma. *Let $\sigma \in \mathrm{Aut}(G)$ and assume that σ fixes all elements of prime order and of order 4 in $[G, \sigma]$. Then $[G, \sigma]$ has exponent dividing the order $o(\sigma)$.*

Proof. This is Theorem A(c) of [5]. □

Somewhat surprisingly, we need the following result, which establishes a *lower* bound on the index of the center of our group. We remark that if G is a p-group and G' is cyclic, then a generator of G' must actually be a commutator in G. The same conclusion therefore holds for finite nilpotent groups with cyclic derived subgroups. (One can work with one Sylow subgroup at a time.)

(3.4) Lemma. *Let G be nilpotent and assume that G' is cyclic and that all elements of order 4 in G' are central in G. Then $|G : \mathbf{Z}(G)| \geq |G'|^2$.*

Proof. As we remarked, some generator of G' must be a commutator, and thus we can choose elements a and b in G such that $\langle [a, b] \rangle = G'$, and we see that $X' = G'$, where $X = \langle a, b \rangle$. Since $|X : \mathbf{Z}(X)| \leq |G : \mathbf{Z}(G)|$, it is no loss to assume that $G = X$, and thus we have $G = \langle a, b \rangle$. Write $Z = \mathbf{Z}(G)$ and let $A = \langle Z, a \rangle$ and $B = \langle Z, b \rangle$. Then both A and B are abelian, and since $G = \langle A, B \rangle$, we see that $A \cap B = Z$.

Consider the inner automorphism σ of G induced by a, and note that the order of σ is exactly $|\langle a \rangle : \langle a \rangle \cap Z| = |A : Z|$. Since $[G, \sigma] = G'$ is cyclic, every subgroup of prime order in $[G, \sigma]$ is normal, and hence is central in G since G is nilpotent. Thus σ fixes all elements of prime order in $[G, \sigma]$ and also, by hypothesis, σ fixes all elements of order 4 in $[G, \sigma]$. Thus Lemma 3.3 applies, and so $|A : Z| = o(\sigma)$ is a multiple of the exponent of $[G, \sigma]$. Since $[G, \sigma] = G'$ is cyclic, its exponent is equal to its order, and we have $|A : Z| \geq |G'|$. Similarly, $|B : Z| \geq |G'|$, and thus $|G : Z| \geq |AB|/|Z| = |A : Z||B : Z| \geq |G'|^2$, as desired. □

The following result is very closely related to Theorem 1 of [2].

(3.5) Lemma. *Let G be nilpotent and assume that G' is cyclic and that all elements of order 4 in G' are central in G. Then there exist subgroups X and Y of G such that $XY = G$, $X' = G'$, $[X, Y] = 1$ and $|G : Y| = |G'|^2$.*

Proof. Since G is nilpotent, we can find (as in the previous proof) a two-generator subgroup $X = \langle a, b \rangle$ of G such that $X' = G'$. Let $Y = \mathbf{C}_G(X)$. By Lemma 3.4, we see that $|G : Y| \geq |X : X \cap Y| = |X : \mathbf{Z}(X)| \geq |X'|^2 = |G'|^2$. But $Y = \mathbf{C}_G(a) \cap \mathbf{C}_G(b)$, and so $|G : Y| \leq |G : \mathbf{C}_G(a)||G : \mathbf{C}_G(b)| = |\operatorname{cl}(a)||\operatorname{cl}(b)| \leq |G'|^2$. It follows that we have equality throughout, and thus in particular, we see that $|G'|^2 = |G : Y| = |X : X \cap Y|$, and it follows that $XY = G$, as required. □

Proof of Theorem C. It suffices to prove the inequality in the statement of the theorem; the fact that equality must hold if G is nilpotent will then follow via Lemma 3.4.

Since G is capable, we can assume that $G = H/Z$, where H is some finite group and $Z = \mathbf{Z}(H)$. We wish to apply Theorem 3.2 to G, and so as in that theorem, we let π be the set of prime divisors of $|G' \cap \mathbf{Z}(G)|$. Then H/Z has a normal π-complement M/Z, and we let $V/M = \mathbf{Z}(H/M)$. By Theorem 3.2(c), we know that

$|G : \mathbf{Z}(G)|$ divides $b\varphi(b)|H : V|$, where b is the π'-part of $|G'|$. Since $b\varphi(b) \le b^2$, we see that it suffices to show that $|H : V|$ is at most equal to the π-part of $|G'|^2$.

Let K/Z be a Hall π-subgroup of H/Z. Then $K/Z \cong H/M$ and it suffices to prove that $|(K/Z) : \mathbf{Z}(K/Z)| \le |(K/Z)'|^2$. By Theorem 3.2(a), we know that K/Z is nilpotent, and thus by Lemma 3.5, there exist subgroups X and Y of K, each of them containing Z, and such that $XY = K$ and $[X, Y] \subseteq Z$. Also $(X/Z)' = (K/Z)'$ and $|K : Y| = |(K/Z)'|^2$.

If Y/Z is abelian, then since Y/Z centralizes X/Z, we see that $Y/Z \subseteq \mathbf{Z}(K/Z)$, and thus $|(K/Z) : \mathbf{Z}(K/Z)| \le |K : Y| = |(K/Z)'|^2$, as required. It suffices to show, therefore, that $Y' \subseteq Z$.

Since $[X, Y] \subseteq Z$, we have $[X, Y, X] = 1$, and thus by the three subgroups lemma, X' centralizes Y, and similarly Y' centralizes X. Since $(X/Z)' = (K/Z)'$, we see that $X'Z = K'Z$, and it follows that K' centralizes Y. But $Y' \subseteq K'$, and thus Y' centralizes both Y and X, and thus $Y' \subseteq \mathbf{Z}(K)$ since $K = XY$.

Recall that our goal is to show that $Y' \subseteq Z$. Since we now know that Y' centralizes K, it suffices to show that Y' also centralizes M. Now $Y'Z/Z \subseteq (H/Z)'$, which is cyclic, and thus $Y'Z \triangleleft H$ and of course, $Y'Z/Z$ is a π-subgroup of H/Z. Thus $Y'Z/Z$ and M/Z are normal subgroups of coprime orders in H/Z, and it follows that $[M, Y'Z] \subseteq Z$. If $m \in M$, it follows that the map $y \mapsto [m, y]$ is a homomorphism from $Y'Z$ into Z, and Z is contained in the kernel of this map. Therefore $|[m, Y'Z]|$ divides $|Y'Z : Z|$, and it follows that $[M, Y'Z]$ is a π-group. Also, we can interchange the roles of M and $Y'Z$ in this argument, and we deduce that $[M, Y'Z]$ is a π'-group. It follows that $[M, Y'] = 1$, and thus $Y' \subseteq \mathbf{Z}(MK) = \mathbf{Z}(H) = Z$, as desired. This completes the proof. ∎

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