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ROUGH SINGULAR INTEGRALS ASSOCIATED TO SURFACES OF REVOLUTION

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ABSTRACT. Let $1 < p < \infty$ and $n \geq 2$. The authors establish the $L^p(\mathbb{R}^{n+1})$-boundedness for a class of singular integral operators associated to surfaces of revolution, $\{(t, \phi(|t|)) : t \in \mathbb{R}^n\}$, with rough kernels, provided that the corresponding maximal function along the plane curve $\{(t, \phi(|t|)) : t \in \mathbb{R}\}$ is bounded on $L^p(\mathbb{R}^2)$.

1. INTRODUCTION

Let $n \geq 2$ and $y \in \mathbb{R}^n$. For the Calderón-Zygmund type kernel

$$K(y) = \frac{\Omega(y)}{|y|^n} b(|y|)$$

and a suitable function ϕ on $[0, \infty)$, we define the singular integral operator T along the surface

$$\Gamma = \{(y, \phi(|y|)) : y \in \mathbb{R}^n\}$$

by

$$(1) \qquad Tf(x, s) = \text{p. v.} \int_{\mathbb{R}^n} f(x - y, s - \phi(|y|)) K(y)\, dy.$$

Here and in what follows, we always assume that b is a measurable function on $[0, \infty)$, Ω is homogeneous of order zero on $\mathbb{R}^n$, integrable on S^{n-1} and satisfies

$$(2) \qquad \int_{S^{n-1}} \Omega(y)\, d\sigma(y) = 0.$$

The kernel $K(y)$, which has radial roughness introduced by the factor $b(|y|)$, was first studied by R. Fefferman in the context of singular integrals on $\mathbb{R}^n$ ([9]).

In [10], Kim, Wainger, Wright and Ziesler proved the following theorem.

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Theorem A ([10]). *Let $\phi \in C^2([0,\infty))$ be convex, increasing and $\phi(0) = 0$. Let $\Omega \in C^\infty(S^{n-1})$ satisfy* (2) *and $b \equiv 1$. Then T in* (1) *is bounded on $L^p(\mathbb{R}^{n+1})$ for $1 < p < \infty$.*

In [3], Chen and Fan generalized the above result by requiring that Ω belongs to a Block space introduced in [11] and $b \in L^\infty([0,\infty))$.

Theorem B ([3]). *Suppose $\Omega \in B_r^\beta(S^{n-1})$ for some $\beta > 0$ and $r > 1$. If the maximal operator ν_ϕ on $\mathbb{R}$ given by*

$$(\nu_\phi g)(x) = \sup_{k\in\mathbb{Z}} \frac{1}{2^k} \int_{2^k}^{2^{k+1}} |g(x - \phi(t))|\, dt$$

is a bounded operator on $L^p(\mathbb{R})$ for $1 < p < \infty$, then T is bounded on $L^p(\mathbb{R}^{n+1})$ for $1 < p < \infty$.

The main purpose of this paper is to consider the L^p boundedness of T when $\Omega \in H^1(S^{n-1})$, the Hardy space on the sphere; see [5] and [4] for the definition. The method that we use in this paper comes from the work of Duoandikoetxea and Rubio de Francia ([6]) and its extension obtained in Fan-Pan ([8]).

To state our main result, we need to introduce the maximal function $\mathcal{M}_\phi$ associated to the plane curve $\{(x, \phi(|x|)) : x \in \mathbb{R}\}$. For any measurable function f on $\mathbb{R}^2$, $\mathcal{M}_\phi f$ is defined by

$$\mathcal{M}_\phi f(x_1, x_2) = \sup_{k\in\mathbb{Z}} \frac{1}{2^k} \int_{2^k}^{2^{k+1}} |f(x_1 - t, x_2 - \phi(|t|))|\, dt. \tag{3}$$

Here is our main theorem.

Theorem 1. *Let $\phi : [0,\infty) \to \mathbb{R}$ be continuously differentiable on $(0,\infty)$ and satisfy*

$$|\phi(t) - \phi(0)| \le Ct^\alpha$$

for some $\alpha > 0$ and small t, where C is a constant independent of t. Let $\Omega \in H^1(S^{n-1})$, $b \in L^\infty([0,\infty))$ and T be given by (1). *Then T is bounded on $L^p(\mathbb{R}^{n+1})$ for $1 < p < \infty$, provided that $\mathcal{M}_\phi$ in* (3) *is bounded on $L^p(\mathbb{R}^2)$.*

The condition imposed on $\phi(t)$ for $t \sim 0$ ensures that the integral in (1) exists in principle-value sense when, say, $f \in \mathcal{S}(\mathbb{R}^{n+1})$.

The $L^p(\mathbb{R}^2)$ boundedness of $\mathcal{M}_\phi$ is known for many ϕ's. Below we shall mention a few prominent cases:

(i) If ϕ is a real-valued polynomial, then $\mathcal{M}_\phi$ is bounded on $L^p(\mathbb{R}^2)$ for $p > 1$; see [13].

(ii) Let $h(t) = t\phi'(t) - \phi(t)$ for $t > 0$. If $\phi : \mathbb{R} \to \mathbb{R}$ is of class $C^2(0,\infty)$, convex on $[0,\infty)$ and $\phi(0) = \phi'(0) = 0$ and there exists an $\varepsilon > 0$ so that for each $t > 0$, $h'(t) > \varepsilon h(t)/t$, then $\mathcal{M}_\phi$ in (3) is bounded on $L^p(\mathbb{R}^2)$ for $p > 1$; see Theorem 1.5 in [2]. Moreover, if ϕ is either even or odd, convex on $[0,\infty)$, and there exists a $0 < C < \infty$ so that for each $t > 0$, $\phi'(Ct) \ge 2\phi'(t)$, then $\mathcal{M}_\phi$ in (3) is bounded on $L^p(\mathbb{R}^2)$ for $p > 1$. For details, see [1] or [2].

(iii) For $\phi(t) = t^\alpha$ with $\alpha \in (0, 1]$, $\mathcal{M}_\phi$ is bounded on $L^p(\mathbb{R}^2)$ for $p > 1$; see [12].

2. Proof of Theorem 1

We begin with the definition of the space $H^1(S^{n-1})$. For $f \in L^1(S^{n-1})$ and $x \in S^{n-1}$, we define

$$P^+ f(x) = \sup_{0<t<1} \left| \int_{S^{n-1}} P_{tx}(y) f(y)\, d\sigma(y) \right|,$$

where

$$P_{tx}(y) = \frac{1-t^2}{|y-tx|^n}$$

for $y \in S^{n-1}$.

Definition 1. An integrable function f on S^{n-1} is in the space $H^1(S^{n-1})$ if and only if

$$\|P^+ f\|_{L^1(S^{n-1})} = \int_{S^{n-1}} |P^+ f(x)|\, d\sigma(x) < \infty$$

and we define

$$\|f\|_{H^1(S^{n-1})} = \|P^+ f\|_{L^1(S^{n-1})}.$$

A very useful characterization of the space $H^1(S^{n-1})$ is its atomic decomposition. Let us first recall the definition of atoms.

Definition 2. A function $a(\cdot)$ on S^{n-1} is a regular atom if there exist $\xi \in S^{n-1}$ and $\rho \in (0,2]$ such that

(i) $\operatorname{supp} a \subset S^{n-1} \cap B(\xi,\rho)$, where $B(\xi,\rho) = \{y \in \mathbb{R}^n : \ |y-\xi| < \rho\}$;
(ii) $\|a\|_{L^\infty(S^{n-1})} \le \rho^{-n+1}$;
(iii) $\displaystyle\int_{S^{n-1}} a(y)\, d\sigma(y) = 0.$

A function $a(\cdot)$ on S^{n-1} is an exceptional atom if $a(\cdot) \in L^\infty(S^{n-1})$ and

$$\|a\|_{L^\infty(S^{n-1})} \le 1.$$

The following can be found in [5] and [4].

Lemma 1. *For any $f \in H^1(S^{n-1})$ there are complex numbers λ_j and atoms (regular or exceptional) a_j such that*

$$f = \sum_j \lambda_j a_j$$

and

$$\|f\|_{H^1(S^{n-1})} \sim \sum_j |\lambda_j|.$$

The following lemma is a simple corollary of Theorem B.

Lemma 2. *Let ϕ be the same as in Theorem 1. Let $\Omega \in L^r(S^{n-1})$ for some $1 < r \le \infty$, $n \ge 2$ and T be given by (1). Then T is bounded on $L^p(\mathbb{R}^{n+1})$ for $1 < p < \infty$, provided that $\mathcal{M}_\phi$ in (3) is bounded on $L^p(\mathbb{R}^2)$.*

Proof. It suffices to show that the $L^p(\mathbb{R}^2)$ boundedess of $\mathcal{M}_\phi$ implies the $L^p(\mathbb{R})$ boundedess of the maximal operator ν_ϕ.

For $N \in \mathbb{N}$, let

$$(\nu_\phi^N g)(x) = \sup_{-\infty<k\le N} \frac{1}{2^k}\int_{2^k}^{2^{k+1}} |g(x-\phi(t))|\,dt.$$

Then for $f(x,y) = \chi_{[0,2^{N+2}]}(x)g(y)$,

$$\chi_{[0,2^{N+1}]}(x)(\nu_\phi^N g)(y) \le (\mathcal{M}_\phi f)(x,y).$$

Thus

$$2^{(N+1)/p}\|\nu_\phi^N g\|_{L^p(\mathbb{R})} \le \|\mathcal{M}_\phi f\|_{L^p(\mathbb{R}^2)} \le C_p\|f\|_{L^p(\mathbb{R}^2)} = C_p 2^{(N+2)/p}\|g\|_{L^p(\mathbb{R})}.$$

By letting $\mathbb{N}\to\infty$ (after dividing both sides by $2^{(N+1)/p}$), we obtain

$$\|\nu_\phi g\|_{L^p(\mathbb{R})} \le C_p 2^{1/p}\|g\|_{L^p(\mathbb{R})}.$$

This finishes the proof of Lemma 2. □

The following lemma in [7] is one of our main tools.

Lemma 3. *Let $l,\ m \in \mathbb{N}$ and $\{\sigma_{s,k}:\ 0\le s\le l$ and $k\in\mathbb{Z}\}$ be a family of measures on $\mathbb{R}^m$ with $\sigma_{0,k}=0$ for every $k\in\mathbb{Z}$. Let $\{\alpha_{sj}:\ 1\le s\le l$ and $1\le j\le 2\}\subset(0,\infty)$, $\{\eta_s : 1\le s\le l\}\subset(0,\infty)\setminus\{1\}$, $\{M_s:\ 1\le s\le l\}\subset\mathbb{N}$, and $L_s:\ \mathbb{R}^m\to\mathbb{R}^{M_s}$ be linear transformations for $1\le s\le l$. Suppose*

(i) *$\|\sigma_{s,k}\|\le 1$ for $k\in\mathbb{Z}$ and $1\le s\le l$;*
(ii) *$|\widehat{\sigma}_{s,k}(\xi)|\le C(\eta_s^k|L_s\xi|)^{-\alpha_{s2}}$ for $\xi\in\mathbb{R}^m,\ k\in\mathbb{Z}$ and $1\le s\le l$;*
(iii) *$|\widehat{\sigma}_{s,k}-\widehat{\sigma}_{s-1,k}(\xi)|\le C(\eta_s^k|L_s\xi|)^{\alpha_{s1}}$ for $\xi\in\mathbb{R}^m, k\in\mathbb{Z}$ and $1\le s\le l$;*
(iv) *For some $q>1$, there exists $A_q>0$ such that*

$$\left\|\sup_{k\in\mathbb{Z}}\big|\,|\sigma_{s,k}|*f\big|\right\|_{L^q(\mathbb{R}^m)} \le A_q\|f\|_{L^q(\mathbb{R}^m)}$$

for all $f\in L^q(\mathbb{R}^m)$ and $1\le s\le l$.

Then for every $p\in(\frac{2q}{q+1},\ \frac{2q}{q-1})$, there exists a positive constant C_p such that

(a)
$$\left\|\sum_{k\in\mathbb{Z}}\sigma_{l,k}*f\right\|_{L^p(\mathbb{R}^m)} \le C_p\|f\|_{L^p(\mathbb{R}^m)}$$

and

(b)
$$\left\|\left(\sum_{k\in\mathbb{Z}}|\sigma_{l,k}*f|^2\right)^{1/2}\right\|_{L^p(\mathbb{R}^m)} \le C_p\|f\|_{L^p(\mathbb{R}^m)}$$

hold for all $f\in L^p(\mathbb{R}^m)$. The constant C_p is independent of the linear transformations $\{L_s\}_{s=1}^l$.

The following result is just Lemma 5.1 in [7], which follows immediately from Lemma 6.2 in [8] and is an extension of an earlier theorem due to Duoandikoetxea and Rubio de Francia in [6].

Lemma 4. *Let $s,\ m\in\mathbb{N},\ \eta\in(0,\infty)\setminus\{1\},\ \delta_1,\ \delta_2>0$, and $L:\ \mathbb{R}^s\to\mathbb{R}^m$ be a linear transformation. Suppose that $\{\sigma_k\}_{k\in\mathbb{Z}}$ is a sequence of measures on $\mathbb{R}^m$*

satisfying:

(i) $\|\sigma_k\| \le 1$ *for* $k \in \mathbb{Z}$;
(ii) $|\widehat{\sigma_k}(\xi)| \le C[min\{(\eta^k|L\xi|)^{\delta_1},\ (\eta^k|L\xi|)^{-\delta_2}\}]$ *for* $\xi \in \mathbb{R}^s$ *and* $k \in \mathbb{Z}$;
(iii) *For some* $q > 1$, *there exists* $A_q > 0$ *such that*

$$\|\sigma^*(f)\|_{L^q(\mathbb{R}^m)} = \left\|\sup_{k\in\mathbb{Z}} |\,|\sigma_k| * f|\,\right\|_{L^q(\mathbb{R}^m)} \le A_q\|f\|_{L^q(\mathbb{R}^m)}$$

for all $f \in L^q(\mathbb{R}^m)$.

Then for $p \in (\frac{2q}{q+1},\ \frac{2q}{q-1})$, *there exists a positive constant* $C_p = C(p, s, m, \eta, \delta_1, \delta_2)$ *such that*

(a) $$\left\|\sum_{k\in\mathbb{Z}} \sigma_k * f\right\|_{L^p(\mathbb{R}^m)} \le C_p\|f\|_{L^p(\mathbb{R}^m)}$$

and

(b) $$\left\|\left(\sum_{k\in\mathbb{Z}} |\sigma_k * f|^2\right)^{1/2}\right\|_{L^p(\mathbb{R}^m)} \le C_p\|f\|_{L^p(\mathbb{R}^m)}$$

hold for all $f \in L^p(\mathbb{R}^m)$. *The constant* C_p *is independent of the linear transformation* L.

In order to handle truncation in the phase space, we need the following useful lemma, which is Lemma 6.4 in [8].

Lemma 5. *For* $s \le d$, *let* $H:\ \mathbb{R}^s \to \mathbb{R}^s$ *and* $G: \mathbb{R}^d \to \mathbb{R}^d$ *be two nonsingular linear transformations and* $\varphi \in \mathcal{S}(\mathbb{R}^s)$. *Define* J *and* $X_r = X_r(\varphi, G, H)$ *by*

$$(Jf)(x) = f(G^t(H^t \otimes id_{\mathbb{R}^{d-s}}))(x)$$

and

$$X_r f(x) = J^{-1}((|\Phi_r| \otimes \delta_{\mathbb{R}^{d-s}}) * Jf)(x),$$

where $x \in \mathbb{R}^d,\ r > 0$, G^t *and* H^t *are respectively the transposes of* G *and* H, $id_{\mathbb{R}^{d-s}}$ *is the identity operator on* $\mathbb{R}^{d-s}$, $\delta_{\mathbb{R}^{d-s}}$ *is the Dirac delta operator on* $\mathbb{R}^{d-s}$, *and* $\Phi \in \mathcal{S}(\mathbb{R}^s)$ *satisfies* $\widehat{\Phi} = \varphi$. *Let* $X = X(\varphi, G, H)$ *be given by*

$$Xf(x) = \sup_{r>0} |X_r f(x)|.$$

Then for $1 < p \le \infty$, *there exists a positive constant* $C_p = C(p, \varphi, s, d)$ *such that*

$$\|Xf\|_{L^p(\mathbb{R}^d)} \le C_p\|f\|_{L^p(\mathbb{R}^d)}$$

for all $f \in L^p(\mathbb{R}^d)$. *The constant* C_p *is independent of the linear transformations* G *and* H.

Now let $\Delta_\gamma(0,\infty)$ denote the set of functions b on $(0,\infty)$ satisfying

$$\sup_{R>0} \frac{1}{R}\int_0^R |b(t)|^\gamma\, dt < \infty.$$

For $y = (y_1, \cdots, y_n) \in \mathbb{R}^n$, let $\widetilde{y} = (y_1, \cdots, y_{n-1}) \in \mathbb{R}^{n-1}$. We denote the north pole $(0, \cdots, 0, 1)$ on S^{n-1} by ρ_1. Let $F:\ \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ be of the form

(4) $$F(t, y) = t^l q(\widetilde{y}) + W_1(t, y) + W_2(t),$$

where $q:\ \mathbb{R}^{n-1}\to\mathbb{R}$ is a polynomial, W_1 satisfies

$$\frac{\partial^l W_1}{\partial t^l}(t,y)\equiv 0, \tag{5}$$

and $W_2(\cdot)$ is an arbitrary function.

The following estimate on the oscillatory integrals is Proposition 5.3 in [8] and is of considerable importance to us.

Lemma 6. *Let $\rho\in(0,1/4)$, $l\in\mathbb{N}$, $m\geq 0$, $q(\widetilde{y})=\sum_{j=0}^m q_j(\widetilde{y})$, where $q_j(\cdot)$ is a homogeneous polynomial of degree j on $\mathbb{R}^{n-1}$ for $0\leq j\leq m$. Let $F(t,y)$ be given by (4) and (5). Suppose that $b\in\Delta_\gamma$ for some $\gamma>1$ and $\Omega(\cdot)$ is a function satisfying*

(a) $\operatorname{supp}(\Omega)\subset B(\rho_1,\rho)$;
(b) $\|\Omega\|_{L^\infty(S^{n-1})}\leq\rho^{-n+1}$.

If we assume $q_m(\widetilde{y})=\sum_{|\beta|=m}\alpha_\beta\widetilde{y}^\beta$ and $\|q_m\|=\sum_{|\beta|=m}|\alpha_\beta|$, then there exists a positive constant C such that

$$\int_{2^k}^{2^{k+1}}\left|\int_{S^{n-1}}e^{iF(t,y)}\Omega(y)\,d\sigma(y)\right|\frac{|b(t)|}{t}\,dt\leq C\left(2^{kl}\rho^m\|q_m\|\right)^{-\frac{1}{4ml\gamma}}.$$

The constant C may depend on l, m, n, and $b(\cdot)$, but it is independent of k, ρ, $W_1(\cdot,\cdot)$, $W_2(\cdot)$, and the coefficients of $q(\cdot)$.

Proof of Theorem 1. Since $\Omega\in H^1(S^{n-1})$ and $\int_{S^{n-1}}\Omega(y)\,d\sigma(y)=0$, there are regular atoms $a_j(\cdot)$ and $\{C_j\}\subset\mathbb{C}$ such that

$$\Omega(y)=\sum_j C_j a_j(y)$$

by Lemma 1.

Therefore, we only need to be concerned with the case where $\Omega(y)$ is a regular atom on S^{n-1}. By Lemma 2 and using a rotation if necessary, we may assume that there is a $\rho\in(0,1/4)$ such that

$$\operatorname{supp}(\Omega)\subset B(\rho_1,\rho),\ \text{where } \rho_1=(0,\cdots,0,1);$$

$$\|\Omega\|_{L^\infty(S^{n-1})}\leq\rho^{-(n-1)};\ \int_{S^{n-1}}\Omega(y)\,d\sigma(y)=0.$$

For any integrable function $a(\cdot)$ on S^{n-1} and a suitable mapping $\Gamma:\ \mathbb{R}^n\to\mathbb{R}^{n+1}$, we define the sequence of measures $\{\sigma_{a,\Gamma,k}\}_{k\in\mathbb{Z}}$ by

$$\int_{\mathbb{R}^{n+1}}F\,d\sigma_{a,\Gamma,k}=\int_{\{y\in\mathbb{R}^n:2^k\leq|y|<2^{k+1}\}}F(\Gamma(y))\frac{a(y)}{|y|^n}b(|y|)\,dy.$$

For $y\in\mathbb{R}^n\setminus\{0\}$, let $\widetilde{y}=(y_1/|y|,\cdots,y_{n-1}/|y|)$. Let $N=[\frac{3(n-1)}{2}]+2$ (this N is chosen so that we can have both (9) and (10) for $j=N$). For $j=1,\cdots,N-2$, let $b_j=(-1)^j\frac{1}{2}(\frac{1}{2}-1)\cdots(\frac{1}{2}-j+1)/j!$. Thus

$$|(1-t)^{1/2}-1-\sum_{l=1}^{j-1}b_lt^l|\leq C_jt^j$$

for $t\in[0,1/4]$.

We now define the mappings $\Gamma_0,\ \Gamma_1,\ \cdots,\ \Gamma_N$ by

$$\Gamma_N(y) = (y,\ \phi(|y|)),$$
$$\Gamma_j(y) = (|y|\widetilde{y},\ |y|(1 + b_1|\widetilde{y}|^2 + \cdots + b_{j-1}|\widetilde{y}|^{2(j-1)}),\ \phi(|y|)),\ j = 2, \cdots, N-1,$$
$$\Gamma_1(y) = (|y|\widetilde{y},\ |y|,\ \phi(|y|)),$$

and

$$\Gamma_0(y) = (0,\ |y|,\ \phi(|y|)).$$

For $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$ and $\eta \in \mathbb{R}$, we shall establish estimates (ii) and (iii) in Lemma 3 for $\{|\widehat{\sigma}_{\Omega,\Gamma_j,k}(\xi,\eta)| :\ 1 \le j \le N \text{ and } k \in \mathbb{Z}\}$. By an inequality on page 551 of [6], we have

$$(6)\qquad \begin{aligned} |\widehat{\sigma}_{\Omega,\Gamma_N,k}(\xi,\eta)| &\le \int_{2^k}^{2^{k+1}} \left| \int_{S^{n-1}} e^{-i[t\xi\cdot y + \eta\phi(t)]}\Omega(y)\,d\sigma(y) \right| |b(t)| \frac{dt}{t} \\ &\le C[2^k|\xi|]^{-1/6}\|\Omega\|_{L^2(S^{n-1})} \\ &\le C[2^k|\rho^{3(n-1)}\xi_n|]^{-1/6}. \end{aligned}$$

One observes that the variable η does not appear in the previous inequality. The same is true for the Fourier estimates obtained from here on.

Now, for $2 \le j \le N-1$, we have

$$\left|\widehat{\sigma}_{\Omega,\Gamma_j,k}(\xi,\eta)\right| \le \int_{2^k}^{2^{k+1}} \left| \int_{S^{n-1}} e^{-it[(\xi_1,\cdots,\xi_{n-1})\cdot\widetilde{y} + \xi_n \sum_{s=1}^{j-1} b_s|\widetilde{y}|^{2s}]}\Omega(y)\,d\sigma(y) \right| |b(t)| \frac{dt}{t}.$$

By applying Lemma 6 with $q(\widetilde{y}) = -[(\xi_1, \cdots, \xi_{n-1}) \cdot \widetilde{y} + \xi_n \sum_{s=1}^{j-1} b_s|\widetilde{y}|^{2s}],\ m = 2(j-1),\ \gamma = 2$ and $l = 1$, we obtain

$$(7)\qquad \left|\widehat{\sigma}_{\Omega,\Gamma_j,k}(\xi,\eta)\right| \le C\left[2^k \left|\rho^{2(j-1)}\xi_n\right|\right]^{-\frac{1}{16(j-1)}}.$$

Finally, by Lemma 6 with $m = 1,\ \gamma = 2$ and $l = 1$, we have

$$(8)\qquad \begin{aligned} |\widehat{\sigma}_{\Omega,\Gamma_1,k}(\xi,\eta)| &\le \int_{2^k}^{2^{k+1}} \left| \int_{S^{n-1}} e^{-it[(\xi_1,\cdots,\xi_{n-1})\cdot\widetilde{y}]}\Omega(y)\,d\sigma(y) \right| |b(t)| \frac{dt}{t} \\ &\le C\left[2^k\left|\rho(\xi_1, \cdots, \xi_{n-1})\right|\right]^{-\frac{1}{8}}. \end{aligned}$$

Let

$$L_1(\xi,\eta) = \rho(\xi_1, \cdots, \xi_{n-1}),\ \theta_1 = \frac{1}{8};$$
$$L_j(\xi,\eta) = \rho^{2(j-1)}\xi_n,\ \theta_j = \frac{1}{16(j-1)},\ 2 \le j \le N-1;$$
$$L_N(\xi,\eta) = \rho^{3(n-1)}\xi_n,\ \theta_N = \frac{1}{6}.$$

Then by (6)–(8), we have

$$(9)\qquad \left|\widehat{\sigma}_{\Omega,\Gamma_j,k}(\xi,\eta)\right| \le C[2^k|L_j(\xi,\eta)|]^{-\theta_j}$$

for $1 \le j \le N,\ k \in \mathbb{Z},\ \xi \in \mathbb{R}^n$ and $\eta \in \mathbb{R}$. Next we shall verify that for $(\xi,\eta) \in \mathbb{R}^{n+1},\ k \in \mathbb{Z}$ and $1 \le j \le N$,

$$(10)\qquad \left|\widehat{\sigma}_{\Omega,\Gamma_j,k}(\xi,\eta) - \widehat{\sigma}_{\Omega,\Gamma_{j-1},k}(\xi,\eta)\right| \le C2^k|L_j(\xi,\eta)|.$$

Let us begin with $j = N$. In this case, we have

$$\begin{aligned}
&\left|\widehat{\sigma}_{\Omega,\Gamma_N,k}(\xi,\eta) - \widehat{\sigma}_{\Omega,\Gamma_{N-1},k}(\xi,\eta)\right| \\
&\quad \le \int_{2^k}^{2^{k+1}} \left| \int_{S^{n-1}} \left(e^{-it\xi_n[(1-|\widetilde{y}|^2)^{1/2}-1-\sum_{s=1}^{N-2} b_s|\widetilde{y}|^{2s}]} - 1 \right) \Omega(y)\, d\sigma(y) \right| |b(t)| \frac{dt}{t} \\
&\quad \le C \int_{2^k}^{2^{k+1}} t|\xi_n|\rho^{2(N-1)} \|\Omega\|_{L^1(S^{n-1})} \frac{dt}{t} \\
&\quad \le C2^k|\rho^{3(n-1)}\xi_n| = C|2^k L_N(\xi,\eta)|.
\end{aligned}$$

For $2 \le j \le N-1$, we have

$$\begin{aligned}
&\left|\widehat{\sigma}_{\Omega,\Gamma_j,k}(\xi,\eta) - \widehat{\sigma}_{\Omega,\Gamma_{j-1},k}(\xi,\eta)\right| \\
&\quad \le \int_{2^k}^{2^{k+1}} \int_{S^{n-1}} \left| e^{-it\xi_n b_{j-1}|\widetilde{y}|^{2(j-1)}} - 1 \right| |\Omega(y)|\, d\sigma(y) |b(t)| \frac{dt}{t} \\
&\quad \le C2^k|\rho^{2(j-1)}\xi_n| = C|2^k L_j(\xi,\eta)|.
\end{aligned}$$

Finally, for $j = 1$, we have

$$\begin{aligned}
&\left|\widehat{\sigma}_{\Omega,\Gamma_1,k}(\xi,\eta) - \widehat{\sigma}_{\Omega,\Gamma_0,k}(\xi,\eta)\right| \\
&\quad \le \int_{2^k}^{2^{k+1}} \int_{S^{n-1}} \left| e^{-it(\xi_1,\cdots,\xi_{n-1})\cdot\widetilde{y}} - 1 \right| |\Omega(y)|\ d\sigma(y) |b(t)| \frac{dt}{t} \\
&\quad \le C2^k\rho|(\xi_1,\cdots,\xi_{n-1})| = C|2^k L_1(\xi,\eta)|.
\end{aligned}$$

This completes the proof of (10). □

We still need to verify condition (iv) in Lemma 3. It suffices to establish the $L^p(\mathbb{R}^n)$ boundedness of the operators $\sigma^*_{\Omega,j}$ defined by

$$\sigma^*_{\Omega,j}(f)(x,s) = \sup_{k\in\mathbb{Z}} |(\sigma_{|\Omega|,\Gamma_j,k} * f)(x,s)|,$$

where $j = 1,\cdots,N,\ x \in \mathbb{R}^n,\ s \in \mathbb{R}$ and $1 < p < \infty$.

Let us begin with $\sigma^*_{\Omega,1}$ which is given by

$$\sigma^*_{\Omega,1}(f)(x,s) = \sup_{k\in\mathbb{Z}} |(\sigma_{|\Omega|,\Gamma_1,k} * f)(x,s)|.$$

Choose $\theta \in C_0^\infty(\mathbb{R}^{n-1})$ such that $\theta(t) \equiv 1$ for $|t| \le 1/2$ and $\theta(t) \equiv 0$ for $|t| \ge 1$. For $k \in \mathbb{Z}$, we define ν_k by

$$\widehat{\nu}_k(\xi,\eta) = \theta(2^k\rho(\xi_1,\cdots,\xi_{n-1}))\widehat{\sigma}_{|\Omega|,\Gamma_0,k}(\xi,\eta)$$

for $\xi \in \mathbb{R}^n$ and $\eta \in \mathbb{R}$. Let $\tau_k = \widehat{\sigma}_{|\Omega|,\Gamma_1,k} - \nu_k$. Then by (10) and $|\widehat{\sigma}_{|\Omega|,\Gamma_0,k}(\xi,\eta)| \le C$, we have

$$\begin{aligned}
|\widehat{\tau}_k(\xi,\eta)| &\le |\widehat{\sigma}_{|\Omega|,\Gamma_1,k}(\xi,\eta) - \widehat{\sigma}_{|\Omega|,\Gamma_0,k}(\xi,\eta)| \\
&\quad + |1 - \theta(2^k\rho(\xi_1,\cdots,\xi_{n-1}))||\widehat{\sigma}_{|\Omega|,\Gamma_0,k}(\xi,\eta)| \\
&\le C\left[|2^k L_1(\xi,\eta)| + |2^k\rho(\xi_1,\cdots,\xi_{n-1})|\right] \\
&= C2^k|L_1(\xi,\eta)|.
\end{aligned}$$

If $2^k|L_1(\xi,\eta)| > 1$, by (9), we have

$$|\widehat{\tau}_k(\xi,\eta)| \le C(2^k|L_1(\xi,\eta)|)^{-1/8}.$$

Thus,

$$|\widehat{\tau}_k(\xi,\eta)| \le C\left[min\{2^k|L_1(\xi,\eta)|,(2^k|L_1(\xi,\eta)|)^{-1}\}\right]^{1/8}. \tag{11}$$

Let

$$\tau^*(f)(x,s)=\sup_{k\in\mathbb{Z}}|(|\tau_k|*f)(x,s)|,\ \ \nu^*(f)(x,s)=\sup_{k\in\mathbb{Z}}|(|\nu_k|*f)(x,s)|$$

and

$$g_\tau(f)(x,s)=\left\{\sum_{k\in\mathbb{Z}}|(\tau_k*f)(x,s)|^2\right\}^{1/2}.$$

Then

$$\sigma^*_{|\Omega|,1}(f)(x,s)\le g_\tau(f)(x,s)+\nu^*(f)(x,s) \tag{12}$$

and

$$\begin{aligned}\tau^*(f)(x,s)&\le \sigma^*_{|\Omega|,1}(|f|)(x,s)+\nu^*(|f|)(x,s)\\&\le g_\tau(|f|)(x,s)+2\nu^*(|f|)(x,s).\end{aligned} \tag{13}$$

By the $L^p(\mathbb{R}^2)$ boundedness of $\mathcal{M}_\phi$ and Lemma 5, for $1<p<\infty$, we have

$$\|\nu^*(|f|)\|_{L^p(\mathbb{R}^{n+1})}\le C_p\|f\|_{L^p(\mathbb{R}^{n+1})}.$$

Also, from (11), it is easy to deduce that

$$\|g_\tau(f)\|_{L^2(\mathbb{R}^{n+1})}\le C\|f\|_{L^2(\mathbb{R}^{n+1})}.$$

Thus, (13) implies that

$$\|\tau^*(f)\|_{L^2(\mathbb{R}^{n+1})}\le C\|f\|_{L^2(\mathbb{R}^{n+1})}.$$

By invoking Lemma 4, we obtain

$$\|g_\tau(f)\|_{L^p(\mathbb{R}^{n+1})}\le C_p\|f\|_{L^p(\mathbb{R}^{n+1})}$$

for $4/3<p<4$. Thus, by (13) again, we obtain

$$\|\tau^*(f)\|_{L^p(\mathbb{R}^{n+1})}\le C_p\|f\|_{L^p(\mathbb{R}^{n+1})} \tag{14}$$

for $4/3<p<4$. By using (14), (13) and repeating the preceding argument, we obtain

$$\|g_\tau(f)\|_{L^p(\mathbb{R}^{n+1})}\le C_p\|f\|_{L^p(\mathbb{R}^{n+1})}$$

for $1<p<\infty$. Now, from (12), it follows that

$$\|\sigma^*_{|\Omega|,1}(f)\|_{L^p(\mathbb{R}^{n+1})}\le C_p\|f\|_{L^p(\mathbb{R}^{n+1})}$$

for $1<p<\infty$.

Similarly, we can show that

$$\|\sigma^*_{|\Omega|,j}(f)\|_{L^p(\mathbb{R}^{n+1})}\le C_p\|f\|_{L^p(\mathbb{R}^{n+1})} \tag{15}$$

for $1\le j\le N$. Now, by (9), (10), (15) and Lemma 3, we have

$$\left\|\sum_{k\in\mathbb{Z}}\sigma_{\Omega,\Gamma_N,k}*f\right\|_{L^p(\mathbb{R}^{n+1})}\le C_p\|f\|_{L^p(\mathbb{R}^{n+1})}$$

for $1 < p < \infty$. Noting that

$$\sum_{k\in\mathbb{Z}} (\sigma_{\Omega,\Gamma_N,k} * f)(x,s) = \int_{\mathbb{R}^n} f(x-y, s-\phi(|y|)) \frac{\Omega(y)}{|y|^n} b(|y|)\, dy,$$

we thus obtain a proof of our theorem.

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