

**A NUMERICAL CONDITION FOR A DEFORMATION
OF A GORENSTEIN SURFACE SINGULARITY TO ADMIT
A SIMULTANEOUS LOG-CANONICAL MODEL**

TOMOHIRO OKUMA

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ABSTRACT. Let $\pi: X \rightarrow T$ be a deformation of a normal Gorenstein surface singularity over the complex number field \mathbb{C} . We assume that T is a neighborhood of the origin of \mathbb{C} . Then we prove that π admits a simultaneous log-canonical model if and only if an invariant $-P_t \cdot P_t$ of each fiber X_t is constant.

1. INTRODUCTION

Let $\pi: X \rightarrow T$ be a deformation of a normal Gorenstein surface singularity over the complex number field \mathbb{C} . We assume that T is a sufficiently small neighborhood of the origin of \mathbb{C} . In his paper [8], Laufer proved that the deformation admits a simultaneous canonical model if and only if $-K_t \cdot K_t$ is constant. In this paper, we prove the log-version of Laufer's results. Instead of $-K \cdot K$, we adopt an invariant $-P \cdot P$. It is a numerical invariant of a normal surface singularity, and its fundamental properties are stated in [14]. For example, the equality below is proved in [14, Introduction]:

$$(1.1) \quad -P \cdot P/2 = \limsup_{m \rightarrow \infty} \delta_m/m^2.$$

In this equality, δ_m denotes the m -th L^2 -plurigenus of the singularity. Since $\delta_m(X_t)$ is upper semicontinuous for any m by [3], it follows from the equality (1.1) that $-P \cdot P$ is upper semicontinuous. Our main theorem is the following:

Theorem 1.1. *The following conditions are equivalent:*

- (1) π admits a simultaneous log-canonical model;
- (2) $-P_t \cdot P_t$ is constant.

The implication (1) \Rightarrow (2) follows from the invariance of the log-plurigenus $\lambda_m(X_t)$ with $m \gg 0$ and an equality similar to (1.1). Let $f: Y \rightarrow X$ be a log-canonical model of X with the exceptional divisor E . Then $f_t: Y_t \rightarrow X_t$ is a log-canonical model for $t \in T \setminus \{0\}$ near 0 (Lemma 3.1). We define sheaves \mathcal{I}_m and \mathcal{A}_m by $\mathcal{I}_m = f_*\mathcal{O}_Y(m(K_Y + E))$ and $\mathcal{A}_m = \psi_*\mathcal{O}_M(m(K_M + A))$, where

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$\psi: (M, A) \rightarrow (X_0, x)$ is a good resolution. Since $Y_0 = \text{Proj} \bigoplus_{m \geq 0} \mathcal{I}_m \otimes \mathbb{C}(0)$, the implication (2) \Rightarrow (1) follows from the claim that if $-P_t \cdot P_t$ is constant, then $\mathcal{I}_m \otimes \mathbb{C}(0) = \mathcal{A}_m$. To show the claim, we will use Izumi's results on the analytic orders [6] (precisely, Ishii's version [4]) and formulas for the plurigenera, and prove the torsion freeness of the sheaf $\mathcal{O}_X(mK_X)/\mathcal{I}_m$ (Lemma 4.8). We will also prove that if a morphism $f: Y \rightarrow X$ is a simultaneous log-canonical model, then it is a log-canonical model of X (Lemma 4.2). Our methods will be used in higher dimensions (Remark 4.12).

We denote by \mathbb{N} , \mathbb{Q} and \mathbb{R} the set of positive integers, the set of rational numbers and the set of real numbers, respectively.

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2. PRELIMINARIES

2.1. Log-canonical models. Let X be a normal variety over \mathbb{C} of dimension $d \geq 2$. Let $D = \sum d_i D_i$ be a \mathbb{Q} -divisor on X , where the D_i are distinct prime divisors and $d_i \in \mathbb{Q}$. We put $D_{red} = \sum_{d_i \neq 0} D_i$. Let $f: Y \rightarrow X$ be a birational morphism of normal varieties and E the maximal reduced exceptional divisor on Y . For a divisor D on X , we denote by $f_*^{-1}(D)$ the strict transform of D under the morphism f . The morphism $f: Y \rightarrow X$ is called a good resolution of a pair (X, D) , if Y is nonsingular and $(f_*^{-1}(D) + E)_{red}$ is a divisor with only simple normal crossings. A \mathbb{Q} -divisor B on X is called a boundary if it satisfies $0 \leq B \leq B_{red}$.

Definition 2.1. Let B be a boundary on X . A divisor $K_X + B$ is said to be log-canonical if the following conditions are satisfied:

- (1) $K_X + B$ is a \mathbb{Q} -Cartier divisor.
- (2) There exists a good resolution $f: Y \rightarrow X$ of (X, B) such that

$$K_Y + f_*^{-1}(B) = f^*(K_X + B) + \sum a_i E_i$$

for $a_i \in \mathbb{Q}$ with the condition that $a_i \geq -1$, where the E_i vary all the exceptional prime divisors on Y .

If $K_X + B$ is log-canonical, then the equality in condition (2) above is satisfied with $a_i \geq -1$ for any resolution of X (see [7, Lemma 0-2-12]). Hence we have the following:

Lemma 2.2. *Let $f: Y \rightarrow X$ be any good resolution of a pair (X, B) with the exceptional divisor E , and let $B_Y = f_*^{-1}(B) + E$. If $K_X + B$ is log-canonical, then we have $f_* \mathcal{O}_Y(m(K_Y + B_Y)) = \mathcal{O}_X(m(K_X + B))$ for any $m \in \mathbb{N}$.*

Definition 2.3. Let $f: Y \rightarrow X$ be a birational morphism with the exceptional divisor E . Then the morphism $f: Y \rightarrow X$ is called a log-canonical model of X , if the divisor $K_Y + E$ is log-canonical and f -ample.

Theorem 2.4 (see [1, 6.16]). *Let X be a normal variety of dimension $d \leq 3$. Then there exists a unique log-canonical model $f: Y \rightarrow X$ of X . In fact, the following morphism is the log-canonical model:*

$$\text{Proj} \bigoplus_{n \geq 0} h_* \mathcal{O}_Z(n(K_Z + F)) \rightarrow X,$$

where $h: Z \rightarrow X$ is any proper birational morphism with the exceptional divisor F such that the divisor $K_Z + F$ is log-canonical.

2.2. Plurigenera. Let (X, x) be a normal isolated singularity and $f: (M, A) \rightarrow (X, x)$ a good resolution of the singularity (X, x) , where A is the exceptional divisor.

Definition 2.5 (cf. [9], [15]). We define the log-plurigenera $\{\lambda_m(X, x)\}_{m \in \mathbb{N}}$ and the L^2 -plurigenera $\{\delta_m(X, x)\}_{m \in \mathbb{N}}$ by

$$\begin{aligned} \lambda_m(X, x) &= \dim_{\mathbb{C}} \mathcal{O}_X(mK_X)/f_*\mathcal{O}_M(m(K_M + A)), \\ \delta_m(X, x) &= \dim_{\mathbb{C}} \mathcal{O}_X(mK_X)/f_*\mathcal{O}_M(m(K_M + A) - A). \end{aligned}$$

These definitions are independent of the choice of a good resolution.

Proposition 2.6. *Let $g: Y \rightarrow X$ be a partial resolution with the exceptional divisor E such that $K_Y + E$ is log-canonical. Then we have*

$$\lambda_m(X, x) = \dim_{\mathbb{C}} \mathcal{O}_X(mK_X)/g_*\mathcal{O}_Y(m(K_Y + E)).$$

Proof. It follows from Lemma 2.2. □

Let (X, x) be a normal surface singularity and $f: (M, A) \rightarrow (X, x)$ the minimal good resolution. Let K be a canonical divisor on M . Let $A = \bigcup_{i=1}^k A_i$ be the decomposition of A into irreducible components. By [12, Theorem A.1], $K + A$ admits a unique Zariski decomposition $P + N$, where P and N are elements of $\sum_{i=1}^k \mathbb{Q}A_i$ satisfying the following:

- (1) $(K + A) \cdot A_i = (P + N) \cdot A_i$ for all i ;
- (2) P is f -nef and N is effective;
- (3) $P \cdot N = 0$.

Theorem 2.7 (see [14], [11]). *For every $m \in \mathbb{N}$, we have*

$$\begin{aligned} \lambda_m(X, x) &= -(P \cdot P)m^2/2 + (P \cdot K)m/2 + b_1(m), \\ \delta_m(X, x) &= -(P \cdot P)(m - 1)^2/2 - (P \cdot K)(m - 1)/2 + b_2(m), \end{aligned}$$

where $b_1(m)$ and $b_2(m)$ are bounded functions of m .

3. DEFORMATIONS AND SOME INVARIANTS

Let $\pi: X \rightarrow T$ be a deformation of a normal Gorenstein surface singularity $(X_0, x) = \pi^{-1}(0)$, where T is a neighborhood of the origin of \mathbb{C} . Then X is a Gorenstein variety. Therefore, for any $t \in T$, we have $\mathcal{O}_{X_t}(mK_X) \cong \mathcal{O}_{X_t}(mK_{X_t})$. For any morphism $h: W \rightarrow X$, we denote by W_t the fiber $(\pi \circ h)^{-1}(t)$ and set $h_t = h|_{W_t}$. Let $f: Y \rightarrow X$ be the log-canonical model of X with the maximal reduced exceptional divisor E on Y . Then $\mathcal{O}_Y(m(K_Y + E))$ is a Cohen-Macaulay \mathcal{O}_Y -module for $m \in \mathbb{N}$. We define sheaves \mathcal{I}_m and \mathcal{Q}_m by $\mathcal{I}_m = f_*\mathcal{O}_Y(m(K_Y + E))$ and $\mathcal{Q}_m = \mathcal{O}_X(mK_X)/\mathcal{I}_m$ for every $m \in \mathbb{N}$. Put $T^* = T \setminus \{0\}$. We assume that T is small enough to satisfy the following conditions:

- (1) there exists a good resolution $g: M \rightarrow Y$ of (Y, E) which induces a good resolution $M_t \rightarrow X_t$ for any $t \in T^*$;
- (2) Y_t is a normal surface with the exceptional divisor $E_t = E \cap Y_t$ for any $t \in T^*$.

Lemma 3.1. *For any $t \in T^*$, the restriction $f_t: Y_t \rightarrow X_t$ is the log-canonical model of X_t .*

Proof. Let t be a point of T^* . Since the normal sheaf of Y_t in Y is trivial and $\mathcal{O}_Y(m(K_Y + E))$ is Cohen-Macaulay, we obtain that

$$\mathcal{O}_{Y_t}(m(K_Y + E)) \cong \mathcal{O}_{Y_t}(m(K_{Y_t} + E_t))$$

for every $m \in \mathbb{N}$. Hence the f_t -ampleness of the divisor $K_{Y_t} + E_t$ follows from the f -ampleness of $K_Y + E$. Let $g: M \rightarrow Y$ be a resolution as in (1) above. Let F be the exceptional divisor of g . Then we obtain that

$$K_M + g_*^{-1}(E) + F = g^*(K_Y + E) + \Delta,$$

where Δ is an effective divisor supported in F . Restricting those divisors to M_t , we see that $K_{Y_t} + E_t$ is log-canonical. \square

Let W be a normal variety whose singular locus W_{sing} is a finite set. Then we put

$$\lambda_m(W) = \sum_{w \in W_{sing}} \lambda_m(W, w) \quad \text{and} \quad \delta_m(W) = \sum_{w \in W_{sing}} \delta_m(W, w).$$

Let $\psi(t): M_t \rightarrow X_t$ be the minimal good resolution of the singularities and K_t the canonical divisor on M_t . Let $A_{t,p}$ be a connected component of the exceptional set A_t on M_t which blows down to $p \in (X_t)_{sing}$. Let $P_{t,p} + N_{t,p}$ be the Zariski decomposition of $K_t + A_{t,p}$, where $P_{t,p}$ and $N_{t,p}$ are \mathbb{Q} -divisors supported in $A_{t,p}$. We define a \mathbb{Q} -divisor P_t on M_t by $P_t = \sum_{p \in (X_t)_{sing}} P_{t,p}$, and regard $-P_t \cdot P_t$ as a function of t .

Proposition 3.2. *For any $m \in \mathbb{N}$, we obtain that*

$$\begin{aligned} \lambda_m(X_t) &= -(P_t \cdot P_t)m^2/2 + (P_t \cdot K_t)m/2 + b_t(m), \\ \delta_m(X_t) &= -(P_t \cdot P_t)(m - 1)^2/2 - (P_t \cdot K_t)(m - 1)/2 + b'_t(m), \end{aligned}$$

where b_t and b'_t are bounded functions of m . Furthermore, $-P_t \cdot P_t$ is upper semi-continuous.

Proof. It follows from Theorem 2.7 and Introduction. \square

Let $\mathbb{C}(t)$ be the residue field of $t \in T$, i.e., $\mathbb{C}(t) = \mathcal{O}_{T,t}/\mathcal{M}_t$, where \mathcal{M}_t is the maximal ideal. We write $\otimes \mathbb{C}(t)$ instead of $\otimes_{\mathcal{O}_T} \mathbb{C}(t)$. By Nakayama's lemma, we obtain the inequality

$$(3.1) \quad \dim_{\mathbb{C}} \mathcal{Q}_m \otimes \mathbb{C}(t) \leq \dim_{\mathbb{C}} \mathcal{Q}_m \otimes \mathbb{C}(0).$$

Let $\mathcal{I}_{m,0}$ be the image of the natural homomorphism $\mathcal{I}_m \otimes \mathbb{C}(0) \rightarrow \mathcal{O}_{X_0}(mK_{X_0})$. Then $\mathcal{Q}_m \otimes \mathbb{C}(0) \cong \mathcal{O}_{X_0}(mK_{X_0})/\mathcal{I}_{m,0}$.

Lemma 3.3. *The following conditions are equivalent:*

- (1) $\dim_{\mathbb{C}} \mathcal{Q}_m \otimes \mathbb{C}(t) = \dim_{\mathbb{C}} \mathcal{Q}_m \otimes \mathbb{C}(0)$ for $t \in T^*$;
- (2) \mathcal{Q}_m is a torsion free \mathcal{O}_T -module;
- (3) $\mathcal{I}_m \otimes \mathbb{C}(0) = \mathcal{I}_{m,0}$.

Lemma 3.4. *There exists a closed analytic subset $S \subset T$ such that $\lambda_m(X_t) = \dim_{\mathbb{C}} \mathcal{Q}_m \otimes \mathbb{C}(t)$ for any $t \in T \setminus S$ and any $m \in \mathbb{N}$.*

Proof. There exists an integer m_0 such that $R^1 f_* \mathcal{O}_Y(m(K_Y + E)) = 0$ for $m > m_0$. Let S be a closed analytic subset of T such that the \mathcal{O}_T -coherent sheaf

$R^1 f_* \mathcal{O}_Y(m(K_Y + E))$ is torsion free outside S for $m \leq m_0$. Then, for any $t \in T \setminus S$, we have the exact sequence

$$\begin{aligned} 0 \rightarrow f_* \mathcal{O}_Y(m(K_Y + E)) &\xrightarrow{\times \tau} f_* \mathcal{O}_Y(m(K_Y + E)) \rightarrow f_* \mathcal{O}_{Y_t}(m(K_{Y_t} + E_t)) \\ &\rightarrow R^1 f_* \mathcal{O}_Y(m(K_Y + E)) \xrightarrow{\times \tau} R^1 f_* \mathcal{O}_Y(m(K_Y + E)), \end{aligned}$$

where τ is a local parameter at t . Since the last arrow is injective, it follows that $f_* \mathcal{O}_{Y_t}(m(K_{Y_t} + E_t))$ is the image of $f_* \mathcal{O}_Y(m(K_Y + E))$. Hence we obtain that

$$\mathcal{Q}_m \otimes \mathbb{C}(t) = \mathcal{O}_{X_t}(mK_{X_t})/f_{t*} \mathcal{O}_{Y_t}(m(K_{Y_t} + E_t)).$$

Now the assertion follows from Proposition 2.6 and Lemma 3.1. □

Let $\psi: (M, A) \rightarrow (X_0, x)$ be a good resolution. For each $m \in \mathbb{N}$, we put $\mathcal{A}_m = \psi_* \mathcal{O}_M(m(K_M + A))$ and define invariants ϵ_m and θ_m by

$$\begin{aligned} \epsilon_m &= \dim_{\mathbb{C}} \mathcal{A}_m / (\mathcal{I}_{m,0} \cap \mathcal{A}_m), \quad \text{and} \\ \theta_m &= \dim_{\mathbb{C}} \mathcal{I}_{m,0} / (\mathcal{A}_m \cap \mathcal{I}_{m,0}), \quad \text{respectively.} \end{aligned}$$

Using Proposition 3.4 and the equality (3.1), we obtain the inequality

$$(3.2) \quad \lambda_m(X_t) \leq \lambda_m(X_0) + \epsilon_m - \theta_m.$$

Lemma 3.5. *There exist $a, b \in \mathbb{Q}$ such that $\epsilon_m \leq am + b$.*

Proof. Let ω be a section of $\psi_* \mathcal{O}_M(m(K_M + A) - A)$. By [3, Theorem 2.1], there exists a section ω' of $f_* \mathcal{O}_Y(m(K_Y + E) - E)$ such that the image of ω' in $\mathcal{O}_{X_0}(mK_{X_0})$ is ω . Since $f_* \mathcal{O}_Y(m(K_Y + E) - E) \subset \mathcal{I}_m$, we see that ω belongs to $\mathcal{I}_{m,0}$. Hence we obtain that $\psi_* \mathcal{O}_M(mK_M + (m - 1)A) \subset \mathcal{I}_{m,0} \cap \mathcal{A}_m$. This implies that

$$\epsilon_m \leq \dim_{\mathbb{C}} \mathcal{A}_m / \psi_* \mathcal{O}_M(mK_M + (m - 1)A) = \delta_m(X_0, x) - \lambda_m(X_0, x).$$

From Theorem 2.7, we obtain the assertion. □

4. THE MAIN THEOREM

In this section, we prove the main theorem. Let $\pi: X \rightarrow T$ be as in the preceding section.

Definition 4.1. Let $f: Y \rightarrow X$ be a birational morphism with the maximal reduced exceptional divisor E . Suppose that $K_Y + E$ is \mathbb{Q} -Cartier. We call f a simultaneous log-canonical model if for any $t \in T$, $f_t: Y_t \rightarrow X_t$ is the log-canonical model of X_t and E_t is a reduced divisor.

A simultaneous log-canonical model of $\pi: X \rightarrow T$ is unique if it exists. In fact, we have the following:

Lemma 4.2. *Let $f: Y \rightarrow X$ be a simultaneous log-canonical model with the exceptional divisor E . Then f is the log-canonical model of X and E_t is the exceptional set on Y_t for any $t \in T$.*

Proof. Let B be the union of the curves C on Y such that $C \not\subset E$ and $f(C) = \{x\}$, the singular point of X_0 . Then $E \cup B$ is the exceptional set of f , and hence $E_0 \cup B$ is the exceptional set of f_0 . Since X is Gorenstein, we may assume that $(K_{Y_0})_{red} \subset E_0$. By assumption, $K_{Y_0} + E_0 + B$ is f_0 -ample. Thus there exists a positive divisor D supported on $E_0 \cup B$ such that $K_{Y_0} + E_0 + B = -D$. Hence B is void, and E_0 is the exceptional set of f_0 . Since $K_{Y_t} + E_t$ is f_t -ample for any $t \in T$, we see that $K_Y + E$ is f -ample. By inversion of adjunction [1, Chap. 17], $K_Y + E$ is log-canonical, since $K_{Y_t} + E_t$ is log-canonical for any $t \in T$. □

Remark 4.3. In the situation above, we see that $h^0(\mathcal{O}_{E_t}) = 1$ for any t . This implies that E_t is connected. Hence X_t has at most one non-log-canonical singularity.

In the following, the notation is the same as in the preceding section, unless otherwise specified; so $f: Y \rightarrow X$ denotes the canonical model of X . We assume that T is sufficiently small in each case. For each $m \in \mathbb{N}$, we regard $\lambda_m(X_t)$ as a function of t .

Proposition 4.4. *Suppose that $f: Y \rightarrow X$ is the simultaneous log-canonical model of the deformation $\pi: X \rightarrow T$. Then $\lambda_m(X_t)$ is constant for $m \gg 0$.*

Proof. Consider the exact sequence

$$0 \rightarrow f_*\mathcal{O}_Y(m(K_Y + E)) \xrightarrow{\times\tau} f_*\mathcal{O}_Y(m(K_Y + E)) \rightarrow f_*\mathcal{O}_{Y_0}(m(K_{Y_0} + E_0)) \rightarrow R^1f_*\mathcal{O}_Y(m(K_Y + E)),$$

where τ is a local parameter at 0. Since $K_Y + E$ is f -ample, $R^1f_*\mathcal{O}_Y(m(K_Y + E)) = 0$ for $m \gg 0$. We fix such an integer m . Then $f_*\mathcal{O}_{Y_0}(m(K_{Y_0} + E_0)) = \mathcal{I}_m \otimes \mathbb{C}(0)$. It is clear that $\mathcal{I}_m \otimes \mathbb{C}(0) = \mathcal{I}_{m,0}$. Thus $\dim_{\mathbb{C}} \mathcal{Q}_m \otimes \mathbb{C}(0) = \lambda_m(X_0)$ by Proposition 2.6. Then Lemma 3.3 and 3.4 imply that $\lambda_m(X_t) = \lambda_m(X_0)$ for $t \in T^*$. \square

In [13], Tomari and Watanabe proved a result on the order of growth of the L^2 -plurigenera by applying Izumi’s results on the analytic orders [6]. We will use their argument. The following lemma is Ishii’s version [4, Lemma 1.5].

Lemma 4.5. *Let (W, w) be a d -dimensional normal isolated singularity, and let $h: W_1 \rightarrow W$ be a resolution of the singularity which factors through the blowing up by the maximal ideal of the singular point. Let $F = \bigcup_{i=1}^k F_i$ be the exceptional divisor on W_1 , where the F_i are irreducible components. Then there exist positive numbers $\beta \in \mathbb{R}$ and $b \in \mathbb{N}$ such that:*

For an \mathcal{O}_W -ideal $J = h_\mathcal{O}_{W_1}(-\sum_{i=1}^k a_i F_i)$ with $a_i > b$ for some i , the inequalities $\dim_{\mathbb{C}} \mathcal{O}_W/J \geq \beta(a_i)^d$ ($i = 1, \dots, k$) hold.*

Lemma 4.6. *Assume that $-P_t \cdot P_t$ is constant. Then $\mathcal{I}_{m,0} \subset \mathcal{A}_m$ for all $m \in \mathbb{N}$.*

Proof. It suffices to show that $\theta_m = 0$ for all $m \in \mathbb{N}$. Note that Proposition 3.2, the inequality (3.2) and Lemma 3.5 imply that θ_m is bounded by a linear function. Assume that $\theta_r \neq 0$. Let ω be a section of $\mathcal{I}_{r,0}$ which does not belong to \mathcal{A}_r . We define a homomorphism $\varphi_m: \mathcal{O}_{X_0} \rightarrow \mathcal{I}_{mr,0}$ by $\varphi_m(s) = s\omega^m$ for every $m \in \mathbb{N}$. We denote by J_m the inverse image $\varphi_m^{-1}(\mathcal{A}_{mr} \cap \mathcal{I}_{mr,0})$. Then we have the injection

$$\mathcal{O}_{X_0}/J_m \rightarrow \mathcal{I}_{mr,0}/(\mathcal{A}_{mr} \cap \mathcal{I}_{mr,0}).$$

We put $a_i = \min\{v_i(\omega) + r, 0\}$, where v_i is the valuation at an irreducible component A_i of A . Then $J_m = \psi_*\mathcal{O}_M(\sum ma_i A_i)$. By the choice of ω , there exists a component A_i such that $a_i < 0$. We may assume that the good resolution $\psi: (M, A) \rightarrow (X_0, x)$ factors through the blowing up by the maximal ideal of the point x . By Lemma 4.5, there exists a positive number $c \in \mathbb{R}$ such that $\theta_{mr} \geq cm^2$ for any $m \in \mathbb{N}$. It contradicts the fact noted above. \square

The technique above is used in the proof of [5, Theorem 1]. In [5], a claim similar to “ $\mathcal{A}_m \subset \mathcal{I}_{m,0}$ ” is proved by using [10, Lemma 1]. The upper semi-continuity of the plurigenus γ_m follows from the claim. Unfortunately we have no log-version of [10, Lemma 1]. However, we can show the claim “ $\mathcal{A}_m \subset \mathcal{I}_{m,0}$ ” under the condition that $-P_t \cdot P_t$ is constant.

Lemma 4.7. *Let F be any prime divisor on Y such that $f(F) = \{x\}$. Then there exists $m > 0$ such that $f_*\mathcal{O}_Y(m(K_Y + E) + F)/\mathcal{I}_m \neq 0$.*

Proof. We note that F is a projective surface. Let \mathcal{I}_F be an \mathcal{O}_Y -ideal of the subvariety F , and let $L_m = m(K_Y + E)$. Since L_1 is f -ample, there exists an integer $n \in \mathbb{N}$ such that $\mathcal{O}_F(L_n)$ is a very ample invertible sheaf and the following sequence is exact for any $m \in \mathbb{N}$:

$$0 \rightarrow f_*(\mathcal{I}_F\mathcal{O}_Y(L_{mn} + F)) \rightarrow f_*\mathcal{O}_Y(L_{mn} + F) \rightarrow H^0(\mathcal{O}_F(L_{mn} + F)) \rightarrow 0.$$

By [2, III, Ex. 5.2], there exists a polynomial q' of degree 2 such that

$$\dim_{\mathbb{C}} f_*\mathcal{O}_Y(L_{mn} + F)/f_*(\mathcal{I}_F\mathcal{O}_Y(L_{mn} + F)) = q'(m)$$

for $m \gg 0$. Since $\mathcal{I}_F\mathcal{O}_Y(L_{mn} + F)$ is isomorphic to $\mathcal{O}_Y(L_{mn})$ outside a one-dimensional subvariety of F , there exists a polynomial q of degree 2 such that $\dim_{\mathbb{C}} f_*\mathcal{O}_Y(L_{mn} + F)/\mathcal{I}_{mn} \geq q(m)$ for $m \gg 0$. \square

Lemma 4.8. *Assume that $-P_t \cdot P_t$ is constant. Then \mathcal{Q}_m is a torsion free \mathcal{O}_T -module for any $m \in \mathbb{N}$.*

Proof. Assume that \mathcal{Q}_m has a torsion element. Then there exists the maximal reduced divisor $D \neq 0$ on Y such that $f(D) = \{x\}$. Let $h: Z \rightarrow Y$ be the blowing up by an ideal sheaf $\mathcal{O}_Y(-D)$. Then h is an isomorphism in codimension one and $h_*^{-1}(D)$ is a Cartier divisor. Thus there exists a prime divisor $F \subset h_*^{-1}(D)$ such that the intersection $F \cap (f \circ h)_*^{-1}(X_0)$ is one-dimensional. By Lemma 4.7, there exist $r > 0$ and $\omega \in \mathcal{O}_X(rK_X)$ with the condition that the image of ω in \mathcal{Q}_r is a torsion element and $v_F(\omega) < -r$, where v_F denotes the valuation at F . Suppose that $v_F(\omega) \leq v_F(\omega')$ for all $\omega' \in \mathcal{O}_X(rK_X)$ with the condition above. We denote by η_1 (resp. η_2) the image of $\eta \in \mathcal{O}_X(mK_X)$ in $\mathcal{Q}_m \otimes \mathbb{C}(0)$ (resp. $\mathcal{O}_{X_0}(mK_{X_0})/\mathcal{A}_m$). We may assume that the good resolution $\psi: (M, A) \rightarrow (X_0, x)$ factors through $(f \circ h)_*^{-1}(X_0) \rightarrow X_0$. We denote by \mathcal{T}_m the torsion submodule of \mathcal{Q}_m for $m \in \mathbb{N}$. By the choice of ω , $\omega_1 \in \mathcal{T}_r \otimes \mathbb{C}(0)$, and ω_1 and ω_2 are nonzero. For each $m \in \mathbb{N}$, define a homomorphism $\varphi_m: \mathcal{O}_{X_0} \rightarrow \mathcal{O}_{X_0}(mrK_{X_0})/\mathcal{A}_{mr}$ by $\varphi_m(s) = (s\omega^m)_2$. By an argument similar to that in the proof of Lemma 4.6, there exists a positive number c such that $\dim_{\mathbb{C}} \varphi_m(\mathcal{O}_{X_0}) \geq cm^2$. By Lemma 4.6, we have the natural surjection $\mathcal{Q}_{mr} \otimes \mathbb{C}(0) \rightarrow \mathcal{O}_{X_0}(mrK_{X_0})/\mathcal{A}_{mr}$. Since the image of $\mathcal{T}_{mr} \otimes \mathbb{C}(0)$ in $\mathcal{O}_{X_0}(mrK_{X_0})/\mathcal{A}_{mr}$ contains $\varphi_m(\mathcal{O}_{X_0})$, we have $\dim_{\mathbb{C}} \mathcal{T}_{mr} \otimes \mathbb{C}(0) \geq cm^2$. Then, for $t \in T^*$, we have the inequality

$$\dim_{\mathbb{C}} \mathcal{Q}_{mr} \otimes \mathbb{C}(t) \leq \dim_{\mathbb{C}} \mathcal{Q}_{mr} \otimes \mathbb{C}(0) - cm^2.$$

This implies the following:

$$\lambda_{mr}(X_t) \leq \lambda_{mr}(X_0) + \varepsilon_{mr} - cm^2.$$

By Proposition 3.2 and Lemma 3.5, we get $-P_t \cdot P_t < -P_0 \cdot P_0$. \square

Remark 4.9. Assume that $-P_t \cdot P_t$ is constant. From the proof above, we see that Y_0 is irreducible. Hence any irreducible component of E dominates T . Since Y_0 is a principal divisor, for any irreducible component F of E , the intersection $F \cap Y_0$ is a one-dimensional variety.

Lemma 4.10. *Assume that $-P_t \cdot P_t$ is constant. Then $\mathcal{I}_{m,0} = \mathcal{A}_m$ for any $m \in \mathbb{N}$.*

Proof. By Lemma 4.6, it suffices to show that $\mathcal{A}_m \subset \mathcal{I}_{m,0}$. Let ω be a section of \mathcal{A}_m and ω' a section of $\mathcal{O}_X(mK_X)$ whose image in $\mathcal{O}_{X_0}(mK_{X_0})$ is ω . We may assume that $\psi: (M, A) \rightarrow (X_0, x)$ factors through $f_0: Y_0 \rightarrow X_0$. If $v_F(\omega') < -m$ for an irreducible component F of E , then there exists an irreducible component A_i of A lying over the variety $F \cap Y_0$ such that $v_{A_i}(\psi^*\omega) < -m$. It contradicts the choice of ω . Hence ω' belongs to \mathcal{I}_m , and ω belongs to $\mathcal{I}_{m,0}$. \square

Now we are ready to prove the main theorem which is the log-version of Laufer's result on simultaneous canonical models.

Theorem 4.11. *The following conditions are equivalent:*

- (1) $\pi: X \rightarrow T$ admits the simultaneous log-canonical model.
- (2) The function $\lambda_m(X_t)$ of t is constant for any $m \in \mathbb{N}$.
- (3) The function $-P_t \cdot P_t$ of t is constant.

Proof. Consider the condition: (2)' $\lambda_m(X_t)$ is constant for $m \gg 0$. By Proposition 3.2 and 4.4, we obtain the implication (1) \Rightarrow (2)' \Rightarrow (3). We assume that $-P_t \cdot P_t$ is constant. Then, from Lemma 3.3, 4.8 and 4.10, we obtain the following equalities for any $m \in \mathbb{N}$:

$$\mathcal{I}_m \otimes \mathbb{C}(0) = \mathcal{I}_{m,0} = \mathcal{A}_m, \quad \dim_{\mathbb{C}} \mathcal{Q}_m \otimes \mathbb{C}(t) = \dim_{\mathbb{C}} \mathcal{Q}_m \otimes \mathbb{C}(0).$$

Now it is clear that (2) holds, and that $Y_0 = \text{Proj}(\bigoplus_{m \geq 0} \mathcal{I}_m \otimes \mathbb{C}(0))$ is the log-canonical model of X_0 . Since $\mathcal{A}_m = f_*\mathcal{O}_{Y_0}(m(K_{Y_0} + E_0))$ for $m \gg 0$ (cf. the proof of Proposition 4.4), and $K_{Y_0} + E_0$ is f_0 -ample, we see that E_0 is reduced. On the other hand, $f_t: Y_t \rightarrow X_t$ is the log-canonical model for $t \in T^*$ by Lemma 3.1. Hence we obtain the condition (1). \square

Remark 4.12. The techniques in this paper can be used in higher dimensions. In fact, we can generalize the main theorem to the higher-dimensional case if the following claims hold:

- (1) there exists a log-canonical model $f: Y \rightarrow X$ of X ;
- (2) for an isolated singularity (W, w) of dimension n , there exists $p = p(W, w) \in \mathbb{R}$ such that $\lambda_m(W, w) = pm^n + O(m^{n-1})$.

Note that $\delta_m(W, w) = \lambda_m(W, w) + O(m^{n-1})$ by an argument in [14, p. 627].

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DEPARTMENT OF MATHEMATICS, GUNMA NATIONAL COLLEGE OF TECHNOLOGY, 580 TORIBA,
MAEBASHI, GUNMA 371, JAPAN

E-mail address: okuma@nat.gunma-ct.ac.jp