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THE NATURAL MAXIMAL OPERATOR ON BMO

WINSTON OU

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ABSTRACT. We introduce a generalization of the Hardy-Littlewood maximal operator, the *natural maximal operator* $M^\natural$, in some sense the maximal operator which most naturally commutes pointwise with the logarithm on A^∞. This commutation reveals the behavior of $M : A^\infty \to A^1$ to directly correspond to that of $M^\natural : BMO \to BLO$; the boundedness of $M : BMO \to BLO$ is an immediate consequence.

In 1981 Bennett, DeVore, and Sharpley [1] proved the Hardy-Littlewood maximal operator M to be bounded on *BMO*. Later [4], this was improved to yield $M : BMO \to BLO$[1] and furthermore was shown to be intimately related to the observation that M maps A^∞ into A^1. In this paper we shall clarify this relationship and in fact demonstrate a precise equivalence.

Let us begin by defining the *natural maximal operator* $M^\natural$ by

$$M^\natural f(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q f.$$

Of course, $M(f)(x) = M^\natural(|f|)(x)$; so to prove the boundedness of M it suffices to show that of $M^\natural$. This is easily accomplished with the following commutation lemma, which allows us to pass between the language of A^∞ and that of *BMO*.

Lemma. *For* $w \in A^\infty$, $0 \le [\log M^\natural - M^\natural \log] w(x) \le \log A^\infty(w)$.[2]

Proof. By Jensen's inequality and the reverse inequality for A^∞,

$$\frac{1}{|Q|} \int_Q w \le A^\infty(w) e^{\frac{1}{|Q|}\int_Q \log w} \le A^\infty(w) \frac{1}{|Q|} \int_Q w.$$

Take the supremum over all $Q \ni x$ and then take log. □

Theorem. $M^\natural$ *maps BMO boundedly into BLO.*

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[1] Recall that *BLO* denotes the functions of bounded lower oscillation, i.e. f such that over all cubes Q, $\frac{1}{|Q|}\int_Q f - \inf_Q f \le C$; we denote by $||f||_{BLO}$ the smallest such C.

[2] $A^\infty(w)$ denotes the smallest C such that for all cubes Q, $\frac{1}{|Q|}\int_Q w \le C e^{\frac{1}{|Q|}\int_Q \log w}$.

Proof. Let $\phi \in BMO$. As a consequence of the John-Nirenberg Inequality, $\phi(x) = \frac{||\phi||_*}{c_n} \log w(x)$ for some $w \in A^2$, with $A^2(w) \leq \sqrt{e}$ and $c_n = \frac{1}{2^{n+1}e}$, where n denotes the dimension. Thus, by the lemma,

$$M^\natural \phi(x) = \frac{||\phi||_*}{c_n}[\log Mw(x) + b(x)]$$

for some $||b||_\infty \leq 1$. Let's consider the $\log Mw$ term: M maps A^∞ into A^1, and log maps A^1 into BLO; so it is in BLO. Further, its norm depends on $A^1(Mw)$, which depends on the reverse Hölder class and norm of w, which in turn depend on the A^p class and A^p norm of w. As noted above, these depend on nothing; thus $||M^\natural \phi||_{BLO} \leq C_n ||\phi||_*$. □

Corollary. *M maps BMO boundedly into BLO.*[3]

Proof. $Mf = M^\natural(|f|) \in BLO$, and $||Mf||_{BLO} = ||M^\natural(|f|)||_{BLO} \leq 2C_n||f||_*$. □

Notice that above, the set inclusion $M(A^\infty) \subset A^1$ implies $M^\natural(BMO) \subset BLO$, and the dependence of $A^1(Mw)$ on $A^p(w)$ and p implies the boundedness of $M^\natural$. Repeated application of the commutation lemma, combined with the observation that $\phi \in BLO$ if and only if $M^\natural \phi(x) \leq \phi(x) + ||\phi||_{BLO}$, yields the following converse (assuming the weak result that $M(A^\infty) \subset A^\infty$ with $A^\infty(Mw)$ dependent on $A^p(w)$ and p):

Theorem. *$M^\natural : BMO \to BLO$ bounded implies $M(A^\infty) \subset A^1$ with $A^1(Mw)$ dependent on $A^p(w)$ and p.*

Proof. Let $w \in A^\infty$. The commutation lemma applied to w implies

$$e^{M^\natural \log w(x)} \leq Mw(x) \leq A^\infty(w) e^{M^\natural \log w(x)};$$

applied twice to Mw it yields

$$e^{M^\natural M^\natural \log w(x)} \leq MMw(x) \leq A^\infty(w) A^\infty(Mw) e^{M^\natural M^\natural \log w(x)}.$$

By hypothesis, $M^\natural \log w \in BLO$; thus

$$MMw(x) \approx e^{M^\natural M^\natural \log w(x)} \leq e^{M^\natural \log w(x) + ||M^\natural \log w||_{BLO}} \approx e^{||M^\natural \log w||_{BLO}} Mw(x),$$

i.e., $Mw \in A^1$, with $A^1(Mw) \leq A^\infty(w) A^\infty(Mw) e^{C_n || \log w||_*}$. □

In other words, the behavior of M on A^∞ corresponds *exactly* to that of $M^\natural$ on BMO. For amusement, one can adapt the original proof of Bennett, DeVore, and Sharpley to prove directly the bound for $M^\natural$ and thus obtain a new, if unwieldy, proof of $M \colon A^\infty \to A^1$.

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[3]Added in proof: Chiarenza and Frasca [2] have also proved this bound directly using the John-Nirenberg inequality and the fact that $(Mf)^{1/2} \in A^1$ for $f \in L^1_{loc}(\mathbb{R}^n)$ [3], though without the commutation or noticing the connection with the behavior on A^∞.

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Department of Mathematics, University of Chicago, Chicago, Illinois 60637
E-mail address: wcwou@math.uchicago.edu