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SOME RESULTS RELATED TO THE LOGVINENKO-SEREDA THEOREM

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Abstract. We prove several results related to the theorem of Logvinenko and Sereda on determining sets for functions with Fourier transforms supported in an interval. We obtain a polynomial instead of exponential bound in this theorem, and we extend it to the case of functions with Fourier transforms supported in the union of a bounded number of intervals.

The purpose of this work is to study the behavior of functions whose Fourier transforms are supported in an interval or in a union of finitely many intervals on "thick" subsets of the real line. A main result of this type was proven by Logvinenko and Sereda.

By a "thick" subset of the real line we mean a measurable set E for which there exist $a > 0$ and $\gamma > 0$ such that

$$|E \cap I| \geq \gamma \cdot a \tag{1}$$

for every interval I of length a.

The Logvinenko-Sereda Theorem. *Let J be an interval with $|J| = b$. If $f \in L^p$, $p \in [1, +\infty]$, and supp $\hat{f} \subset J$, and if a measurable set E satisfies* (1)*, then*

$$\|f\|_{L^p(E)} \geq \exp(-C \cdot \frac{(ab+1)}{\gamma}) \cdot \|f\|_p. \tag{2}$$

It is a well-known fact that condition (1) is also necessary for an inequality of the form

$$\|f\|_{L^p(E)} \geq C \cdot \|f\|_p$$

to hold. See for example [3], p. 113.

We will improve the estimate (2) by getting a polynomial dependence on γ and show that our estimate is optimal except for the constant C:

Theorem 1.

$$\|f\|_{L^p(E)} \geq \left(\frac{\gamma}{C}\right)^{C\cdot(ab+1)} \cdot \|f\|_p.$$

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We will also generalize the Logvinenko-Sereda theorem to functions whose Fourier transforms are supported on a union of finitely many intervals:

Theorem 2. *Let* J_k *be intervals with* $|J_k| = b$. *If* $f \in L^p$, $p \in [1, +\infty]$, *and supp* $\hat{f} \subset \bigcup\limits_{1}^{n} J_k$, *and if a measurable set* E *satisfies* (1), *then*

$$\|f\|_{L^p(E)} \geq c(\gamma, n, ab, p) \cdot \|f\|_p$$

where $c(\gamma, n, ab, p) = \left(\frac{C}{\gamma}\right)^{-ab\left(\frac{C}{\gamma}\right)^n - n + \frac{p-1}{p}}$ *depends only on the number of intervals but not how they are placed.*

Note that the constant C below is not fixed and varies appropriately from one equality or inequality to another even without mentioning it.

Proof of Theorem 1. First we treat the case when $p \in [1, +\infty)$. Without loss of generality we can always assume that $J = [-\frac{b}{2}, \frac{b}{2}]$. By considering $f(\frac{x}{a})$ instead of f we can also assume that $|E \cap [x, x+1]| \geq \gamma \ \forall x$ and supp $\hat{f} \subset [-\frac{ab}{2}, \frac{ab}{2}]$, just say supp $\hat{f} \subset [-\frac{b}{2}, \frac{b}{2}]$. Bernstein's inequality ([1], Theorem 11.3.3) gives that

$$\int |f^{(\alpha)}|^p \leq (C \cdot b)^{\alpha p} \cdot \int |f|^p$$

with $C = \frac{1}{2}$.

Divide the whole $\mathbb{R}$ into intervals of length 1. Choose $A > 1$. Call an interval I bad if $\exists \alpha \geq 1$ such that

$$\int\limits_I |f^{(\alpha)}|^p \geq A^{\alpha p} (C \cdot b)^{\alpha p} \cdot \int\limits_I |f|^p.$$

Then

$$\begin{aligned} \int\limits_{\bigcup\limits_{I \text{ is bad}} I} |f|^p &\leq \int\limits_{\bigcup\limits_{I \text{ is bad}} I} \sum_{\alpha=1}^{\infty} \frac{1}{A^{\alpha p}(C \cdot b)^{\alpha p}} |f^{(\alpha)}|^p \\ &= \sum_{\alpha=1}^{\infty} \frac{1}{A^{\alpha p}(C \cdot b)^{\alpha p}} \int\limits_{\bigcup\limits_{I \text{ is bad}} I} |f^{(\alpha)}|^p \\ &\leq \sum_{\alpha=1}^{\infty} \frac{1}{A^{\alpha p}(C \cdot b)^{\alpha p}} \int |f^{(\alpha)}|^p \\ &\leq \sum_{\alpha=1}^{\infty} \frac{1}{A^{\alpha p}} \int |f|^p \\ &= \frac{1}{A^p - 1} \int |f|^p. \qquad (3) \end{aligned}$$

Choose $A = 3$ and apply (3). So

$$\int\limits_{\bigcup\limits_{I \text{ is bad}} I} |f|^p \leq \frac{1}{2} \int |f|^p.$$

Therefore,

$$\int\limits_{\bigcup_{I\ is\ good} I} |f|^p \geq \frac{1}{2}\int |f|^p. \tag{4}$$

We claim that $\exists B > 1$ such that if I is a good interval, then $\exists x \in I$ with the property that

$$|f^{(\alpha)}(x)|^p \leq 2 \cdot B^{\alpha p}(C \cdot b)^{\alpha p} \cdot \int\limits_I |f|^p \quad \forall \alpha \geq 0.$$

Suppose towards a contradiction that this is not true. Then

$$2 \cdot \int\limits_I |f|^p \leq \sum_{\alpha=0}^{\infty} \frac{1}{B^{\alpha p}(C \cdot b)^{\alpha p}} |f^{(\alpha)}(x)|^p \quad \forall x \in I. \tag{5}$$

Integrate both sides of (5) over I:

$$\begin{aligned} 2 \cdot \int\limits_I |f|^p &\leq \sum_{\alpha=0}^{\infty} \frac{1}{B^{\alpha p}(C \cdot b)^{\alpha p}} \int\limits_I |f^{(\alpha)}(x)|^p \\ &\leq \sum_{\alpha=0}^{\infty} \frac{1}{B^{\alpha p}} \int\limits_I |f|^p \\ &= \frac{1}{1 - \left(\frac{1}{B}\right)^p} \int\limits_I |f|^p. \end{aligned} \tag{6}$$

Choose $B = 3$ and apply (6). So

$$2 \cdot \int\limits_I |f|^p \leq \frac{3}{2} \int\limits_I |f|^p.$$

This contradiction proves our claim.

We will need to prove the following local estimate:

$$\int\limits_{E \cap I} |f|^p \geq \left(\frac{\gamma}{C}\right)^{Cbp+2} \int\limits_I |f|^p$$

for every good interval I. Without loss of generality we can assume that $I = [-\frac{1}{2}, \frac{1}{2}]$ by considering a shift $f(x-n)$ which has $\mathrm{supp}\widehat{f(x-n)} \subset [-\frac{b}{2}, \frac{b}{2}]$. Therefore if $y \in D(0,R) \subset D(x, R+\frac{1}{2})$, then

$$\begin{aligned} |f(y)| &\leq \sum_{\alpha=0}^{\infty} \frac{|f^{(\alpha)}(x)|}{\alpha!} \cdot |y-x|^\alpha \\ &\leq \sum_{\alpha=0}^{\infty} 2^{\frac{1}{p}} \frac{(R+\frac{1}{2})^\alpha \cdot (Cb)^\alpha}{\alpha!} \|f\|_{L^p(I)} \\ &= 2^{\frac{1}{p}} \exp(Cb(R+\frac{1}{2})) \cdot \|f\|_{L^p(I)}. \end{aligned} \tag{7}$$

Now we will give a local estimate for analytic functions.

Lemma 1. *Let $\phi(z)$ be analytic in $D(0,5)$ and let I be an interval of length* 1 *such that $0 \in I$ and let $E \subset I$ be a measurable set of positive measure. If $|\phi(0)| \geq 1$ and $M = \max_{|z|\leq 4} |\phi(z)|$, then*

$$\sup_{x\in I} |\phi(x)| \leq \left(\frac{C}{|E|}\right)^{\frac{\ln M}{\ln 2}} \sup_{x\in E} |\phi(x)|. \tag{8}$$

Proof of Lemma 1. Let $a_1, a_2, \ldots, a_n$ be the zeros of ϕ in $D(0,2)$. If

$$g(z) = \phi(z) \cdot \prod_{k=1}^{n} \frac{4 - \bar{a}_k z}{2(a_k - z)} = \phi(z) \cdot \frac{Q(z)}{P(z)},$$

then $|g(0)| \geq 1$ and $\max_{|z|\leq 2} |g(z)| \leq M$ by the property of Blaschke products. Applying Harnack's inequality to the positive harmonic function $\ln M - \ln |g(z)|$ in $D(0,2)$ we have

$$\max_{|z|\leq 1} (\ln M - \ln |g(z)|) \leq 3 \ln M.$$

Therefore,

$$\min_{|z|\leq 1} |g(z)| \geq M^{-2}.$$

This gives us

$$\frac{\max_{x\in I} |g(x)|}{\min_{x\in I} |g(x)|} \leq M^3.$$

We can give a similar estimate for Q:

$$\frac{\max_{x\in I} |Q(x)|}{\min_{x\in I} |Q(x)|} \leq \frac{\max_{|z|\leq 1} \prod_{k=1}^{n} |4 - \bar{a}_k z|}{\min_{|z|\leq 1} \prod_{k=1}^{n} |4 - \bar{a}_k z|} \leq 3^n.$$

From the Remez inequality for polynomials ([2], Theorem 5.1.1) it follows that

$$\sup_{x\in I} |P(x)| \leq \left(\frac{4}{|E|}\right)^n \cdot \sup_{x\in E} |P(x)|.$$

Therefore,

$$\begin{aligned} \sup_{x\in I} |\phi(x)| &\leq \max_{x\in I} |g(x)| \cdot \frac{\max_{x\in I} |P(x)|}{\min_{x\in I} |Q(x)|} \\ &\leq M^3 \cdot 3^n \cdot \left(\frac{C}{|E|}\right)^n \cdot \min_{x\in I} |g(x)| \cdot \frac{\sup_{x\in E} |P(x)|}{\max_{x\in I} |Q(x)|} \\ &\leq M^3 \cdot \left(\frac{C}{|E|}\right)^n \cdot \sup_{x\in E} |\phi(x)|. \end{aligned}$$

From Jensen's formula it follows that $n \leq \frac{\ln M}{\ln 2}$. Therefore,

$$\sup_{x\in I} |\phi(x)| \leq \left(\frac{C}{|E|}\right)^{\frac{\ln M}{\ln 2}} \sup_{x\in E} |\phi(x)|.$$

□

Corollary. *If $p \in [1,\infty)$, then*

$$\|\phi\|_{L^p(I)} \le \left(\frac{C}{|E|}\right)^{\frac{\ln M}{\ln 2}+\frac{1}{p}} \|\phi\|_{L^p(E)}. \tag{9}$$

It follows from (8) *that*

$$|\{x \in I : |\phi(x)| < \left(\frac{\epsilon}{C}\right)^{\frac{\ln M}{\ln 2}} \|\phi\|_{L^\infty(I)}\}| \le \epsilon, \quad \epsilon > 0.$$

If we put $\epsilon = \frac{|E|}{2}$, then

$$|\{x \in I : |\phi(x)| < \left(\frac{|E|}{2C}\right)^{\frac{\ln M}{\ln 2}} \|\phi\|_{L^\infty(I)}\}| \le \frac{|E|}{2}.$$

Therefore,

$$\begin{aligned}\int_E |\phi|^p &\ge \int_E \chi_{|\phi| \ge \left(\frac{|E|}{2C}\right)^{\frac{\ln M}{\ln 2}} \cdot \|\phi\|_{L^\infty(I)}} \cdot |\phi|^p \\ &\ge \frac{|E|}{2} \cdot \left(\frac{|E|}{2C}\right)^{p\frac{\ln M}{\ln 2}} \cdot \|\phi\|^p_{L^\infty(I)} \\ &\ge \left(\frac{|E|}{2C}\right)^{p\frac{\ln M}{\ln 2}+1} \cdot \int_I |\phi|^p.\end{aligned}$$

□

Now we are in a position to proceed with the proof of our theorem. We can assume that $\int_I |f|^p = 1$. Then $\exists x_0 \in I$ such that $|f(x_0)| \ge 1$. Applying (9) to $\phi(z) = f(z + x_0)$, $I - x_0$ and $(E \cap I) - x_0$ we have

$$\int_{E\cap I} |f|^p \ge \left(\frac{|E \cap I|}{C}\right)^{p\frac{\ln M}{\ln 2}+1} \int_I |f|^p.$$

Apply (7) to get

$$\begin{aligned}M &\le \max_{|z| \le 4+\frac{1}{2}} |f(z)| \\ &\le 2^{\frac{1}{p}} \exp(5Cb).\end{aligned}$$

Therefore, we have

$$\int_{E\cap I} |f|^p \ge \left(\frac{\gamma}{C}\right)^{Cbp+2} \int_I |f|^p. \tag{10}$$

Summing (10) over all good intervals and applying (4) we have

$$\begin{aligned}\int_E |f|^p &\ge \int_{E\cap \bigcup_{I \text{ is good}} I} |f|^p \\ &\ge \left(\frac{\gamma}{C}\right)^{Cbp+2} \cdot \int_{\bigcup_{I \text{ is good}} I} |f|^p \\ &\ge \frac{1}{2}\left(\frac{\gamma}{C}\right)^{Cbp+2} \cdot \int |f|^p.\end{aligned}$$

Replacing b with ab and choosing a new C we have

$$\int\limits_E |f|^p \geq \left(\frac{\gamma}{C}\right)^{Cabp+2} \cdot \int |f|^p.$$

If $p = \infty$, then the proof is almost the same: $\|f\|_{L^\infty(\bigcup\limits_{I \text{ is good}} I)} = \|f\|_\infty$. If I is good, then $\|f\|_{L^\infty(E \cap I)} \geq \left(\frac{\gamma}{C}\right)^{Cb+1} \cdot \|f\|_{L^\infty(I)}$. Hence,

$$\|f\|_{L^\infty(E)} \geq \left(\frac{\gamma}{C}\right)^{Cb+1} \cdot \|f\|_\infty.$$

The proof of Theorem 1 is complete. □

If we keep track of all the constants and do the calculations more accurately, then we can get that if $p \in [1, \infty)$,

$$\|f\|_{L^p(E)} \geq \left(\frac{\gamma}{300}\right)^{33ab+\frac{2}{p}} \cdot \|f\|_p;$$

if $p = \infty$,

$$\|f\|_{L^\infty(E)} \geq \left(\frac{\gamma}{100}\right)^{33ab+1} \cdot \|f\|_\infty.$$

However, if we try to minimize the factor in front of ab, then we can get the following estimate:

$$\|f\|_{L^p(E)} \geq \left(\frac{\gamma}{C}\right)^{(\frac{(1+\epsilon)}{2}+\epsilon)\cdot ab + A(\epsilon)} \cdot \|f\|_p \quad \forall \epsilon > 0.$$

The following example suggests that the right behavior of the estimate in the Logvinenko-Sereda theorem is γ to the power of a linear function of ab and how far we are from the exact factor in front of ab.

Let E be a 1-periodic set such that

$$E \cap [-\frac{1}{2}, \frac{1}{2}] = [-\frac{1}{2}, -\frac{1}{2} + \frac{\gamma}{2}] \cap [\frac{1}{2} - \frac{\gamma}{2}, \frac{1}{2}]$$

and let

$$f(x) = \left(\frac{\sin(2\pi x)}{x}\right)^{[\frac{b}{4\pi}]}.$$

If b is large enough, we have

$$\|f\|_{L^p(E)} \leq \left(\frac{\gamma}{C}\right)^{\frac{b}{4\pi}-1} \|f\|_p$$

and $\text{supp}\hat{f} \subset [-\frac{b}{2}, \frac{b}{2}]$.

Remark 1. When ab is sufficiently small, the proof of the theorem is much simpler: if $ab \leq 1$, then $\|f\|_{L^p(E)} \geq \frac{\gamma^{\frac{1}{p}}}{2} \|f\|_p$. This can be proven very easily. If $p \in [1, +\infty)$, we have

$$\begin{aligned} |f(x)|^p = |f(y) - \int\limits_x^y f'(t)dt|^p &\geq \frac{|f(y)|^p}{2^{p-1}} - |\int\limits_x^y f'(t)dt|^p \\ &\geq \frac{|f(y)|^p}{2^{p-1}} - \int\limits_I |f'|^p \cdot a^{p-1} \end{aligned}$$

where $x, y \in I$, $|I| = a$. Hence,

$$\begin{aligned} a \cdot \int\limits_{E\cap I} |f(x)|^p dx &= \int\limits_I \left(\int\limits_{E\cap I} |f(x)|^p dx \right) dy \\ &\geq |E \cap I| \cdot \left(\frac{1}{2^{p-1}} \int\limits_I |f|^p - a^p \int\limits_I |f'|^p \right). \end{aligned}$$

Therefore, $\frac{1}{\gamma} \cdot \int\limits_{E\cap I} |f|^p \geq \frac{1}{2^{p-1}} \int\limits_I |f|^p - a^p \int\limits_I |f'|^p$. Summing over all intervals I we have

$$\begin{aligned} \frac{1}{\gamma} \cdot \int\limits_E |f|^p &\geq \frac{1}{2^{p-1}} \int |f|^p - a^p \int |f'|^p \\ &\geq \frac{1}{2^{p-1}} \int |f|^p - \left(\frac{b}{2}\right)^p a^p \int |f|^p \\ &\geq \frac{1}{2^p} \cdot \int |f|^p. \end{aligned}$$

Using $\|(f^p)'\|_1 \leq \frac{pb}{2}\|f^p\|_1$ we can get a similar result. The proof for $p = \infty$ is even easier.

In a similar way we can treat the case when $1 - \gamma$ is sufficiently small depending on ab: if $p \in [1, \infty)$ and $1 - \gamma \leq \frac{1}{2+pab}$, then $\|f\|^p_{L^p(E)} \geq \frac{1}{2}\|f\|^p_p$.

Proof of Theorem 2. Let $J_k = [\lambda_k - \frac{b}{2}, \lambda_k + \frac{b}{2}]$, $k = 1, 2, ..., n$. First we will prove a special case of Theorem 2:

Theorem 2′. *If* $\lambda_{k+1} - \lambda_k \geq 2b > 0$ $(k = 1, 2, ..., n-1)$, *then*

$$\|f\|_{L^p(E)} \geq c(\gamma, n, ab, p) \cdot \|f\|_{L^p}$$

where $c(\gamma, n, ab, p) = \left(\frac{\gamma}{C}\right)^{ab\left(\frac{C}{\gamma}\right)^n + n - \frac{p-1}{p}}$.

Proof of Theorem 2′. Let $\hat{f}(x) = \sum\limits_{k=1}^n \hat{f}_k(x - \lambda_k)$ where $\operatorname{supp}\hat{f}_k \subset [-\frac{b}{2}, \frac{b}{2}]$ and $f(x) = \sum\limits_{k=1}^n f_k(x)e^{i\lambda_k x}$. The following lemma gives an estimate of $\|f_k\|_p$ from above.

Lemma 2.

$$\|f_k\|_p \leq C\|f\|_p \quad (k = 1, 2, ..., n). \tag{11}$$

Proof of Lemma 2. Let ϕ be a Schwartz function such that supp $\hat{\phi} \subset [-1, 1]$ and $\hat{\phi}(x) = 1$ for $x \in [-\frac{1}{2}, \frac{1}{2}]$. Then $\hat{f}_k(x) = \hat{f} \cdot \hat{\phi}(\frac{x-\lambda_k}{b})$. Therefore, $f_k = f * (b\phi(bx)e^{i\lambda_k x})$. Applying Young's inequality we have $\|f_k\|_p \leq \|f\|_p \cdot \|\phi\|_1$. ☐

We will also need the following auxiliary lemma:

Lemma 3. *If* $r(x) = \sum\limits_{k=1}^n p_k(x)e^{i\lambda_k x}$, *where* $p_k(x)$ *is a polynomial of degree* $\leq m-1$ *and* $E \subset I$ *is measurable with* $|E| > 0$, *then*

$$\|r\|_{L^p(I)} \leq \left(\frac{C|I|}{|E|}\right)^{nm - \frac{(p-1)}{p}} \cdot \|r\|_{L^p(E)}. \tag{12}$$

Proof of Lemma 3. First we prove the statement for pure trigonometric polynomials, i.e., if $g(x) = \sum\limits_{k=1}^{n} c_k e^{i\lambda_k x}$, then

$$\|g\|_{L^p(I)} \le \left(\frac{C|I|}{|E|}\right)^{n-\frac{(p-1)}{p}} \cdot \|g\|_{L^p(E)}. \tag{13}$$

This follows from a theorem on trigonometric polynomials by F. Nazarov ([4], Theorem 1.5) saying that

$$\|g\|_{L^\infty(I)} \le \left(\frac{C|I|}{|E|}\right)^{n-1} \cdot \|g\|_{L^\infty(E)}. \tag{14}$$

An argument similar to the proof of the Corollary to Lemma 1 shows that (13) follows from (14).

If $p(x) = \sum\limits_{l=0}^{m-1} a_l x^l$ is a polynomial of degree $m-1$, then it can be approximated uniformly on an interval with a trigonometric polynomial of order $\le m$

$$\tilde{p}(x) = \sum_{l=0}^{m-1} a_l \left(\frac{e^{i\lambda x} - 1}{i\lambda}\right)^l = \sum_{l=0}^{m-1} \tilde{a}_l e^{il\lambda x}$$

because $x = \lim\limits_{\lambda \to 0} \frac{e^{i\lambda x}-1}{i\lambda}$ uniformly on an interval. Applying (13) to the trigonometric polynomial of order mn,

$$\tilde{r}(x) = \sum_{k=1}^{n} \tilde{p}_k(x) e^{i\lambda_k x}$$

and taking the limit we have the desired result

$$\|r\|_{L^p(I)} \le \left(\frac{C|I|}{|E|}\right)^{nm-\frac{(p-1)}{p}} \cdot \|r\|_{L^p(E)}.$$

□

We will need the Taylor formula

$$\begin{aligned} g(x) &= \sum_{l=0}^{m-1} \frac{g^l(s)}{l!}(x-s)^l + \frac{1}{(m-1)!}\int_s^x g^{(m)}(t)(x-t)^{m-1}dt \\ &= p(x) + \frac{1}{(m-1)!}\int_s^x g^{(m)}(t)(x-t)^{m-1}dt \end{aligned}$$

where $p(x)$ is a polynomial of degree $m-1$.

Now we are in a position to proceed with the proof of Theorem 2′.

First we assume that $p \in [1, \infty)$. Divide the whole $\mathbb{R}$ into intervals of length a each. Consider one of them: $I = [s, s+a]$. Then

$$\begin{aligned} f(x) &= \sum_{k=1}^{n} f_k(x) e^{i\lambda_k x} \\ &= \sum_{k=1}^{n} p_k(x) e^{i\lambda_k x} + \frac{1}{(m-1)!} \sum_{k=1}^{n} e^{i\lambda_k x} \int\limits_s^x f_k^{(m)}(t)(x-t)^{m-1} dt \\ &= r(x) + T(x) \end{aligned}$$

Applying Holder's inequality we have

$$\begin{aligned} \int\limits_I |T(x)|^p dx &\leq \frac{n^{p-1}}{[(m-1)!]^p} \sum_{k=1}^{n} \int\limits_I |\int\limits_s^x f_k^{(m)}(t)(x-t)^{m-1} dt|^p dx \\ &\leq \frac{n^{p-1} a^{pm}}{[m!]^p} \sum_{k=1}^{n} \int\limits_I |f_k^{(m)}|^p. \end{aligned} \tag{15}$$

$$\begin{aligned} \int\limits_I |f|^p &\leq 2^{p-1} \int\limits_I |r|^p + 2^{p-1} \int\limits_I |T|^p \\ &\leq \left(\frac{C|I|}{|E \cap I|} \right)^{pnm-(p-1)} \cdot \int\limits_{E\cap I} |r|^p + 2^{p-1} \int\limits_I |T|^p \\ &\leq \left(\frac{C}{\gamma} \right)^{pnm-(p-1)} \cdot \left(2^{p-1} \int\limits_{E\cap I} |f|^p + 2^{p-1} \int\limits_{E\cap I} |T|^p \right) + 2^{p-1} \int\limits_I |T|^p \\ &\leq \left(\frac{C}{\gamma} \right)^{pnm-(p-1)} \cdot \int\limits_{E\cap I} |f|^p + \left(\frac{C}{\gamma} \right)^{pnm-(p-1)} \cdot \int\limits_I |T|^p \\ &\leq \left(\frac{C}{\gamma} \right)^{pnm-(p-1)} \cdot \int\limits_{E\cap I} |f|^p + \left(\frac{C}{\gamma} \right)^{pnm-(p-1)} \cdot \frac{n^{p-1} a^{pm}}{[m!]^p} \sum_{k=1}^{n} \int\limits_I |f_k^{(m)}|^p. \end{aligned}$$

The second inequality is based on Lemma 3. The last follows from (15).

Summing over all intervals I we have

$$\begin{aligned} \int |f|^p &\leq \left(\frac{C}{\gamma} \right)^{pnm-(p-1)} \cdot \int\limits_E |f|^p + \left(\frac{C}{\gamma} \right)^{pnm-(p-1)} \frac{n^{p-1} a^{pm}}{[m!]^p} \sum_{k=1}^{n} \int |f_k^{(m)}|^p \\ &\leq \left(\frac{C}{\gamma} \right)^{pnm-(p-1)} \cdot \int\limits_E |f|^p + \left(\frac{C}{\gamma} \right)^{pnm-(p-1)} \frac{n^{p-1} a^{pm} (Cb)^{pm}}{[m!]^p} \sum_{k=1}^{n} \int |f_k|^p \\ &\leq \left(\frac{C}{\gamma} \right)^{pnm-(p-1)} \cdot \int\limits_E |f|^p + \left(\frac{C}{\gamma} \right)^{pnm-(p-1)} \frac{n^p (ab)^{pm}}{[m!]^p} \int |f|^p \\ &\leq \left(\frac{C}{\gamma} \right)^{pnm-(p-1)} \cdot \int\limits_E |f|^p + \left(\frac{C}{\gamma} \right)^{pnm} \frac{(ab)^{pm}}{m^{pm}} \int |f|^p. \end{aligned}$$

The second inequality follows from Bernstein's Theorem. The third is an application of (11). The last inequality is due to Stirling's formula for $m!$ and the fact that $n \leq 2^n$.

Choose m such that it is a positive integer and $\left(\frac{C}{\gamma}\right)^n \frac{ab}{m} \leq \frac{1}{2}$, e.g., $m = 1 + [\left(\frac{C}{\gamma}\right)^n \cdot ab]$ for some $C > 0$. Therefore,

$$\begin{aligned} \int |f|^p &\leq \left(\frac{C}{\gamma}\right)^{pn(1+\left(\frac{C}{\gamma}\right)^n \cdot ab)-(p-1)} \cdot \int\limits_E |f|^p \\ &\leq \left(\frac{C}{\gamma}\right)^{p\left(\frac{C}{\gamma}\right)^n \cdot ab + pn - (p-1)} \cdot \int\limits_E |f|^p. \end{aligned}$$

The proof for $p = \infty$ is similar and even simpler. The proof of Theorem $2'$ is complete. □

Now we can proceed with the proof of Theorem 2. We will apply induction on n. For $n = 1$ the theorem follows from Theorem $2'$ or the usual Logvinenko-Sereda Theorem. Suppose the statement is true for $n \leq m$. Let $n = m + 1$.

If $\lambda_{k+1} - \lambda_k \geq 2b > 0$ $(k = 1, 2...)$, then the result follows from Theorem $2'$.

If $0 < \lambda_{k+1} - \lambda_k < 2b$ for some k, then we can replace b with $3b$ reducing the number of frequencies λ_k. Therefore, by induction

$$\begin{aligned} \|f\|_{L^p(E)} &\geq \left(\frac{C}{\gamma}\right)^{-3ba\left(\frac{C}{\gamma}\right)^m - m + \frac{p-1}{p}} \cdot \|f\|_p \\ &\geq \left(\frac{C}{\gamma}\right)^{-ab\left(\frac{C}{\gamma}\right)^{(m+1)} - (m+1) + \frac{p-1}{p}} \cdot \|f\|_p. \end{aligned}$$

The proof of Theorem 2 is complete. □

The purpose of this theorem is to prove the existence of a constant $c(\gamma, n, ab, p) > 0$ depending only on the number of intervals and not how they are placed rather than to get the best possible estimate.

Final remark. By a "thick" subset of $\mathbb{R}^d$ we mean a measurable set E for which there exist a parallelepiped Π with sides of length $a_1, a_2, ..., a_d$ parallel to coordinate axes and $\gamma > 0$ such that

$$|E \cap (\Pi + x)| \geq \gamma |\Pi| \tag{16}$$

for every $x \in \mathbb{R}^d$. Theorems 1 and 2 can be easily extended to higher dimensions with polynomial dependence on γ for the former. The proofs are analogous to the previous proofs. We can assume that Π is a unit cube. Define good cubes in a similar way. The main issue is how to obtain a local estimate for good cubes. If $|f|$ attains its maximum in a cube Π at $y \in \Pi$, then following an idea of F. Nazarov we can use spherical coordinates centered at y to find a segment I in Π such that $y \in I$ and $\frac{|E \cap I|}{|I|} \geq C(d)\gamma$, and reduce our problem to a 1-dimensional one. In case of Theorem 1 we can define an analytic function of one complex variable which coincides with f on I. In case of Theorem 2 we will approximate f on I with a polynomial defined on I.

Theorem 3. *Let J be a parallelepiped with sides of length $b_1, b_2, ..., b_d$ parallel to coordinate axes. If $f \in L^p(\mathbb{R}^d)$, $p \in [1, +\infty]$, and supp $\hat{f} \subset J$, and if a measurable, set E satisfies* (16), *then*

$$\|f\|_{L^p(E)} \geq \left(\frac{\gamma}{C^d}\right)^{C(d+\sum\limits_{k=1}^{d} a_k b_k)} \|f\|_p.$$

By an example similar to the one after Theorem 1 (with supp$\hat{f}$ in a neighborhood of a main diagonal of J with the direction of $\mathbf{b} = (b_1, ..., b_d)$ and E periodic along the same direction with period $\sim \mathbf{a} \cdot \mathbf{b}/|\mathbf{b}|$) we can show that this estimate is optimal except for the constant C.

Theorem 4. *Let J_l be parallelepipeds with sides of length $b_1, b_2, ..., b_d$ parallel to coordinate axes. If $f \in L^p(\mathbb{R}^d)$, $p \in [1, +\infty]$, and supp $\hat{f} \subset \bigcup\limits_{1}^{n} J_l$, and if a measurable set E satisfies* (16), *then*

$$\|f\|_{L^p(E)} \geq c(\gamma, n, \mathbf{a} \cdot \mathbf{b}, p, d)\|f\|_p$$

where $c(\gamma, n, \mathbf{a} \cdot \mathbf{b}, p, d) = \left(\frac{C^d}{\gamma}\right)^{-\left(\frac{C^d}{\gamma}\right)^n \cdot \sum\limits_{k=1}^{d} a_k b_k - n + \frac{p-1}{p}}$ *depends only on the number of parallelepipeds but not how they are placed.*

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References

[1] R.P. Boas, *Entire functions*, Academic Press Inc., New York, 1954. MR **16:**914f

[2] P. Borwein, T. Erdelyi, *Polynomials and polynomial inequalities*, Springer-Verlag, New York, 1995. MR **97e:**41001

[3] V. P. Havin, B. Joricke, *The Uncertainty Principle in Harmonic Analysis*, Springer-Verlag, Berlin Heidelberg, 1994. MR **96c:**42001

[4] F. L. Nazarov, *Local estimates of exponential polynomials and their application to inequalities of uncertainty principle type*, St. Petersburg Math. J. 5(1994), 663-717.

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