

LOCAL SOLVABILITY FOR POSITIVE COMBINATIONS
 OF GENERALIZED SUB-LAPLACIANS
 ON THE HEISENBERG GROUP

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ABSTRACT. As one step in a program to understand local solvability of complex coefficient second order differential operators on the Heisenberg group in a complete way, solvability of operators of the form $\Delta_{S,\alpha} = \Delta_S + i\alpha U$, where the leading term Δ_S is a “positive combination of generalized and degenerate generalized sub-Laplacians”, has been studied in a recent article by M. Peloso, F. Ricci and the first-named author (J. Reine Angew Math. **513** (1999)). It was shown that there exists a discrete set of “critical” values $E \subset \mathbb{C}$, such that solvability holds for $\alpha \notin E$. The case $\alpha \in E$ remained open, and it is the purpose of this note to close this gap. Our results extend corresponding results in another article by the above-mentioned authors (J. Funct. Anal. **148** (1997)), by means of an even simplified approach which should allow for further generalizations.

1. INTRODUCTION

Let ω denote the symplectic form on \mathbb{R}^{2n} given by

$$\omega(z, z') := {}^t z' J z, \quad J := \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.$$

The Heisenberg group \mathbb{H}_n is $\mathbb{R}^{2n} \times \mathbb{R}$, endowed with the group law

$$(1.1) \quad (z, u)(z', u') := (z + z', u + u' - \frac{1}{2}\omega(z, z')).$$

If we write $z = (x, y) = (x_1, \dots, x_n, y_1, \dots, y_n)$, a basis for the Lie algebra \mathfrak{h}_n of \mathbb{H}_n is given by the left-invariant vector fields

$$\begin{aligned} X_j &:= \frac{\partial}{\partial x_j} - \frac{1}{2}y_j \frac{\partial}{\partial u}, & j = 1, \dots, n, \\ Y_j &:= \frac{\partial}{\partial y_j} + \frac{1}{2}x_j \frac{\partial}{\partial u}, & j = 1, \dots, n, \\ U &:= \frac{\partial}{\partial u}. \end{aligned}$$

The non-trivial commutation relations among these vector fields are

$$[X_j, Y_j] = U, \quad j = 1, \dots, n.$$

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Denote by $\mathfrak{sp}(n, \mathbb{C})$ the symplectic Lie algebra, consisting of all complex $2n \times 2n$ -matrices S satisfying

$${}^tSJ + JS = 0.$$

Let $S \in \mathfrak{sp}(n, \mathbb{C})$. Following the notation of [5] and [4], we denote by $A = (a_{jk})$ the symmetric matrix $A := SJ$, and put

$$(1.2) \quad \Delta_S := \sum_{j,k=1}^{2n} a_{jk} V_j V_k,$$

where $V_j := X_j$, $V_{n+j} := Y_j$, $j = 1, \dots, n$.

In [4], the situation where the matrix S assumes a block diagonal form

$$(1.3) \quad S = \begin{pmatrix} \gamma_1 S_{(1)} & & & \\ & \gamma_2 S_{(2)} & & \\ & & \ddots & \\ & & & \gamma_m S_{(m)} \end{pmatrix}$$

with respect to a suitable decomposition of \mathbb{R}^{2n} into symplectic subspaces has been studied, under the assumptions that $\gamma_j \in \mathbb{C}^\times = \mathbb{C} \setminus \{0\}$ and $S_{(j)}^2 = -I$, $j = 1, \dots, m$. By means of Hörmander's criterion, it has been shown that for "most" of these matrices S , the operators $\Delta_S + \text{lower order terms}$ are locally non-solvable.

There are only five exceptional classes of operators of the above type to which Hörmander's criterion does not apply and which are listed in [4]. In all these classes, we may assume that each of the block matrices $S_{(j)}$ is of size 2×2 . Moreover, according to the classification of normal forms in [6], after applying a suitable symplectic change of coordinates, we may assume that $S_{(j)}$ is either of the form

$$(1.4) \quad S_{(j)} = \begin{pmatrix} i\varepsilon_j \lambda_j & \lambda_j^2 - 1 \\ 1 & -i\varepsilon_j \lambda_j \end{pmatrix}, \quad \text{"Type 1"}$$

with $\lambda_j \in \{-1\} \cup [0, \infty[$ and $\varepsilon_j = 1$ if $|\lambda_j| \leq 1$, and $\varepsilon_j = \pm 1$ if $\lambda_j > 1$, or of the form

$$S_{(j)} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \quad \text{"Type 3"}.$$

Observe that the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ can be conjugated into the matrix $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ by means of the real symplectic matrix $\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \in \text{Sp}(2, \mathbb{R})$. We may therefore assume as well that blocks of Type 3 are given by

$$(1.5) \quad S_{(j)} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix},$$

which turns out to be more convenient for our purposes.

Denote by σ_S the principal symbol of $-\Delta_S$. If we assume that $\text{Re } \sigma_S \geq 0$, then Δ_S belongs to one of the major exceptional classes listed in [4], and it follows from [4, Thm. 6.1] and (1.5) that Δ_S is of the form

$$(1.6) \quad \Delta_S = \sum_{j=1}^r \gamma_j [(1 - \lambda_j^2) X_j^2 + Y_j^2 + i\lambda_j (X_j Y_j + Y_j X_j)] + i \sum_{j=r+1}^n \gamma_j (X_j Y_j + Y_j X_j),$$

where $0 \leq r \leq n$, $|\lambda_j| \leq 1$, $\gamma_j \in \mathbb{C}^\times$ for $j = 1, \dots, r$, $\gamma_j > 0$ for $j = r + 1, \dots, n$, and where for each $j = 1, \dots, r$ and every $\xi_j, \eta_j \in \mathbb{R}$

$$(1.7) \quad \operatorname{Re} [\gamma_j[(1 - \lambda_j^2)\xi_j^2 + \eta_j^2 + 2i\lambda_j\xi_j\eta_j]] \geq 0,$$

provided we choose appropriate coordinates.

Let us assume in the sequel that Δ_S is of this form, and put $\Delta_{S,\alpha} := \Delta_S + i\alpha U$, $\alpha \in \mathbb{C}$. Set

$$E^\pm := \left\{ \pm \sum_{j=1}^r \gamma_j(2\ell_j + 1) : \ell_1, \dots, \ell_r \in \mathbb{N} \right\},$$

and put $E := E^+ \cup E^-$. Moreover, denote by n_1, n_2^+ and n_2^- the number of ‘‘Type 1’’ blocks $S_{(j)}$ in S with $|\lambda_j| < 1$, $\lambda_j = 1$ and $\lambda_j = -1$, respectively, and by n_3 the number of ‘‘Type 3’’ blocks. We shall prove the following extension of [3, Thm. 3.2] and [4, Thms. 7.1, 7.9].

Theorem 1.1. *Let Δ_S be as described by (1.6) and (1.7), and assume that $\operatorname{Re} \gamma_j > 0$ for $j = 1, \dots, r$. Then E is discrete in \mathbb{C} , and the following holds:*

- (i) *If $\alpha \notin E$, then $\Delta_{S,\alpha}$ is locally solvable.*
- (ii) *If $\alpha \in E^-$, then $\Delta_{S,\alpha}$ is locally solvable if and only if $n_2^+ + n_3 > 0$.*
- (iii) *If $\alpha \in E^+$, then $\Delta_{S,\alpha}$ is locally solvable if and only if $n_2^- + n_3 > 0$.*

Remark 1.2. (1.7) implies that $\operatorname{Re} \gamma_j \geq 0$ for $j = 1, \dots, r$. Moreover, if $\operatorname{Re} \gamma_j = 0$, then $\lambda_j = 0$, again by (1.7). Therefore, if $\operatorname{Re} \gamma_j = 0$ for every j , then Δ_S is a multiple of a real-coefficient operator, and local solvability of $\Delta_{S,\alpha}$ can in this case be described by means of [5]. Notice that in this case the exceptional set may be non-discrete. We do not know at present what will happen if $\operatorname{Re} \gamma_j = 0$ only for some $j \in \{1, \dots, r\}$.

2. PROOF OF THEOREM 1.1

The negative results are proved in the same way as Theorem 7.1 in [4]. For instance, if $\alpha \in E^+$ and $n_2^- = n_3 = 0$, then

$$\Delta_S = \sum_{j=1}^n \gamma_j \Delta_{S_{(j)}} = \sum_{j=1}^n \gamma_j [(1 - \lambda_j^2)X_j^2 + Y_j^2 + i\lambda_j(X_j Y_j + Y_j X_j)]$$

where $|\lambda_j| < 1$ or $\lambda_j = 1$. Let π_μ denote the Schrödinger representation of \mathbb{H}_n with parameter $\mu \in \mathbb{R}^\times$, acting on the Hilbert space $L^2(\mathbb{R}^n)$.

If $|\lambda_j| < 1$, then there exist Schwartz functions $\varphi_{\ell_j}^\mu = \varphi_{\ell_j}^\mu(t_j)$ on \mathbb{R} such that

$$d\pi_\mu(\Delta_{S_{(j)}})\varphi_{\ell_j}^\mu = -|\mu|(2\ell_j + 1)\varphi_{\ell_j}^\mu$$

for every $\ell_j \in \mathbb{N}$ (compare [4, Prop. 7.2]).

And, if $\lambda_j = 1$, then

$$d\pi_\mu(\Delta_{S_{(j)}} - iU) = -\mu t_j \left(2 \frac{\partial}{\partial t_j} + \mu t_j \right);$$

hence

$$d\pi_\mu(\Delta_{S_{(j)}} - iU)e^{-\mu t_j^2/4} = 0.$$

Choosing $\mu > 0$ and putting $\varphi_0^\mu(t_j) := e^{-\mu t_j^2/4}$, we see that $\varphi_0^\mu \in \mathcal{S}(\mathbb{R})$ and

$$d\pi_\mu(\Delta_{S_{(j)}} - iU)\varphi_0^\mu = 0.$$

More generally, following [3], define operators $A := -\mu t_j$, $B := 2\frac{\partial}{\partial t_j} + \mu t_j$, and put $\varphi_{\ell_j}^\mu := A^{\ell_j} \varphi_0^\mu$. Since $d\pi_\mu(\Delta_{S(j)} - iU) = AB$ and $[B, A] = -2\mu$, by induction over $\ell_j \in \mathbb{N}$ one obtains

$$d\pi_\mu(\Delta_{S(j)} - iU)\varphi_{\ell_j}^\mu = -\mu 2\ell_j \varphi_{\ell_j}^\mu;$$

hence

$$d\pi_\mu(\Delta_{S(j)})\varphi_{\ell_j}^\mu = -|\mu|(2\ell_j + 1)\varphi_{\ell_j}^\mu,$$

where $\varphi_{\ell_j}^\mu \in \mathcal{S}(\mathbb{R})$. Consequently, if we define $\varphi_\ell^\mu(t_1, \dots, t_n) := \varphi_{\ell_1}^\mu(t_1) \dots \varphi_{\ell_n}^\mu(t_n)$, for $\ell = (\ell_1, \dots, \ell_n) \in \mathbb{N}^n$, then $\varphi_\ell^\mu \in \mathcal{S}(\mathbb{R}^n)$, and

$$d\pi_\mu({}^t\Delta_{S,\alpha})\varphi_\ell^\mu = -|\mu|(\sum_{j=1}^n \gamma_j(2\ell_j + 1) - \alpha)\varphi_\ell^\mu,$$

provided $\mu > 0$. This shows that $d\pi_\mu({}^t\Delta_{S,\alpha})$ annihilates a non-trivial Schwartz function; hence $\Delta_{S,\alpha}$ is locally non-solvable.

Consider next the case $\alpha \notin E$. Put

$$A := \sum_j \operatorname{Re} \gamma_j, \quad a := \min_j \operatorname{Re} \gamma_j,$$

and define

$$(2.1) \quad \Gamma_{t,S}^\mu(z) := \frac{|\mu|^n}{(4\pi)^n \prod_{j=1}^n \sinh(\gamma_j t)} e^{-\frac{|\mu|}{4}(\sum_{j=1}^n \coth(\gamma_j t) {}^t z_j J_1 S(j) z_j)}$$

for $z = (z_1, \dots, z_n) \in \mathbb{R}^{2n}$, where $z_j = (x_j, y_j)$ and $J_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. It follows from the arguments in [4, Section 7] that for $\alpha \in \mathbb{C}$ with $|\operatorname{Re} \alpha| < A$, a tempered distribution F_α is defined on \mathbb{H}_n by means of the formula

$$(2.2) \quad \langle F_\alpha, \varphi \rangle := -\frac{1}{2\pi} \int_0^\infty \int_{\mathbb{R}^\times} e^{-\alpha(\operatorname{sgn} \mu)t} \langle \Gamma_{t,S}^\mu, \varphi^{-\mu} \rangle \frac{i\mu}{|\mu|} d\mu dt, \quad \varphi \in \mathcal{S}(\mathbb{H}_n),$$

which satisfies

$$(2.3) \quad \Delta_{S,\alpha} F_\alpha = U\delta_0.$$

Here, $\langle \cdot, \cdot \rangle$ denotes the pairing between \mathcal{S}' and \mathcal{S} , and φ^μ the partial Fourier transform

$$\varphi^\mu(z) := \int_{\mathbb{R}} \varphi(z, u) e^{-i\mu u} du, \quad \mu \in \mathbb{R},$$

of φ along the center.

Moreover, the family of distributions $\{F_\alpha\}_{|\operatorname{Re} \alpha| < A}$ is analytic.

As for values of α with $|\operatorname{Re} \alpha| > A$, fix $m \in \mathbb{N}$ and put $G_\alpha = G_{\alpha,m} := U^m F_\alpha$, first for $|\operatorname{Re} \alpha| < A$, i.e.

$$(2.4) \quad \langle G_\alpha, \varphi \rangle = -\frac{1}{2\pi} \int_0^\infty \int_{\mathbb{R}^\times} e^{-\alpha(\operatorname{sgn} \mu)t} \langle \Gamma_{t,S}^\mu, \varphi^{-\mu} \rangle \frac{(i\mu)^{m+1}}{|\mu|} d\mu dt.$$

The arguments in [4, Section 7] can easily be adapted to prove the following:

Proposition 2.1. *The family of distributions G_α admits an analytic extension to a family of distributions $\{G_\alpha\}_{\alpha \in D_m \setminus E}$, where D_m denotes the domain $D_m := \{\alpha \in \mathbb{C} : |\operatorname{Re} \alpha| < A + 2ma\}$. Moreover,*

$$(2.5) \quad \Delta_{S,\alpha} G_\alpha = U^m \delta_0.$$

This result implies in particular that $\Delta_{S,\alpha}$ is locally solvable for $\alpha \notin E$.

There remains the case where $\alpha \in E^\pm$. Then choose $m \in \mathbb{N}$ sufficiently large, so that $\alpha \in D_m$. For $\beta \in D_m \setminus E$ sufficiently close to α , we have

$$(2.6) \quad \Delta_{S,\alpha}G_\beta = U^{m+1}\delta_0 + i(\alpha - \beta)UG_\beta.$$

The crucial point in proving local solvability for $\Delta_{S,\alpha}$ is the following:

Lemma 2.2. *If $\alpha \in E^\pm$ and $n_2^\mp + n_3 > 0$, then there exists a right-invariant locally solvable differential operator $D^{(r)}$ on \mathbb{H}_n , such that the family of distributions $\{D^{(r)}G_\beta\}_{\beta \in D_m \setminus E}$ extends analytically into the point α .*

In fact, given this lemma, we may adapt the proof of [3, Lemma 2.2] as follows. Define the distribution K_α by $\langle K_\alpha, \varphi \rangle := \lim_{\beta \rightarrow \alpha} \langle D^{(r)}G_\beta, \varphi \rangle$. Since, according to (2.6),

$$\Delta_{S,\alpha}(D^{(r)}G_\beta) = D^{(r)}(\Delta_{S,\alpha}G_\beta) = U^{m+1}D^{(r)}\delta_0 + i(\alpha - \beta)U(D^{(r)}G_\beta),$$

we obtain in the limit as $\beta \in D_m \setminus E$ tends to α (taken in the distributional sense) that

$$\Delta_{S,\alpha}K_\alpha = U^{m+1}D^{(r)}\delta_0.$$

Since U^{m+1} and $D^{(r)}$ are locally solvable, this implies local solvability of $\Delta_{S,\alpha}$.

There remains to prove Lemma 2.2. To this end, let us first consider G_β for $|\operatorname{Re} \beta| < A$, so that G_β is given by

$$(2.7) \quad \langle G_\beta, \varphi \rangle = -\frac{1}{2\pi} \int_0^\infty \int_{\mathbb{R}^\times} e^{-\beta(\operatorname{sgn} \mu)t} \langle \Gamma_{t,S}^\mu, \varphi^{-\mu} \rangle \frac{(i\mu)^{m+1}}{|\mu|} d\mu dt.$$

Moreover, if we put $\check{g}(z) := g(-z)$, then

$$(2.8) \quad \langle \Gamma_{t,S}^\mu, \varphi^{-\mu} \rangle = \langle \widehat{\Gamma}_{t,S}^\mu, \widehat{\check{\varphi}}^{-\mu} \rangle,$$

where by [4] $\widehat{\Gamma}_{t,S}^\mu$ is given by

$$(2.9) \quad \widehat{\Gamma}_{t,S}^\mu(\zeta) = \frac{1}{\sigma(t)} e^{-\frac{1}{|\mu|}q_t(\zeta)},$$

with

$$(2.10) \quad \begin{aligned} \sigma(t) &:= \prod_{j=1}^n \cosh(\gamma_j t), \\ q_t(\zeta) &:= \sum_{j=1}^r \tanh(\gamma_j t) [(1 - \lambda_j^2)\xi_j^2 + \eta_j^2 + 2i\lambda_j \xi_j \eta_j] + \sum_{j=r+1}^n \tanh(\gamma_j t) 2i\xi_j \eta_j, \end{aligned}$$

if $\zeta_j = (\xi_j, \eta_j)$.

Consider the right-invariant real vector fields

$$X_j^{(r)} := \frac{\partial}{\partial x_j} + \frac{1}{2}y_j \frac{\partial}{\partial u}, \quad Y_j^{(r)} := \frac{\partial}{\partial y_j} - \frac{1}{2}x_j \frac{\partial}{\partial u},$$

and put

$$q_j(\xi_j, \eta_j) := \begin{cases} (1 - \lambda_j^2)\xi_j^2 + \eta_j^2 + 2i\lambda_j \xi_j \eta_j & \text{if } j = 1, \dots, r, \\ 2i\xi_j \eta_j & \text{if } j = r + 1, \dots, n. \end{cases}$$

For $k \in \mathbb{N}$, by (2.8) we have

$$\begin{aligned} \langle \Gamma_{t,S}^\mu, ((-X_j^{(r)})^k \varphi)^{-\mu} \rangle &= \langle (\frac{\partial}{\partial x_j} + i\frac{\mu}{2}y_j)^k \Gamma_{t,S}^\mu, \varphi^{-\mu} \rangle \\ &= \langle (-\frac{\mu}{2} \frac{\partial}{\partial \eta_j} + i\xi_j)^k \widehat{\Gamma}_{t,S}^\mu, \widehat{\varphi}^{-\mu} \rangle. \end{aligned}$$

And, if $j \geq r + 1$, then $-\frac{\mu}{2} \frac{\partial}{\partial \eta_j} q_j = -i\mu\xi_j$, so that

$$(2.11) \quad (-\frac{\mu}{2} \frac{\partial}{\partial \eta_j} + i\xi_j)^k \widehat{\Gamma}_{t,S}^\mu = (i\xi_j)^k (1 + (\operatorname{sgn} \mu) \tanh(\gamma_j t))^k \widehat{\Gamma}_{t,S}^\mu.$$

Similarly,

$$\langle \Gamma_{t,S}^\mu, ((-Y_j^{(r)})^k \varphi)^{-\mu} \rangle = \langle (\frac{\mu}{2} \frac{\partial}{\partial \xi_j} + i\eta_j)^k \widehat{\Gamma}_{t,S}^\mu, \widehat{\varphi}^{-\mu} \rangle,$$

and $\frac{\mu}{2} \frac{\partial}{\partial \xi_j} q_j = i\mu\lambda_j\eta_j$, if $j \leq r$ and $|\lambda_j| = 1$, $\frac{\mu}{2} \frac{\partial}{\partial \xi_j} q_j = i\mu\eta_j$, if $j \geq r + 1$, so that

$$(2.12) \quad \begin{aligned} &(\frac{\mu}{2} \frac{\partial}{\partial \xi_j} + i\eta_j)^k \widehat{\Gamma}_{t,S}^\mu \\ &= \begin{cases} (i\eta_j)^k (1 - \lambda_j (\operatorname{sgn} \mu) \tanh(\gamma_j t))^k \widehat{\Gamma}_{t,S}^\mu & \text{if } j \leq r, |\lambda_j| = 1, \\ (i\eta_j)^k (1 - (\operatorname{sgn} \mu) \tanh(\gamma_j t))^k \widehat{\Gamma}_{t,S}^\mu & \text{if } j \geq r + 1. \end{cases} \end{aligned}$$

Now assume first that $n_3 > 0$, so that $r < n$. By (2.10), we have

$$|e^{-\beta(\operatorname{sgn} \mu)t} \widehat{\Gamma}_{t,S}^\mu(\zeta)| \leq 2^n e^{-(\operatorname{Re} \beta(\operatorname{sgn} \mu) + A)t},$$

so that by (2.11),

$$(2.13) \quad \begin{aligned} &|e^{-\beta(\operatorname{sgn} \mu)t} \langle \Gamma_{t,S}^\mu, ((-X_n^{(r)})^k \varphi)^{-\mu} \rangle| \\ &\leq \begin{cases} e^{(-\operatorname{Re} \beta - A)t} \|\varphi^{-\mu}\|_{\mathcal{S}} & \text{if } \mu > 0, \\ e^{(\operatorname{Re} \beta - A - 2k\gamma_n)t} \|\varphi^{-\mu}\|_{\mathcal{S}} & \text{if } \mu < 0, \end{cases} \end{aligned}$$

where $\|\cdot\|_{\mathcal{S}}$ denotes a suitable continuous norm on $\mathcal{S}(\mathbb{R}^{2n})$. Since $\langle (X_n^{(r)})^k G_\beta, \varphi \rangle = \langle G_\beta, (-X_n^{(r)})^k \varphi \rangle$, this shows that the family of distributions $D^{(r)} G_\beta$, with $D^{(r)} := (X_n^{(r)})^k$, extends analytically to the region $-A < \operatorname{Re} \beta < A + 2k\gamma_n$.

Similarly, from (2.12) we find that $D^{(r)} G_\beta$ extends analytically to the region $-A - 2k\gamma_n < \operatorname{Re} \beta < A$, if we choose $D^{(r)} := (Y_n^{(r)})^k$. This proves Lemma 2.2 for the case $n_3 > 0$.

Assume finally that $n_3 = 0$, and that $\alpha \in E^+$ and $n_2^- > 0$ (the case $\alpha \in E^-$ and $n_2^+ > 0$ can be treated in an analogous way). We may then assume that $\lambda_n = -1$.

Choosing $D^{(r)} := (Y_n^{(r)})^k$ and arguing as before, we deduce from (2.12) that we get the analogue to estimate (2.13), only with $X_n^{(r)}$ replaced by $Y_n^{(r)}$. Thus, the family of distributions $D^{(r)} G_\beta$ extends analytically to the region $-A < \operatorname{Re} \beta < A + 2k \operatorname{Re} \gamma_n$, which contains the point α , if k is sufficiently large.

This concludes the proof of Lemma 2.2, and hence of Theorem 1.1.

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